Representing the geometry of domains by wavelets with applications to partial differential equations

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## ABSTRACT

## 1 INTRODUCTION

The recently introduced compactly supported wavelets of Daubechies have proven to be very useful in various aspects of signal processing, notably in image compression (see, e.g., [23]). The fundamental idea in the application to image compression was to take a digitized image and to use wavelets to provide a multiscale representation of it, and to then discard some information at some scales, and leave information more intact at other scales. The Daubechies wavelets have differentiability properties in addition to their compact support and orthogonality properties. A number of authors have used such wavelet systems for solving various problems in differential and integral equations (see [7,12,2,1,19,17]).

An additional point of view for approaching boundary value problems in partial differential equations was introduced in [18], in which the boundary, the boundary data, and the unknown solution (of a boundary value problem) on the interior of a domain are all uniformly represented in terms of compactly supported wavelet functions in an extrinsic ambient Euclidean space. In this paper we describe multiscale representations of domains and their boundaries and obtain multiscale representations of some of the basic elements of geometric calculus (line integrals, surface measures, etc.) which are then in turn useful for specific numerical calculations in problems in approximate solutions of differential equations. This is similar to the spectral method in solving differential equations (essentially using an orthonormal expansion), but here we use the localization property of wavelets to extend this orthonormal representation to boundary data and geometric boundaries. In this paper we want to indicate how to carry this out.

We illustrate this principle for the Dirichlet problem for a classical elliptic differential equation:

$$\begin{aligned} -\Delta u + u &= f \text{ in } \Omega, \\ u|_{\partial\Omega} &= g, \end{aligned} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbf{R}^n$, with a boundary $\partial\Omega$ satisfying a suitable regularity condition and where $f$ and $g$ are suitable given functions. The basic principle divides into several basic points:

1. Represent the geometric region for the boundary value problem in terms of wavelets. For this we use the characteristic function of the domain $\chi_\Omega$ and let $\nabla\chi_\Omega$ represent its boundary as a measure on $\mathbf{R}^n$. Both of these can be represented in terms of a wavelet series, using differentiable wavelets and connection coefficients (the latter are useful for calculating derivatives of wavelet expansions, see [11]).

2. Represent the functions defined on the boundary and on the interior of the region in terms of wavelet series defined in a rectangular region containing the domain. For this we observe that approximate solutions of differential equations are restrictions of functions defined on all of $\mathbf{R}^n$, while any function defined on the boundary in some smoothness class admits extensions (not unique) to the ambient space.

3. Convert the differential equation to some weak form extended to the larger rectangular region (fictitious domain approach), involving integration over the interior and along the boundary, all of which are to be represented in terms of wavelets.

4. Formulate and solve a multiscale wavelet-Galerkin problem for the domain and the differential equation, using localized wavelets as an orthonormal basis, and representing both the unknown functions and the test functions in terms of the same basis.

In this paper we want to indicate how these steps can be carried out in a somewhat generic fashion, and will close with a summary of some explicit results which have been carried out in a series of longer papers. In particular we want to indicate how the multiscale representation can be used at different scales in different parts of the problem, and how the Mallat algorithm allows one to go from one scale to another with ease.

Our basic computational strategy in approaching a boundary value problem from a wavelet multiscale point of view involves the following basic steps (borrowed from image processing!).

1. Scan the given data and the characterisic function of the domain at the finest scale available in the problem (the *pixel scale*). Using connection coefficients at this fine scale, compute the gradient of the characteristic function in terms of wavelets. This is a one-time calculation independent of the differential equation system.

2. Use the Mallat algorithm to convert the data to coarser intermediate scales, when needed. The coarser scales correspond to finite element scales or the mesh size of a finite difference scheme ("$h$"), at which we solve a (possibly in a multiscale or multigrid version) wavelet-Galerkin problem (keeping as much fine-scale information as possible).

3. Use the multiscale property of the data and the problem representation to develop algorithms which do more intensive calculations at a range of scales where this is necessary (localized in either space or time), and choose scales where the calculations are less intensive when this is possible (this part is analogous to the quantization process in image compression algorithms).

4. Use the Mallat algorithm to go back to the fine (pixel) scale from the intermediate scales where calculations have been made, and use this scale to calculate errors or to represent the solution numerically or graphically.

