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Abstract

In the last decade a number of perfect reconstruction filter bank structures have been proposed [11]. Of these the modulated filter banks are the easiest to design and implement [7, 6, 4]. However, the filters in a modulated filter bank *cannot* be linear phase. Recently, unitary FIR filter banks with linear-phase have been completely parameterized [10]. In that paper, the eigenstructure of the exchange matrix played an important role. In this correspondence we give a new characterization of the polyphase matrix of a filter bank with filters of various types of symmetry. Using matrix extensions of the well-known hyperbolic and orthogonal lattices, we give a alternative proof for the parameterization of linear-phase unitary filter banks. Moreover, a complete parameterization of FIR unitary filter banks with each of the different types of symmetries considered (not just linear-phase) is also given. We also mention how the above results can be used to generate non-unitary filter banks with symmetries, though no completeness results are obtained. In some cases implicit, and in others explicit parameterization of wavelet tight frames associated with these filter banks are also given. In this paper, we only consider the case of M even. For odd M a similar theory can be developed.

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1 Introduction

In a filter bank (see Fig. 1), the passbands of the filters must approximate ideal filters. Since the ideal filters occupy adjacent bands, a number of transformations relate these filter responses. For example, the filters could be related by modulation as seen in the theory of modulated filter banks [6, 4]. From both design and implementation points of view unitary modulated filter banks are

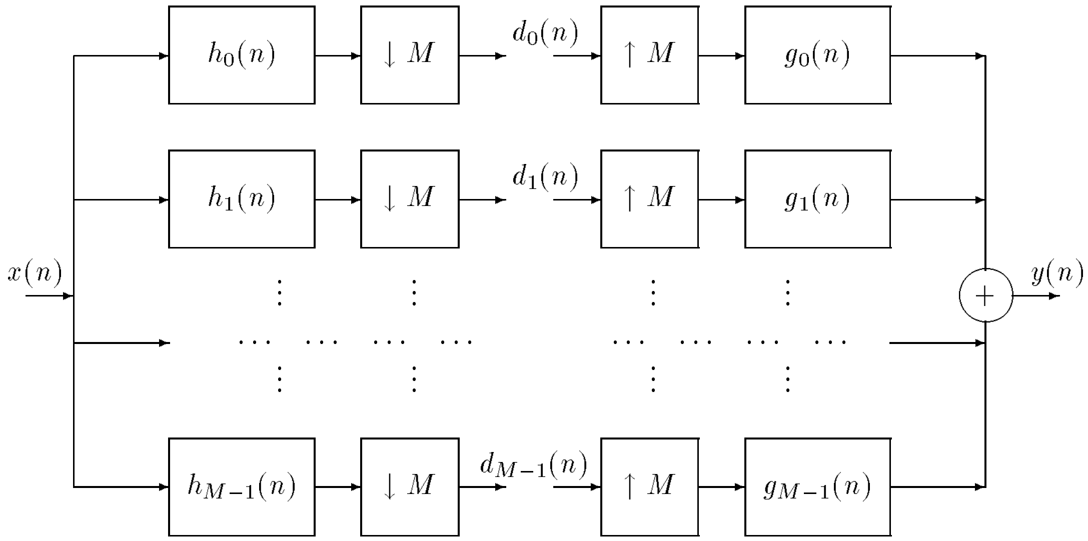

Figure 1: An M-channel Filter Bank

the most efficient. However, it has been proved that filters in a unitary modulated filter bank (FIR) *cannot* be linear-phase. Recently, a complete parameterization of unitary filter banks with linear-phase has been obtained [10].

In this paper we consider the following types of symmetries in filter banks: In Fig. 2, the response of the $M-1-i^{th}$ filter can be obtained by shifting the response of the i^{th} filter by π. Therefore, they could be related as,

$$h_{M-1-i}(n) = (-1)^n h_i(-n) \;\; ; \;\; H_{M-1-i}(z) = H_i(-z) \;\; ; \; H_{M-1-i}(\omega) = H_i(\omega+\pi) \tag{1}$$

or,

$$h_{M-1-i}(n) = (-1)^n h_i(N-1-n) \;\; ; \;\; H_{M-1-i}(z) = H_i^R(-z) \;\; ; \; H_{M-1-i}(\omega) = H_i^\star(\omega+\pi) \tag{2}$$