This process is similar to image processing, in that all of the "image" (the complete boundary value problem in this case) is represented in terms of wavelets, and these are then processed, and then the wavelet representation is converted back to the normal functional representation at the scanning or pixel level. For representation of video images, it is the coarser scale information which is manipulated ("quantized" in the case of image compression, i.e., some of the wavelet coefficients are thrown out and a new image is constructed from the remainder). Here we have to solve for new coefficients by using the equations which arise, instead of simply throwing some of the coefficients away in a systematic fashion. By using connection coefficients again, a nonlinear differential equation is reduced to a system of nonlinear algebraic equations which can be solved by iterative methods. Moreover, the wavelet structure allows for a natural multigrid solution of the corresponding algebraic equations.

One important property of this numerical scheme is that the coding for the solution of the boundary value problem is *independent* of the geometry of the boundary. The geometry of the boundary is encoded in the scanning and digitizing at the pixel scale of the geometric region under consideration; this is a simple input to the problem, and it can be changed as simply as the boundary data itself can be changed. Normally in finite element approaches to such a boundary value problem, one has a finite element grid which is *adapted* to the boundary. This is not necessary for this scheme, which gives it great flexibility in applications.

In Section 2 we find a review and a summary of the basic information about wavelets which we need for this paper. In Section 3.1 we give some of the theoretical background for the geometric-measure theoretic objects which are of interest in this paper. This includes arc-length (more generally surface measure in the higher-dimensional case) as well as Radon measures representing integration of a function over the same boundary. In Section 3.2 we find wavelet representations of these same objects, using the characteristic function of the domain as the principal way in which the geometry is expressed. We show that natural truncations of these wavelet representation formulas converge at specific and calculable rates. Finally, in Section 4 we give examples of a number of different solution methodologies for numerical solutions of specific partial differential equations which have been formulated and solved by these methods.

## 2 WAVELET MULTISCALE REPRESENTATION OF FUNCTIONS

In this section we will briefly recall the Daubechies compactly supported wavelets which are differentiable and which will be used in this paper. Let $g$ be a positive integer, and consider a $2 \times 2g$ matrix of the form

$$a = \begin{pmatrix} a_0^0 & \dots & a_{2g-1}^0 \\ a_0^1 & \dots & a_{2g-1}^1 \end{pmatrix}, \tag{2.2}$$

where

$$\sum_{k=0}^{2g-1} a_k^r a_{k+2l}^s = 2\delta_{l,0}\delta^{r,s}, \tag{2.3}$$

$$\sum_{k=0}^{2g-1} a_k^r = 2\delta^{r,0}, \tag{2.4}$$

where

$$\delta^{l,m} = \delta_{l,m} = \begin{cases} 0 & l \neq m \\ 1 & l = m, \quad l, m \in \mathbf{Z} \end{cases}$$

is the Kronecker delta function. This is a *wavelet matrix* of *rank* 2 and *genus* $g$ (see [10]). By choosing

$$a_k^1 = (-1)^{k+1} a_{2g-1-k}^0,$$

we see that the orthogonality conditions in (2.3) and (2.4) are satisfied if they are satisfied for the first row. Given any wavelet matrix of the form (2.2), we can find compactly supported scaling functions and wavelet functions which are in $L^2(\mathbf{R})$ and which satisfy

$$\varphi(x) = \sum_{k=0}^{2g-1} a_k^0 \varphi(2x-k),$$

$$\psi(x) = \sum_{k=0}^{2g-1} a_k^1 \varphi(2x-k).$$

The function $\varphi(x)$ is called the *scaling function* associated with $a$, and $\psi(x)$ is called the *wavelet function* associated with $a$. There is a $(g-1)$-dimensional parameter space of choice of wavelet matrices ([16], [20]). The functions $\varphi$ and $\psi$ both have support in the interval $[0, 2g-1]$, and we define the *wavelet system* associated with $a$ by

$$\begin{aligned} \varphi_k^j(x) &:= 2^{\frac{j}{2}} \varphi(2^j x - k), \\ \psi_k^j(x) &:= 2^{\frac{j}{2}} \psi(2^j x - k), \end{aligned} \tag{2.5}$$

where $k \in \mathbf{Z}$, $j \in \mathbf{Z}^+$ (the nonnegative integers). The fundamental theorem of Daubechies and Lawton is that

$$\varphi_k^0(x), \psi_k^j(x), \quad k \in \mathbf{Z}, j \in \mathbf{Z}^+, \tag{2.6}$$

is a tight frame for $L^2(\mathbf{R})$ for any wavelet matrix $a$ (see [3], [13]), i.e., any $f \in L^2(\mathbf{R})$ has an $L^2$ expansion of the form

$$f = \sum f_k \varphi_k^0(x) + \sum f_k^j \psi_k^j(x), \tag{2.7}$$

where

$$f_k = \int f(x)\varphi_k^0(x)dx, \qquad f_k^j = \int f(x)\psi_k^j(x)dx. \tag{2.8}$$

For a generic wavelet matrix, (2.6) is an orthonormal system, and (2.7) is the usual orthonormal expansion (cf. Lawton [14]).

Daubechies chooses a very specific wavelet matrix $a_{Dg}$, for each genus $g \geq 1$, which maximizes the vanishing moments of $\psi$, i.e., she chooses $a$ so that

$$\int_{\mathbf{R}} x^n \psi(x)dx = 0, \quad m = 0, \ldots, g-1.$$

For each $g$, there are only a finite number of such choices, and we use the choice made by Daubechies. Her choice leads to a specific countable family of scaling functions and wavelet functions

$$\varphi_{Dg}, \psi_{Dg}, \quad g = 1, 2, \ldots,$$

with a number of properties. In Daubechies' paper she denoted these functions by ${}_N\varphi(x)$, ${}_N\psi(x)$, where $N = g$. We call $\varphi_{Dg}$ the *Daubechies scaling function* of genus $g$, and $\psi_{Dg}$ the *Daubechies wavelet function* of genus $g$, for $g \in \mathbf{N}$ (the positive integers). For simplicity we let $\varphi$ and $\psi$ denote the Daubechies scaling and wavelet functions throughout the remainder of this paper, and the corresponding wavelet system is defined by (2.5) (we choose the support of $\varphi(x)$ and $\psi(x)$ to be $[0, 2g-1]$). In this case (2.6) is an orthonormal basis for $L^2(\mathbf{R})$, and (2.7) and (2.8) hold as before. In Daubechies' new book [4] is a thorough discussion of all of these issues. In particular, the Daubechies wavelets have specific differentiability properties, namely, $\varphi \in C^{\alpha(g)}$, where $\alpha$ is affine linear, and for $g \geq 3$, $\varphi \in C^1$, which is all we will need in this paper.

The Daubechies wavelet system gives a specific multiresolution analysis of $L^2(\mathbf{R})$ (cf. Mallat [15]). Namely, if we let (in the topology of $L^2(\mathbf{R})$)

$$V_j := \text{closure } (\text{span } \{\varphi_k^j : k \in \mathbf{Z}\}), \tag{2.9}$$

and let $W_j$ be the orthogonal complement of $V_j$ in $V_{j+1}$, then for fixed $j \in \mathbf{Z}$, $\{\varphi_k^j\}_{k\in\mathbf{Z}}$, is an orthonormal basis for $V_j$, and $\{\psi_k^j\}_{k\in\mathbf{Z}}$, is an orthonormal basis for $W_j$. Moreover, for any $j \in \mathbf{Z}$,

$$L^2(\mathbf{R}) = V_j \oplus \bigoplus_{n\geq j} W_n = \bigoplus_{n\in\mathbf{Z}} W_n.$$

In addition, if we let $P_j$ be the orthonormal projection $L^2(\mathbf{R}) \xrightarrow{P_j} V_j$, then we see that

$$\begin{aligned} \bigcup_{j\in\mathbf{Z}} V_j &= L^2(\mathbf{R}), \\ \bigcap_{j\in\mathbf{Z}} V_j &= 0, \end{aligned}$$

in the sense that

$$\begin{aligned} \lim_{j\to\infty} P_j f &= f, \\ \lim_{j\to-\infty} P_j f &= 0. \end{aligned}$$

The Mallat identity asserts that for a fixed *level* $J$ (corresponding to a scale width of $2^{-J}$),

$$V_J = V_0 \oplus W_0 \oplus \cdots \oplus W_{J-1},$$

where $W_j$ is the orthogonal complement of $V_j$ in $V_{j-1}$, and one has that $W_j$ are spanned by $\psi_k^j$. The formulas relating these two representations are given by ([15], see also [22]): if we assume an expansion of the form

$$f^j(x) = \sum_{k\in\mathbf{Z}} f_k^{j,0} \varphi_k^j(x) \in V_j, \tag{2.10}$$

(where $f_*^{*,0}$ denotes coefficients of $\varphi$ and $f_*^{*1}$ denotes coefficients of $\psi$),

$$f^j(x) = \sum_{k\in\mathbf{Z}} f_k^{0,0} \varphi_k(x) + \sum_{i=0}^{j-1} \sum_{k\in\mathbf{Z}} f_k^{i,1} \psi_{ik}(x) \in V_0 \oplus W_0 \oplus \ldots + W_{j-1}, \tag{2.11}$$

where

$$\begin{aligned} f_k^{i,0} &= \frac{1}{\sqrt{2}} \sum_l f_l^{i+1,0} a_{l-2k}, \quad 0 \leq i \leq j, \\ f_k^{i,1} &= \frac{1}{\sqrt{2}} \sum_l f_l^{i+1,0} b_{l-2k}, \quad 0 \leq i \leq j. \end{aligned} \tag{2.12}$$