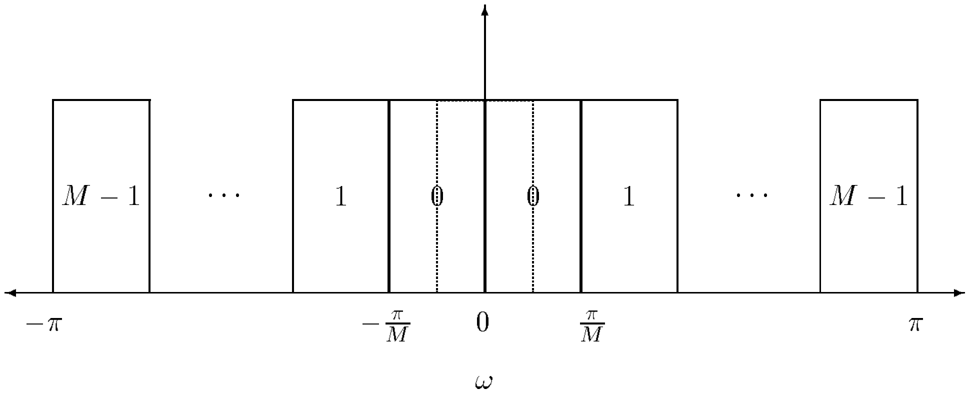

Figure 2: Ideal Frequency Responses in M-channel Filter Bank

where for causal $H(z)$, $H^R(z)$ denotes its reflection (i.e causal time-reversed sequence). The former will be called *pairwise-shift* (or PS) symmetry (also known as pairwise-mirror image symmetry [8]) , while the latter will be called *pairwise-conjugated-shift* (or PCS) symmetry (also known as pairwise-symmetry [8]). Both these symmetries relate pairs of filters in the filter bank. Another type of symmetry occurs when the filters themselves are symmetric or linear-phase. The only type of linear-phase symmetry we will consider is,

$$h_i(n) = \pm h_i(N-1-n) \quad ; \quad H_i(z) = \pm H_i^R(z) \tag{3}$$

where the filters are all of fixed length N, and the symmetry is about $\frac{N-1}{2}$. For these three symmetries, the next section studies the form of the (first-orthant or Type 1 [11, 1]) polyphase component matrix, $H_p(z)$, of the analysis bank.

2 Characterization of Filter Banks with Symmetry

We will only consider the case of even M. For even M, if the filters are linear-phase half the filters are symmetric, while the other half are anti-symmetric [10]. Let J denote the *exchange matrix* with ones on the anti-diagonal. Post-multiplying a matrix A by J is equivalent to reversing the order of the columns of A, and pre-multiplying is equivalent to reversing the order of the rows of A. Let V denote the *sign-alternating* matrix, the diagonal matrix of alternating ± 1's. Post-multiplying by V, alternates the signs of the columns of A, while premultiplying alternates the signs of the rows of A. The polyphase components of $H(z)$ are related to the polyphase components of $H^R(z)$ by reflection and reversal of the ordering of the components. For, if $H(z)$ is of length Mm, and

$H(z) = \sum_{l=0}^{M-1} z^{-l} H_l(z^M)$, then,

$$\begin{aligned} H^R(z) &= z^{-Mm+1} \sum_{l=0}^{M-1} z^l H_l(z^{-M}) \\ &= \sum_{l=0}^{M-1} z^{-(M-1-l)} (z^{-mM+M} H_l(z^{-M})) \\ &= \sum_{l=0}^{M-1} z^{-l} H^R_{M-1-l}(z^M) \end{aligned} \tag{4}$$

Therefore,

$$(H^R)_l(z) = (H_{M-1-l})^R(z) \tag{5}$$

and for linear-phase $H(z)$, since $H^R(z) = \pm H(z)$,

$$H_l(z) = \pm (H^R)_{M-1-l}(z). \tag{6}$$

Lemma 1 *For even M, $H_p(z)$ is of the form,*

PS Symmetry

$$\begin{bmatrix} W_0(z) & W_1(z) \\ JW_0(z)V & (-1)^{M/2} JW_1(z)V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0(z) & W_1(z) \\ W_0(z)V & (-1)^{M/2} W_1(z)V \end{bmatrix} \tag{7}$$

PCS Symmetry

$$\begin{bmatrix} W_0(z) & W_1(z)J \\ JW_1^R(z)V & (-1)^{M/2} JW_0^R(z)JV \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0(z) & W_1(z)J \\ W_1^R(z)V & (-1)^{M/2} W_0^R(z)JV \end{bmatrix} \tag{8}$$

Linear Phase

$$\begin{bmatrix} W_0(z) & D_0 W_0^R(z)J \\ W_1(z) & D_1 W_1^R(z)J \end{bmatrix} = \begin{bmatrix} W_0(z) & D_0 W_0^R(z) \\ W_1(z) & D_1 W_1^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \tag{9}$$

or

$$Q \begin{bmatrix} W_0(z) & W_0^R(z)J \\ W_1(z) & -W_1^R(z)J \end{bmatrix} = Q \begin{bmatrix} W_0(z) & W_0^R(z) \\ W_1(z) & -W_1^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \tag{10}$$

Linear Phase and PCS

$$\begin{bmatrix} W_0(z) & DW_0^R(z)J \\ JDW_0(z)V & (-1)^{M/2} JW_0^R(z)JV \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0(z) & DW_0^R(z)J \\ DW_0(z)V & (-1)^{M/2} W_0^R(z)JV \end{bmatrix} \tag{11}$$

Linear Phase and PS

$$\begin{bmatrix} W_0(z) & DW_0^R(z)J \\ JW_0(z)V & (-1)^{M/2}JDW_0^R(z)JV \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & JD \end{bmatrix} \begin{bmatrix} W_0(z) & DW_0^R(z)J \\ DW_0(z)V & (-1)^{M/2}W_0^R(z)JV \end{bmatrix} \tag{12}$$