Conversely, if we know the coefficients of the expansion (2.11), then the coefficients in (2.10) are determined recursively by

$$f_k^{i,0} = \frac{1}{\sqrt{2}} \sum_l \left( f_l^{i-1,0} a_{k-2l} + f_L^{i-1,1} b_{k-2l} \right), \quad 0 \leq i \leq j. \tag{2.13}$$

Tensor products of the $L^2$ basis $\varphi_n(x), \psi_k^j(x)$ are used in the 2-dimensional setting, and tensor products of these formulas are used as well.

## 3 REPRESENTATION OF DOMAINS AND BOUNDARIES BY WAVELETS

### 3.1 Elements of Geometric Measure Theory

In this subsection, we will briefly review some basic measure-theoretic properties concerning sets of finite perimeter. Readers are referred to [5] or [6] for more details on geometric measure theory. The properties developed will include specific formulas from geometric measure theory for boundary integrals and related quantities. In the next subsection we will represent these same quantities in terms of wavelets.

We will say that a Lebesgue integrable function $u$ is of *bounded variation* over a domain $U \subset \mathbf{R}^n$ if

$$\sup_{g \in C_0^1(U, \mathbf{R}^n), |g| \leq 1} \int_U u \mathrm{div} g \, dx_1 \ldots dx_n \leq +\infty.$$

Suppose that $u$ is of bounded variation in $U \subset \mathbf{R}^n$. Then, using the Reisz representation theorem, there exist a Radon measure $\mu$ and a $\mu$-measurable function $\nu$ with $|\nu(x)| = 1$, for $\mu$-almost every $x \in U$, so that, for any $g \in C_0^1(U, \mathbf{R}^n)$,

$$\int_U u \mathrm{div} g \, dx_1 \ldots dx_n = \int_U g \cdot \nu \, d\mu.$$

We will denote by $|\nabla u|$ the Radon measure $\mu$ and by $\nabla u$ the vector-valued measure $\nu \cdot \mu$. Note that if $u$ is continuously differentiable, then $\mu$ is the 2-dimensional Lebesgue measure weighted by the norm of the gradient of $u$ and $\nu = \dfrac{\nabla u}{|\nabla u|}$.

We will say that an open set $\Omega$ is of *finite perimeter in $U$* if its characteristic function $\chi_\Omega$ is of bounded variation in $U$. In particular, we say $\Omega$ is of *finite perimeter* if $\chi_\Omega$ is a function of bounded variation in $\mathbf{R}^n$.

**Proposition 3.1** *For any open set $U$, if $\Omega$ is of finite perimeter in $U$, then*

$$\int_U |\nabla \chi_\Omega| = \sup_{g \in C_0^1(U, \mathbf{R}^n), |g| \leq 1} \int_\Omega \mathrm{div} g \, dx_1 \ldots dx_n. \tag{3.1}$$

*In particular, if $\partial\Omega$ is piecewise smooth, then*

$$\int_U |\nabla \chi_\Omega| = A(\partial\Omega \cap U), \tag{3.2}$$

*where $A(\partial\Omega \cap U)$ is the $n-1$-dimensional area (volume) of $\partial\Omega$ inside $U$ (for $n = 2$, $A(\partial\Omega \cap U)$ is the arc length of $\partial\Omega \cap U$).*

Again, by the Reisz representation theorem, there exists a vector-valued $|\nabla \chi_\Omega|$-measurable function $\nu$ with $|\nu| = 1$, $|\nabla \chi_\Omega|$-almost everywhere. In the case that $\partial\Omega$ is a level set of some Lipschitz function $F$, then $\nu = \dfrac{\nabla F}{|\nabla F|}$ almost everywhere along the boundary $\partial\Omega$, and $\int \nu \cdot \nabla \chi_\Omega$ is the area of $\partial\Omega$.

Assume $\Omega$ is a bounded domain in $\mathbf{R}^n$ whose boundary $\partial\Omega$ admits a representation of the form

$$\partial\Omega = \{x \in \mathbf{R}^n : F(x) = c\}$$

for some Lipschitz function $F$ and constant $c$. Then the unit normal vector along the boundary $\partial\Omega$, $\vec{n}$, can be written as

$$\vec{n} = \frac{\nabla F}{|\nabla F|}. \tag{3.3}$$

Hence by extending $\vec{n}$ to $\mathbf{R}^n$ "smoothly", we formally derive the following formula for the area of $\partial\Omega$,

$$\begin{aligned} A(\partial\Omega) &= \int_{\partial\Omega} ds = \int_{\partial\Omega} \vec{n}\cdot\vec{n}\, ds = \int_{\Omega} \mathrm{div}\vec{n}\, dx_1 \ldots dx_n \\ &= \int_{\mathbf{R}^n} \chi_\Omega \mathrm{div}\vec{n}\, dx_1 \ldots dx_n = -\int_{\mathbf{R}^n} \nabla\chi_\Omega \cdot \vec{n}\, dx_1 \ldots dx_n, \end{aligned}$$

where $\chi_\Omega$ is the characteristic function of $\Omega$. So we have

$$A(\partial\Omega) = -\int_{\mathbf{R}^n} \nabla\chi_\Omega \cdot \vec{n}\, dx_1 \ldots dx_n. \tag{3.4}$$

More generally, by the same derivation, one has for any integrable function $f$ defined on $\partial\Omega$, after extending $f$ to $\mathbf{R}^n$,

$$\int_{\partial\Omega} f\, ds = -\int_{\mathbf{R}^n} f\, ||\partial\Omega||, \tag{3.5}$$

where we use $||\partial\Omega||$ to denote this numerical boundary measure $-\nabla\chi_\Omega\cdot\vec{n}$.

The formulas (3.4) and (3.5) represent the area of $\partial\Omega$ and boundary integrals over $\partial\Omega$ in terms of *extrinsic* data, namely the function $F(x)$, and does not depend on a local coordinate parametrization of $\partial\Omega$.

### 3.2 A Wavelet Multiresolution of Geometric Measure Integrals

For simplicity we restrict our attention to domains in $\mathbf{R}^2$, but the results are valid in $\mathbf{R}^n$ with a suitable change in the formulas. Let $\Omega$ be an open bounded set in $\mathbf{R}^2$ whose boundary is defined by

$$\partial\Omega = \{(x,y) \in \mathbf{R}^2 : F(x,y) = t\},$$

for some Lipschitz function $F$ defined in $\mathbf{R}^2$ of $\partial\Omega$ and for some constant $t$. It follows that $\Omega$ has finite perimeter, and all of the analysis in Section 3.1 is valid. In addition, let $\{\varphi_k^j(x), \psi_k^j(x)\}$ be a Daubechies wavelet system of genus $g$ in $\mathbf{R}$.

We can interpolate the characteristic function $\chi_\Omega$ and the defining function $F$ in terms of scaling functions at a certain level $j$ by using the following interpolation given in [22], where $c$ is a suitable constant (which in practice can be taken to be zero). We set

$$I_j(\chi_\Omega)(x,y) = \sum_{p,q} \chi_{p,q}^j \varphi_p^j(x)\varphi_q^j(y), \tag{3.6}$$

$$I_j(F)(x,y) = \sum_{p,q} F_{p,q}^j \varphi_p^j(x)\varphi_q^j(y), \tag{3.7}$$

where $\chi_{p,q}^j = \frac{1}{2^j}\chi_\Omega(\frac{p+c}{2j}, \frac{q+c}{2j})$, and $F_{p,q}^j = \frac{1}{2^j}F(\frac{p+c}{2j}, \frac{q+c}{2j})$.

If $F$ is $C^2$, then by [22], the interpolation (3.7) of $F$ will converge to $F$ in $H^1$ as $j \to +\infty$. In fact, by a little more careful analysis, we can prove the following theorem (see [21])

**Theorem 3.2** *Assume $\Omega$ is a bounded set of finite perimeter in $\mathbf{R}^2$, then $\chi_\Omega^j$ converges to $\chi_\Omega$ in $L^2(\mathbf{R}^2)$ as $j \to +\infty$.*