Proof: For PS symmetry, since $H_{M-1-i}(z) = H_i(-z)$, $H_{M-1-i,l}(z) = (-1)^l H_{i,l}(z)$. Thus if $[W_0(z)\ W_1(z)]$ is the first $M/2$ rows of $H_p(z)$, the rest is given by reversing the order of rows of $[W_0(z)\ W_1(z)]$ and alternating the signs of the columns. Hence the result follows. For PCS symmetry, since $H_{M-1-i}(z) = H_i^R(-z)$, from Eqn. 5,

$$H_{i,l}(z) = (-1)^l H^R_{M-1-i,M-1-l}(z). \tag{13}$$

Therefore if $H_p(z)$ is a 2×2 block matrix, the 10 and 11 blocks are obtained by reversing the order of rows and columns of the 00 and 01 blocks respectively and alternating signs of columns, which is equivalent to Eqn. 8. For linear-phase, from Eqn. 6, $H_{i,l}(z) = \pm H_{i,M-1-l}(z)$, and therefore Eqn. 9 follows with D_0 and D_1 diagonal matrices of ± 1's to account for the type of symmetry. From the fact that half the filters are symmetric and the other half anti-symmetric, by a permutation Q, we can stack the symmetric filters in the first $M/2$ rows and the anti-symmetric filters in the rest, giving Eqn. 10 The combination of linear-phase and PCS (or PS) is then immediate. Notice also, that one needs to consider only one case (i.e linear phase and PCS or linear phase and PS). $\square$

Thus in order to *generate* $H_p(z)$ for all symmetries *other than PS*, we need a mechanism that generates a pair of matrices and their reflection (i.e $W_0(z)$, $W_1(z)$ $W_0^R(z)$ and $W_1^R(z)$). In the scalar case, there are two well-known lattice structures, that generate such pairs. The first case is the power-complementary (or orthogonal lattice) [12], while the second is the linear-prediction (or hyperbolic) lattice [9]. A K^{th} order power-complementary lattice is generated by the product,

$$\left\{\prod_{i=0}^{K} \begin{bmatrix} a_i & z^{-1}b_i \\ -b_i & z^{-1}a_i \end{bmatrix}\right\} \begin{bmatrix} a_0 & b_0 \\ -b_0 & a_0 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} Y_0(z) & Y_1(z) \\ -Y_1^R(z) & Y_0^R(z) \end{bmatrix} \stackrel{\text{def}}{=} X(z). \tag{14}$$

This lattice is always invertible (unless a_i and b_i are both zero!), and the inverse is anti-causal since,

$$\begin{bmatrix} a_i & z^{-1}b_i \\ -b_i & z^{-1}a_i \end{bmatrix}^{-1} = \frac{1}{a_1^2 + b_i^2} \begin{bmatrix} a_i & -b_i \\ zb_i & za_i \end{bmatrix}. \tag{15}$$

This lattice plays a fundamental role in the theory of FIR unitary modulated filter banks, and many other areas. The hyperbolic lattice of order K, generates the product,

$$\left\{\prod_{i=0}^{K} \begin{bmatrix} a_i & z^{-1}b_i \\ b_i & z^{-1}a_i \end{bmatrix}\right\} \begin{bmatrix} a_0 & b_0 \\ b_0 & a_0 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} Y_0(z) & Y_1(z) \\ Y_1^R(z) & Y_0^R(z) \end{bmatrix} \stackrel{\text{def}}{=} X(z). \tag{16}$$

where $Y_0(z)$ and $Y_1(z)$ are of order K. This lattice is invertible only when $a_i^2 \neq b_i^2$ (or equivalently $(a_i + b_i)/2$ and $(a_i - b_i)/2$ are non-zero), in which case, the inverse is non-causal, since,

$$\begin{bmatrix} a_i & z^{-1}b_i \\ b_i & z^{-1}a_i \end{bmatrix}^{-1} = \frac{1}{a_1^2 - b_i^2} \begin{bmatrix} a_i & -b_i \\ -zb_i & za_i \end{bmatrix}. \tag{17}$$

Since the matrix $\begin{bmatrix} a_i & b_i \\ b_i & a_i \end{bmatrix}$ can be orthogonal iff $\{a_i, b_i\} = \{\pm 1, 0\}$, or $\{a_i, b_i\} = \{0, \pm 1\}$, the (2×2) matrix generated by the lattice can never be unitary.