By differentiating the sums on the right-hand sides of (3.6) and (3.7), and using the connection coefficients defined by (see [11] for a discussion of connection coefficients)

$$\Gamma_m^l =: \int_{\mathbf{R}} \varphi_m' \varphi_l \, dx,$$

one derives the expansions for the partial derivatives of $\chi_\Omega$ and $F$,

$$\frac{\partial \chi_\Omega^j}{\partial x}(x,y) = \sum_{p.q} \left(\frac{\partial \chi}{\partial x}\right)_{p,q}^j \varphi_p^j(x)\varphi_q^j(y), \tag{3.8}$$

$$\frac{\partial \chi_\Omega^j}{\partial y}(x,y) = \sum_{p.q} \left(\frac{\partial \chi}{\partial y}\right)_{p,q}^j \varphi_p^j(x)\varphi_q^j(y), \tag{3.9}$$

$$\frac{\partial F^j}{\partial x}(x,y) = \sum_{p.q} \left(\frac{\partial F}{\partial x}\right)_{p,q}^j \varphi_p^j(x)\varphi_q^j(y), \tag{3.10}$$

$$\frac{\partial F^j}{\partial y}(x,y) = \sum_{p.q} \left(\frac{\partial F}{\partial y}\right)_{p,q}^j \varphi_p^j(x)\varphi_q^j(y). \tag{3.11}$$

By Theorem 3.2, we have the following

**Corollary 3.3** *Under the assumption in Theorem* 3.2, $\nabla \chi_\Omega^j$ *converges to* $\nabla \chi_\Omega$ *(as distributions), as* $j \to +\infty$, *and the support of* $\nabla \chi_\Omega^j$ *is contained in* $U_\Omega^j$.

*Proof:* For $g \geq 3$, the scaling function $\varphi$ is differentiable, hence for any continuously differentiable vector-valued function $\vec{\xi}$, using integration by parts, one has

$$\int \vec{\xi} \cdot \nabla \chi_\Omega^j \, dxdy = \int \operatorname{div}(\vec{\xi}) \chi_\Omega^j \, dxdy.$$

Then the corollary follows readily from Theorem 1 by letting $j \to +\infty$. The second conclusion in the corollary is clear. □

Now we approximate the normal vector $\vec{n}$. For $j$ large, we approximate the values of $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ at the integer points by their corresponding expansion coefficients, that is,

$$\left(\frac{\partial F}{\partial x}\right)_{p,q}^j \approx \frac{\partial F}{\partial x}(p,q)/2^j,$$

and

$$\left(\frac{\partial F}{\partial y}\right)_{p,q}^j \approx \frac{\partial F}{\partial y}(p,q)/2^j.$$

So

$$|\nabla F|(p.q) \approx 2^j \sqrt{\left(\left(\frac{\partial F}{\partial x}\right)_{p,q}^j\right)^2 + \left(\left(\frac{\partial F}{\partial y}\right)_{p,q}^j\right)^2}. \tag{3.12}$$

If we write $\vec{n} = (n_x, n_y)$, then again,

$$n_x(p,q) \approx \frac{\left(\frac{\partial F}{\partial x}\right)_{p,q}^j}{\sqrt{\left(\left(\frac{\partial F}{\partial x}\right)_{p,q}^j\right)^2 + \left(\left(\frac{\partial F}{\partial y}\right)_{p,q}^j\right)^2}};$$

therefore, $n_x$ admits an expansion at level $j$,

$$n_x^j(x,y) = \sum_{p,q} \frac{\left(\frac{\partial F}{\partial x}\right)_{p,q}^j}{2^j\sqrt{\left(\left(\frac{\partial F}{\partial x}\right)_{p,q}^j\right)^2 + \left(\left(\frac{\partial F}{\partial y}\right)_{p,q}^j\right)^2}} \varphi_p^j(x)\varphi_q^j(y). \tag{3.13}$$

Similarly,

$$n_y^j(x,y) = \sum_{p,q} \frac{\left(\frac{\partial F}{\partial y}\right)_{p,q}^j}{2^j\sqrt{\left(\left(\frac{\partial F}{\partial x}\right)_{p,q}^j\right)^2 + \left(\left(\frac{\partial F}{\partial y}\right)_{p,q}^j\right)^2}} \varphi_p^j(x)\varphi_q^j(y). \tag{3.14}$$

Using the continuity of $\nabla F$, we have the following lemma.

**Lemma 3.4** *Assume that $F$ is twice differentiable, then $\vec{n}^j \equiv (n_x^j, n_y^j)$ converges to $\vec{n}$ in $H^1_{loc}(\mathbf{R}^2)$ as $j \to +\infty$.*

Using the formula (3.4), and taking (3.8), (3.9), (3.12), (3.13) and (3.14) into account, let us set

$$L^j(\partial\Omega) = -\left[\sum_{p,q}\left(\left(\frac{\partial\chi}{\partial x}\right)_{p,q}^j \left(\frac{\partial F}{\partial x}\right)_{p,q}^j + \left(\frac{\partial\chi}{\partial y}\right)_{p,q}^j \left(\frac{\partial F}{\partial y}\right)_{p,q}^j\right)/|\nabla F|(p,q)\right]. \tag{3.15}$$

Then we have the following theorem which follows directly from the above results.

**Theorem 3.5** *Suppose that $\Omega$ is a bounded set of finite perimeter, and that the function $F$ is twice differentiable, then $L^j(\partial\Omega)$ converges to the length of $\partial\Omega$ as $j \to +\infty$.*

For a more general domain, we also have the following theorem which is proved in detail in [21].

**Theorem 3.6** *Assume that $\partial\Omega$ is Lipschitz and piecewise smooth with a finite number of nonsmooth points. Then, as $j \to +\infty$, $L^j(\partial\Omega)$ converges to $L(\partial\Omega)$, the length of $\partial\Omega$.*

In order to calculate the general boundary integral of any given integrable function over $\partial\Omega$, it is useful to calculate the *numerical boundary measure* $||\partial\Omega||$ defined in (3.5). Using the connection coefficients

$$\Gamma_{i,j}^k =: \int_{\mathbf{R}} \varphi_k\varphi_i\varphi_j \, dx,$$

one obtains that

$$||\partial\Omega||^j = \sum_{p,q\in\Lambda} (||\partial\Omega||)_{p,q}^j \varphi_p^j(x)\varphi_q^j(y)\, dxdy, \tag{3.16}$$

where for each $(p,q) \in \Lambda_\Omega^j$,

$$(||\partial\Omega||)_{p.q}^j = \sum_{k,l,s,t} \frac{\left(\left(\frac{\partial\chi}{\partial x}\right)_{k,l}\left(\frac{\partial F}{\partial x}\right)_{s,t} + \left(\frac{\partial\chi}{\partial y}\right)_{k,l}\left(\frac{\partial F}{\partial y}\right)_{s,t}\right)\Gamma_{k,s}^p\Gamma_{l,t}^q}{2^j\sqrt{\left(\frac{\partial F}{\partial x}\right)_{s,t}^2 + \left(\frac{\partial F}{\partial y}\right)_{s,t}^2}}.$$

Then we have another formula for calculating the arclength,

$$L^j(\partial\Omega) = -\sum_{p,q\in\Lambda} (||\partial\Omega||)_{p,q}^j/2^j. \tag{3.17}$$

In *higher dimensions* we obtain specific formulas of the same type for surface measures as well. We recall that if we have a parametrized curve in $\mathbf{R}^2$, there are simple formulas for the arclength, and the approach we take here does not *assume* such a parametrization, but only requires *knowledge* of the chracteristic function of the domain (at all scales theoretically, at a finite range of scales computationally). For higher-dimensional boundaries of domains, there is no global parametrization of a boundary as in the case of a planar curve, but local coordinate patches must be used to locally parametrize an arbitrary higher-dimensional surface (submanifold) in $\mathbf{R}^n$. Our approach yields explicit computational formulas for the surface area (in $\mathbf{R}^3$) or higher-dimensional $(n-1)$-dimensional hypersurface measure for a hypersurface of dimension $n-1$ in $\mathbf{R}^n$.

## 4 APPLICATIONS TO BOUNDARY VALUE PROBLEMS IN PARTIAL DIFFERENTIAL EQUATIONS

In this section we want to consider two different methods of numerically solving (1.1) using the wavelet representation of the geometry of the boundary of the domain as discussed in the earlier sections. We consider (1.1) in $\mathbf{R}^2$, namely,

$$\begin{aligned} -\triangle u + u &= f, \quad \text{in } \Omega \subset \mathbf{R}^2, \\ u &= g, \quad \text{on } \partial\Omega, \end{aligned} \tag{4.18}$$

where $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$ are given functions on $\Omega$ and $\partial\Omega$, respectively, and where $u \in H^1(\Omega)$ is the sought for solution to the problem (here $H^p(X)$ denotes the Sobolev space of $L^2$ functions in $X$ whose $p$th derivative is also in $L^2(X)$ (see, e.g., [21]). Now we formulate two integral equations on a square $D$ which is assumed to contain $\bar{\Omega}$.