We now propose the following matrix generalizations of these two lattices, which will play a fundamental role in the characterization of $H_p(z)$ for the various symmetries we considered earlier. In the matrix case, it turns out that both the lattices, can generate unitary matrices, and this will lead to a parameterization of FIR unitary $H_p(z)$ for PCS, linear-phase, and PCS+linear-phase symmetries. We prefer to call the generalization of the orthogonal lattice, the *anti-symmetric* lattice and the generalization of the hyperbolic lattice the *symmetric* lattice, which should be obvious from the form of the product. The reason for this is that the anti-symmetric lattice *may not* generate a unitary matrix transfer function (in the scalar case, the 2×2 transfer function generated is always unitary). The anti-symmetric lattice is defined by the product,

$$X(z) \stackrel{\text{def}}{=} \left\{ \prod_{i=1}^{K} \begin{bmatrix} A_i & z^{-1}B_i \\ -B_i & z^{-1}A_i \end{bmatrix} \right\} \begin{bmatrix} A_0 & B_0 \\ -B_0 & A_0 \end{bmatrix} \tag{18}$$

where A_i and B_i are constant square matrices of size $M/2 \times M/2$. It is readily verified that $X(z)$ is of the form

$$X(z) = \begin{bmatrix} Y_0(z) & Y_1(z) \\ -Y_1^R(z) & Y_0^R(z) \end{bmatrix} \tag{19}$$

Given $X(z)$, its invertibilty is equivalent to the invertibility of the constant matrices,

$$\begin{bmatrix} A_i & B_i \\ -B_i & A_i \end{bmatrix} \quad \text{since} \quad \begin{bmatrix} A_i & z^{-1}B_i \\ -B_i & z^{-1}A_i \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ -B_i & A_i \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & z^{-1}I \end{bmatrix}, \tag{20}$$

which, in turn is related to the invertibility of the *complex* matrices $C_i = (A_i + \imath B_i)$ and $D_i = (A_i - \imath B_i)$, since,

$$\frac{1}{2} \begin{bmatrix} I & I \\ \imath I & -\imath I \end{bmatrix} \begin{bmatrix} C_i & 0 \\ 0 & D_i \end{bmatrix} \begin{bmatrix} I & -\imath I \\ I & \imath I \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ -B_i & A_i \end{bmatrix} \tag{21}$$

Moreover, the orthogonality of the matrix is equivalent to the unitariness of the complex matrix C_i (since D_i is just its Hermitian conjugate). Since an arbitrary complex matrix, of size $M/2 \times M/2$, is determined by precisely $2 \binom{M/2}{2}$ parameters, each of the matrices, $\begin{bmatrix} A_i & B_i \\ -B_i & A_i \end{bmatrix}$ has that many degrees of freedom. Clearly when these matrices are orthogonal, $X(z)$ is unitary (on the unit circle) and $X^T(z^{-1})X(z) = I$. For unitary $X(z)$, the converse is also true as we will shortly prove.

The symmetric lattice is defined by the product,

$$X(z) \stackrel{\text{def}}{=} \left\{ \prod_{i=1}^{K} \begin{bmatrix} A_i & z^{-1}B_i \\ B_i & z^{-1}A_i \end{bmatrix} \right\} \begin{bmatrix} A_0 & B_0 \\ B_0 & A_0 \end{bmatrix} \tag{22}$$

Once again A_i and B_i are constant square matrices, and it is readily verified that $X(z)$ written as a product above is of the form

$$X(z) = \begin{bmatrix} Y_0(z) & Y_1(z) \\ Y_1^R(z) & Y_0^R(z) \end{bmatrix} \tag{23}$$

The invertibility of $X(z)$ is equivalent to the invertibility of

$$\begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix} \quad \text{since} \quad \begin{bmatrix} A_i & z^{-1}B_i \\ B_i & z^{-1}A_i \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & z^{-1}I \end{bmatrix}, \tag{24}$$

which, in turn is equivalent to the invertibility of $C_i = (A_i + B_i)$ and $D_i = (A_i - B_i)$, since,

$$\frac{1}{2} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} C_i & 0 \\ 0 & D_i \end{bmatrix} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix} \tag{25}$$

Once again, the orthogonality of the constant matrix, is equivalent to the orthogonality of the *real* matrices C_i and D_i, and since each real orthogonal matrix of size $M/2 \times M/2$ is determined by $\binom{M/2}{2}$ parameters, the constant orthogonal matrices have $2\binom{M/2}{2}$ degrees of freedom. Clearly, when the matrices are orthogonal, and $X^T(z^{-1})X(z) = I$. For the hyperbolic lattice too, the converse is true.

Theorem 1 *Let $X(z)$ be an FIR, order K, $M \times M$, and unitary. Depending on whether $X(z)$ is of the form in Eqn. 19, or Eqn. 23, it is generated by an order K anti-symmetric or symmetric lattice.*

Proof: In both cases, we prove the results by induction on the order K. First consider $X(z)$ as in Eqn. 19. When $K = 0$, the result is evident. For any $X(z)$ of order K, we reduce its order by 1 to complete the proof. Let

$$X(z) = \sum_{i=0}^{K} x(i) z^{-i} = \sum_{i=0}^{K} z^{-i} \begin{bmatrix} y_0(i) & y_1(i) \\ -y_1(K-i) & y_0(K-i) \end{bmatrix} \tag{26}$$

Notice that it suffices to find an orthogonal matrix $\begin{bmatrix} A_K^T & -B_K^T \\ B_K^T & A_K^T \end{bmatrix}$ such that,

$$\begin{bmatrix} A_K^T & -B_K^T \end{bmatrix} x(K) = \begin{bmatrix} A_K^T & -B_K^T \end{bmatrix} \begin{bmatrix} y_0(K) & y_1(K) \\ -y_1(0) & y_0(0) \end{bmatrix} = 0 \tag{27}$$

because, then,

$$\begin{bmatrix} A_K^T & -B_K^T \\ zB_K^T & zA_K^T \end{bmatrix} X(z) = \sum_{i=0}^{K-1} z^{-i} \begin{bmatrix} A_K^T y_0(i) + B^T y_1(K-i) & A_K^T y_1(i+1) - B_K^T y_0(K-1-i) \\ -A_K^T y_1(K-i) + B_K^T y_0(i) & A_K^T y_0(K-1-i) + B_K^T y_1(i+1) \end{bmatrix}$$

gives the desired order reduction. The right hand side is unitary (since it is a product of unitary matrices), of degree $K-1$ and of the form in Eqn. 19. Also $X(z)$ is $\begin{bmatrix} A_K & z^{-1}B_K \\ -B_K & z^{-1}A_K \end{bmatrix}$ times the reduced order unitary matrix. We now construct such an orthogonal matrix, using the crucial assumption of *unitariness*. Since $X^T(z^{-1})X(z) = I$, the coefficient of z^K in the product is 0. That is,

$$x^T(0)x(K) = \begin{bmatrix} y_0^T(0) & -y_1^T(K) \\ y_1^T(0) & y_0^T(K) \end{bmatrix} \begin{bmatrix} y_0(K) & y_1(K) \\ -y_1(0) & y_0(0) \end{bmatrix} = 0 \tag{28}$$

and therefore the rank of $x(K)$ (= rank of $x(0)$) is some $P \leq M/2$. Let T be an $M \times P$ matrix that orthogonalizes (Gram-Schmidt) the columns of this rank P matrix.

$$x(K)T = x(K)\begin{bmatrix} T_0 \\ T_1 \end{bmatrix} = \begin{bmatrix} y_0(K) & y_1(K) \\ -y_1(0) & y_0(0) \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} U_0 \\ U_1 \end{bmatrix} \text{ and } \begin{bmatrix} U_0^T & U_1^T \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \end{bmatrix} = I \tag{29}$$

We will complete $\begin{bmatrix} U_0 \\ U_1 \end{bmatrix}$ by appending $M-P$ orthogonal columns to give the desired orthogonal matrix. Notice that

$$x(0)\begin{bmatrix} T_1 \\ -T_0 \end{bmatrix} = \begin{bmatrix} y_0(0) & y_1(0) \\ -y_1(K) & y_0(K) \end{bmatrix} \begin{bmatrix} T_1 \\ -T_0 \end{bmatrix} = \begin{bmatrix} U_1 \\ -U_0 \end{bmatrix} \tag{30}$$

and therefore from Eqn. 28,

$$\begin{bmatrix} U_1^T & -U_0^T \end{bmatrix} x(K) = 0 = \begin{bmatrix} U_1^T & -U_0^T \end{bmatrix} \begin{bmatrix} U_1 \\ U_0 \end{bmatrix} \tag{31}$$

Therefore, $U_e = (U_0 + \imath U_1)$ and $U_o = (U_0 - \imath U_1)$ are $M/2 \times P$, matrices of unitary columns, that are Hermitian conjugates. Let $[U_e V_e]$ and $[U_o V_o]$ be unitary completions to square $M/2 \times M/2$ unitary matrices (Gram-Schmidt again, though *complex*!!). Then let

$$\frac{1}{2}\begin{bmatrix} I & I \\ \imath I & -\imath I \end{bmatrix} \left[\begin{array}{cc|cc} U_e & V_e & 0 & 0 \\ \hline 0 & 0 & U_o & V_o \end{array}\right] \left[\begin{array}{cc|cc} I & 0 & -\imath I & 0 \\ 0 & I & 0 & -\imath I \\ \hline I & 0 & \imath I & 0 \\ 0 & I & 0 & \imath I \end{array}\right] = \left[\begin{array}{cc|cc} U_1 & V_1 & U_0 & V_0 \\ \hline -U_0 & -V_0 & U_1 & V_1 \end{array}\right] \stackrel{\text{def}}{=} \left[\begin{array}{c|c} A_K & B_K \\ \hline -B_K & A_K \end{array}\right]. \tag{32}$$

Clearly this matrix is orthogonal, and moreover, satisfies Eqn. 27.