The first is the *penalty method*, with a penalty parameter $\varepsilon > 0$. Namely, for any $\varepsilon > 0$, consider the variational problem

$$\int_D \nabla u_\varepsilon \cdot \nabla v \, dxdy + \int_D u_\varepsilon v \, dxdy + \frac{1}{\varepsilon}\int_{\partial\Omega} u_\varepsilon v \, ds = \int_D fv \, dxdy + \frac{1}{\varepsilon}\int_{\partial\Omega} gv \, ds, \tag{4.19}$$

where $u_\varepsilon \in H'(D)$ is the desired solution in $D$ and (4.19) is satisfied for all $v \in H^1(D)$. It is proved in [21] that the solution $u_\varepsilon$ of (4.19) *restricted to* $\Omega$ converges in $H^1(\Omega)$ to the unique solution $u$ of (4.18). We can then try to find numerical approximations to solutions of (4.19) in $D$ (parametrized by $\varepsilon$) and restrict them to $\Omega$ to get approximate solutions to (4.18) on $\Omega$.

The second is the *Lagrange multiplier method*, which involves an additional function $\lambda$ (a Lagrange multiplier) defined on a new auxiliary boundary $\partial S$ squeezed between $\partial D$ and $\partial\Omega$. This was first exploited numerically by Qian and Weiss [17] and then developed further in [9]. In this case, we consider the square $D$ containing $\Omega$ as before and an additional open square $S$ satisfying $\Omega \subset\subset S \subset\subset D$, i.e., $\bar{\Omega} \subset S$, $\bar{S} \subset D$. Then we have the following two variational equations on the square $D$ (we let $H^1_P(D)$ denote *periodic functions* (in the square) in the Sobolev space $H^1(D)$), for $f \in L^2(D), g \in H^{1/2}(\partial\Omega)$,

$$\begin{gathered} \int_D (Du\nabla v + uv)dxdy = \int_D fv \, dxdy + \int_{\partial S} \lambda v dS, \\ u \in H^1_P(D), \qquad \lambda \in L^2(\partial s) \quad \text{for all } v \in H^1_P(D), \end{gathered} \tag{4.20}$$

$$\begin{gathered} \int_D \nabla p \nabla v + pv = \int_{\partial\Omega} (u-g) v ds \\ p \in H^1_P(D), \quad \text{for all } v \in H^1_P(D). \end{gathered} \tag{4.21}$$

In [9] we find a proof that if $(u, \lambda)$ satisfy (4.20), then $u\big|_\Omega$ satisfies $-\Delta u + u = f$. In addition, if $(u, \lambda, p)$ satisfy (4.20) and (4.21), and $p\big|_{\partial S} = 0$, then $u = g$ on $\partial\Omega$.

The function $\lambda$ is a Lagrange multiplier type of additional unknown function (defined on $\partial S$). We set up a system of equivalent equations (see [17,9]) so that we have to solve an equation involving given data on $\partial\Omega$ with the solution on $\partial S$. This will involve a nonsquare matrix in general when discretized. This (lower-dimensional) problem is solved first, and then for this given $\lambda$, we solve the periodic boundary problem (4.20) on $D$ using a fast periodic solver (which comes from the wavelet representation in the discrete version which can be solved using an FFT). The scaling spaces $V_J = V_0 \oplus \cdots \oplus W_{J-1}$ on the square $D$ are used (using tensor products of one-dimensional wavelet functions).

In the papers [21,8,17] and [9] one can find details of this analysis. The wavelet geometric calculus developed in the earlier part of this paper is used in [21,8] and [9], and the main virtue is that one doesn't need to know the detailed nature of the location of $\partial\Omega$ to make the calculation. Moreover, the accuracy in this method is spectral with respect to the genus of the wavelet system used, and the solution is quite fast with the periodic solver.

Both of these methods have their value and are being tested on a variety of problems at the present time. It seems that for some problems (in particular for (4.18) above), the Lagrange multiplier method is quite superior with respect to speed and accuracy.

These ideas have also been carried out for nonlinear equations (Euler flow, plasticity equations, Navier–Stokes equations; see the reference above), as well as for the Neumann version of the linear elliptic problem (4.18 above).

The principal advantage of these scalable wavelet-Galerkin methods consists of:

- the coding is independent of the geometric shape of the boundary;
- the method works in any number of dimensions;
- the multiscale representation admits readily multigrid-type iterative methods for solving the associated algebraic problems after wavelet discretization;
- the extension to many nonlinear problems is straightforward, provided the nonlinearity is of a simple polynomial character.

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