Now consider the case when $X(z)$ is of the form in Eqn. 23. Once gain for $K = 0$ the result is evident. We reduce the order of a degree K, $X(z)$ and will be done. Let

$$X(z) = \sum_{i=0}^{K} x(i)z^{-i} = \sum_{i=0}^{K} z^{-i} \begin{bmatrix} y_0(i) & y_1(i) \\ y_1(K-i) & y_0(K-i) \end{bmatrix} \tag{33}$$

It suffices to find an orthogonal matrix $\begin{bmatrix} A_K^T & B_K^T \end{bmatrix}$ such that,

$$\begin{bmatrix} A_K^T & B_K^T \\ B_K^T & A_K^T \end{bmatrix} x(K) = \begin{bmatrix} A_K^T & B_K^T \\ B_K^T & A_K^T \end{bmatrix} \begin{bmatrix} y_0(K) & y_1(K) \\ y_1(0) & y_0(0) \end{bmatrix} = 0 \tag{34}$$

because, then,

$$\begin{bmatrix} A_K^T & B_K^T \\ zB_K^T & zA_K^T \end{bmatrix} X(z) = \sum_{i=0}^{K-1} z^{-i} \begin{bmatrix} A_K^T y_0(i) + B^T y_1(K-i) & A_K^T y_1(i+1) + B_K^T y_0(K-1-i) \\ A_K^T y_1(K-i) + B_K^T y_0(i) & A_K^T y_0(K-1-i) + B_K^T y_1(i+1) \end{bmatrix}$$

gives the desired order reduction. The right hand side is unitary (since it is a product of unitary matrices), of degree $K-1$ and of the form in Eqn. 23. Also $X(z)$ is $\begin{bmatrix} A_K & z^{-1}B_K \\ B_K & z^{-1}A_K \end{bmatrix}$ times the reduced order unitary matrix. We now exhibit such an orthogonal matrix. Since $X^T(z^{-1})X(z) = I$, the coefficient of z^K in the product is 0. That is,

$$x^T(0)x(K) = \begin{bmatrix} y_0^T(0) & y_1^T(K) \\ y_1^T(0) & y_0^T(K) \end{bmatrix} \begin{bmatrix} y_0(K) & y_1(K) \\ y_1(0) & y_0(0) \end{bmatrix} = 0 \tag{35}$$

and therefore the rank of $x(K)$ (= rank of $x(0)$) is some $P \leq M/2$. Let T be an $M \times P$ matrix that orthogonalizes (Gram-Schmidt) the columns of this rank P matrix.

$$x(K)T = x(K) \begin{bmatrix} T_0 \\ T_1 \end{bmatrix} = \begin{bmatrix} U_0 \\ U_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U_0^T & U_1^T \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \end{bmatrix} = I \tag{36}$$

We will complete $\begin{bmatrix} U_0 \\ U_1 \end{bmatrix}$ by appending $M-P$ orthogonal columns to give the desired orthogonal matrix. From Eqn. 35 it follows that,

$$x(0) \begin{bmatrix} T_1 \\ T_0 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_0 \end{bmatrix} \tag{37}$$

and therefore,

$$\begin{bmatrix} U_1^T & U_0^T \end{bmatrix} x(K) = 0 = \begin{bmatrix} U_1^T & U_0^T \end{bmatrix} \begin{bmatrix} U_1 \\ U_0 \end{bmatrix} \tag{38}$$

Therefore, $U_e = (U_0 + U_1)$ and $U_o = (U_0 - U_1)$ are $M/2 \times P$, matrices of orthonormal columns. Let $[U_e V_e]$ and $[U_o V_o]$ be completions to square $M/2 \times M/2$ orthogonal matrices (Gram-Schmidt again!). Then let

$$\frac{1}{2}\begin{bmatrix} I & I \\ I & -I \end{bmatrix} \left[\begin{array}{cc|cc} U_e & V_e & 0 & 0 \\ \hline 0 & 0 & U_o & V_o \end{array}\right] \left[\begin{array}{cc|cc} I & 0 & I & 0 \\ 0 & I & 0 & I \\ \hline I & 0 & -I & 0 \\ 0 & I & 0 & -I \end{array}\right] = \left[\begin{array}{cc|cc} U_1 & V_1 & U_0 & V_0 \\ \hline U_0 & V_0 & U_1 & V_1 \end{array}\right] = \left[\begin{array}{c|c} A_K & B_K \\ \hline B_K & A_K \end{array}\right]. \tag{39}$$

Clearly this matrix is orthogonal, and moreover, satisfies Eqn. 34 $\square$

2.1 PS Symmetry

The form of $H_p(z)$ for PS symmetry in Eqn. 7 can be simplified by a permutation. Let P be the permutation matrix, that exchanges the first column with the last column, the third column with the last but third etc. That is,

$$P = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} \tag{40}$$

Then,

$$\begin{bmatrix} W_0(z) & W_1(z) \\ W_0(z)V & (-1)^{M/2}W_1(z)V \end{bmatrix} \tag{41}$$

that occurs in the form of $H_p(z)$ with PS symmetry (Eqn. 7) can be written as,

$$\frac{1}{\sqrt{2}} \begin{bmatrix} W_0'(z) & W_1'(z) \\ -W_0'(z) & W_1'(z) \end{bmatrix} P \tag{42}$$

Therefore,

$$\begin{aligned} H_p(z) &= \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0(z) & W_1(z) \\ W_0(z)V & (-1)^{M/2}W_1(z)V \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0'(z) & W_1'(z) \\ -W_0'(z) & W_1'(z) \end{bmatrix} P \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \begin{bmatrix} W_0'(z) & 0 \\ 0 & W_1'(z) \end{bmatrix} P \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -J & J \end{bmatrix} \begin{bmatrix} W_0'(z) & 0 \\ 0 & W_1'(z) \end{bmatrix} P \end{aligned} \tag{43}$$

Therefore we have the following theorem:

Theorem 2 (Unitary PS Symmetry) *$H_p(z)$ of order K forms a unitary PR filter bank with PS symmetry iff there exist unitary, order K, $M/2 \times M/2$ matrices $W_0'(z)$ and $W_1'(z)$, such that*

$$H_p(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -J & J \end{bmatrix} \begin{bmatrix} W_0'(z) & 0 \\ 0 & W_1'(z) \end{bmatrix} P \tag{44}$$

A unitary H_p, with PS symmetry is determined by precisely $2(M/2-1)K + 2\binom{M/2}{2}$ parameters.

2.2 PCS Symmetry

In this case,

$$\begin{aligned} H_p(z) &= \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0(z) & W_1(z)J \\ W_1^R(z)V & (-1)^{M/2} W_0^R(z)JV \end{bmatrix} \\ &\stackrel{\text{def}}{=} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0' & W_1'J \\ -(W_1')^R & (W_0')^R J \end{bmatrix} P \\ &= \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0' & W_1' \\ -(W_1')^R & (W_0')^R \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} P \end{aligned} \tag{45}$$

Therefore from Lemma 1 we have the following theorem:

Theorem 3 *$H_p(z)$ forms an order K, unitary filter bank with PCS symmetry iff*

$$H_p(z) = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \left\{ \prod_{i=1}^{K} \begin{bmatrix} A_i & z^{-1}B_i \\ -B_i & z^{-1}A_i \end{bmatrix} \right\} \begin{bmatrix} A_0 & B_0 \\ -B_0 & A_0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} P \tag{46}$$

where $\begin{bmatrix} A_i & B_i \\ -B_i & A_i \end{bmatrix}$ are constant orthogonal matrices. $H_p(z)$ is characterized by $2K\binom{M/2}{2}$ parameters.

2.3 Linear Phase

For the linear-phase case,

$$\begin{aligned} H_p(z) &= Q \begin{bmatrix} W_0(z) & W_0^R(z) \\ W_1(z) & -W_1^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \\ &= \frac{1}{2} Q \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} W_0(z) + W_1(z) & W_0^R(z) - W_1^R(z) \\ W_0(z) - W_1(z) & W_0^R(z) + W_1^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \\ &\stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} Q \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} W_0'(z) & W_1'(z) \\ (W_1')^R(z) & (W_0')^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \end{aligned} \tag{47}$$

Therefore, we have the following Theorem:

Theorem 4 *$H_p(z)$ of order K, forms a unitary filter bank with linear-phase filters iff*

$$H_p(z) = \frac{1}{\sqrt{2}} Q \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \left\{ \prod_{i=1}^{K} \begin{bmatrix} A_i & z^{-1}B_i \\ B_i & z^{-1}A_i \end{bmatrix} \right\} \begin{bmatrix} A_0 & B_0 \\ B_0 & A_0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \tag{48}$$

where $\begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix}$ are constant orthogonal matrices. $H_p(z)$ is characterized by $2K\binom{M/2}{2}$ parameters.

2.4 Linear Phase and PCS

In this case, $H_p(z)$ is given by,

$$
\begin{aligned}
H_p(z) &= \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0(z) & DW_0^R(z)J \\ DW_0(z)V & (-1)^{M/2}W_0^R(z)JV \end{bmatrix} \\
&\stackrel{\mathrm{def}}{=} \frac{1}{\sqrt{2}} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0'(z) & D(W_0')^R(z)J \\ -DW_0'(z) & (W_0')^R(z)J \end{bmatrix} P \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} I & D \\ -D & I \end{bmatrix} \begin{bmatrix} W_0'(z) & 0 \\ 0 & W_0')^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} P \qquad (49)
\end{aligned}
$$

Therefore we have proved the following Theorem:

Theorem 5 *$H_p(z)$ of order K forms a unitary filter bank with linear-phase and PCS filters iff there exists a unitary, order K, $M/2 \times M/2$ matrix $W_0'(z)$ such that*

$$
H_p(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} I & D \\ -JD & J \end{bmatrix} \begin{bmatrix} W_0'(z) & 0 \\ 0 & (W_0')^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} P. \qquad (50)
$$

In this case $H_p(z)$ is determined by precisely $(M/2-1)K + \binom{M/2}{2}$ parameters.

2.5 Linear Phase and PS Symmetry

From the previous result it it immediately follows that,

Theorem 6 *$H_p(z)$ of order K forms a unitary filter bank with linear-phase and PS filters iff there exists a unitary, order K, $M/2 \times M/2$ matrix $W_0'(z)$ such that*

$$
H_p(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} I & D \\ -J & JD \end{bmatrix} \begin{bmatrix} W_0'(z) & 0 \\ 0 & (W_0')^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} P. \qquad (51)
$$

H_p is determined by precisely $(M/2-1)K + \binom{M/2}{2}$ parameters.

Notice that Theorem 2-6 gives a completeness characterization for unitary filter banks with the symmetries in question. However, if one requires the matrices $W_0'(z)$ and $W_1'(z)$ in the above theorems to be invertible on the unit circle (and not unitary), then the above results give a parameterization of non-unitary PR filter banks with the symmetries considered.

3 Associated Wavelet Tight Frames

A necessary and sufficient condition for a a unitary (FIR) filter bank to give rise to a compactly supported wavelet tight frame (WTF) is that the lowpass filter h_0 in the filter bank satisfies the linear constraint [3],

$$\sum_n h_0(n) = \sqrt{M}. \tag{52}$$

Together with the following quadratic constraint that comes from the unitariness of the filter bank,

$$\sum_n h_0(n)h_0(n + Ml) = \delta(l), \tag{53}$$

this implies that $H_0(z)$ is of the form [2],

$$H_0(z) = \left(\frac{1 + z^{-1} + \ldots + z^{-(M-1)}}{M}\right)^K Q(z) \tag{54}$$

for $K = 1$ and some polynomial Q. If $H_0(z)$ satisfies Eqn. 54 for maximal possible K, the WTF is said to be K-regular. K-regularity can be shown to be $(M-1)(K-1)$ linear constraints $H_0(z)$ or equivalently $(M-1)(K-1)$ non-linear constraints on the parameterization of $H_p(z)$. Therefore, K-regular WTFs associated with the filter banks discussed in this paper can be constructed by solving a set of non-linear equations.

We now show how 1 regular (which is required for a WTF to exist) WTFs can be constructed for the filter banks considered in this paper.

First consider the case of PS symmetry. It is easily seen that Eqn. 52 is equivalent to

$$\text{first row of } H_p(z) \text{ for } z = 1 = \left[\begin{array}{ccc} 1/\sqrt{M} & \ldots & 1/\sqrt{M} \end{array}\right]. \tag{55}$$

In Eqn. 44 since P permutes the columns, the first row is unaffected. Therefore, it is readily seen that Eqn. 55 is equivalent to the first rows of both $W_0'(z)$ and $W_1'(z)$ when $z = 1$ is given by

$$\left[\begin{array}{ccc} \sqrt{2/M} & \ldots & \sqrt{2/M} \end{array}\right]. \tag{56}$$

Notice that this is precisely the condition to be satisfied by a WTF of multiplicity $M/2$. Therefore both $W_0'(z)$ and $W_1'(z)$ give rise to multiplicity $M/2$ compactly supported WTFs. Moreover, they are parameterized each by $\binom{M/2-1}{2} + (M/2-1)K$ parameters [3]. In summary, a WTF with PS symmetry can be *explicitly* parameterized by $2\binom{M/2-1}{2} + 2(M/2-1)K$ parameters. PS symmetery does not reflect itself as any simple property of the scaling function $\psi_0(t)$ and wavelets

$\psi_i(t)$, $i \in \{1, \ldots, M-1\}$ of the WTF. However, from design and implementation points of view [5], the PS symmetry is useful in reducing computation.

Next consider PCS symmetry. In this case, from Eqn. 46 it is easily verified that Eqn. 55 is equivalent to, the first rows of the matrices A and B defined by,

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix} = \left\{ \prod_{i=K}^{0} \begin{bmatrix} A_i & B_i \\ -B_i & A_i \end{bmatrix} \right\} \tag{57}$$

are of the form $\begin{bmatrix} 1/\sqrt{M} & \ldots & 1/\sqrt{M} \end{bmatrix}$. Here we only have an implicit parameterization of WTFs, unlike the case of PS symmetry. As in the case of PS symmetery there is no simple symmetry relationships between the wavelets.

Now consider the case of linear-phase. In this case it can be seen [10] that the wavelets are also linear-phase. If we define

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} = \left\{ \prod_{i=K}^{0} \begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix} \right\}, \tag{58}$$

then it can be verified that one of the rows of the matrix $A+B$ has to be of the form, $\begin{bmatrix} \sqrt{2/M} & \ldots & \sqrt{2/M} \end{bmatrix}$. Notice that this is an implicit parameterization of the WTF.

Consider the last case of linear-phase with PCS symmetry. As before, in this case the wavelets are also linear-phase. From Eqn. 50, it can be verified that we have a WTF iff the first row of $W_0'(z)$ for $z = 1$, evaluates to the vector $\begin{bmatrix} \sqrt{2/M} & \ldots & \sqrt{2/M} \end{bmatrix}$. Equivalently, $W_0'(z)$ gives rise to a multiplicity $M/2$ WTF. In this case the WTF is parameterized by precisely $\binom{M/2-1}{2} + (M/2-1)K$ parameters.

4 Conclusion

In this paper we give a characterization of the the polyphase component matrix of a filter bank with various types of symmetry restrictions on the filters. We then specialize to the case of unitary FIR filter banks and give complete parameterizations of filter banks with above symmetries. A fundamental tool we use to obtain the results is a generalization of the scalar hyperbolic (or linear-prediction) and orthogonal lattices. Though completeness is not guaranteed, one can use the above results to generate non-unitary PR filter banks also. In some cases implicit, and in other explicit parameterizations of the associated wavelet tight frames are also given.

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