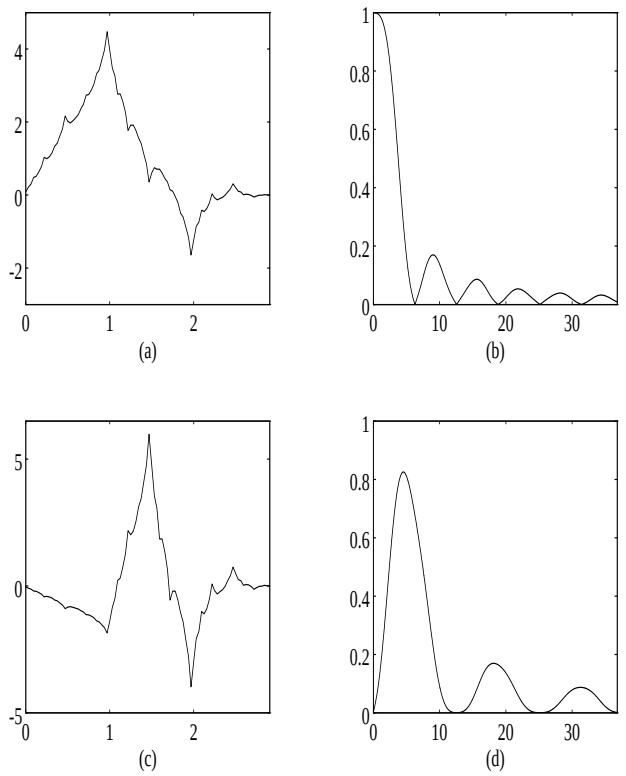


Figure 5. MWTF Example: Type 1, $M = 2$, $N = 4$. (a) Scaling Function $\psi_0(t)$ (b) Fourier transform $\hat{\psi}_0(\omega)$ (c) Wavelet $\psi_1(t)$ and (d) Fourier Transform $\hat{\psi}_0(\omega)$

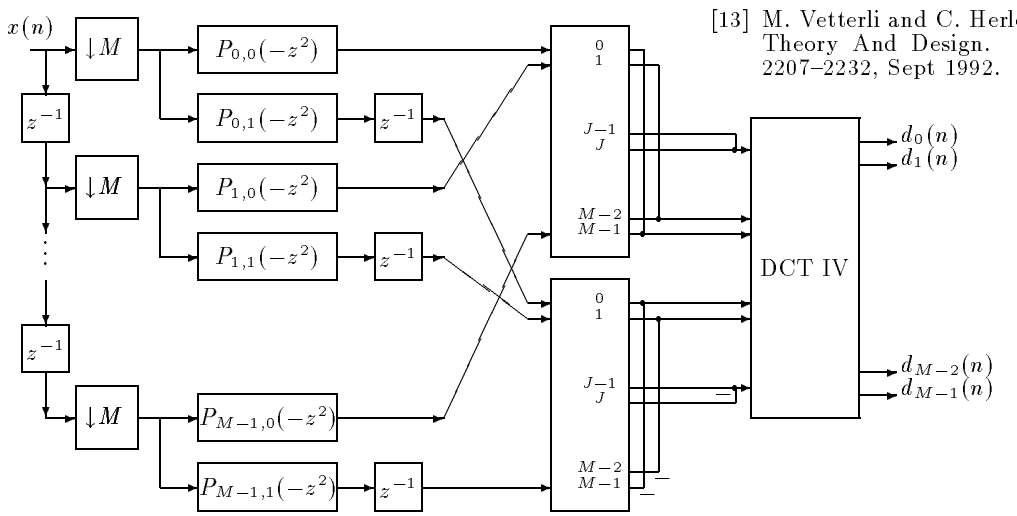


Figure 6. MFB Analysis Bank: Type 1, even M

REFERENCES

[1] R. A. Gopinath. *Wavelet And Filter Banks - New Results And Applications.* PhD thesis, Rice University, Houston, TX-77251, August 1992.

[2] R. A. Gopinath and C. S. Burrus. Wavelets and Filter Banks. Tech. Rept. CML TR91-20, Rice University, September 1991. Appears in "Wavelets - A Tutorial in Theory and Applications", edited by C.K.Chui, published by Academic Press.

[3] R. A. Gopinath and C. S. Burrus. On Cosine-Modulated orthonormal wavelet bases. In *Proceedings of Fifth DSP Workshop, Starved Rock Lodge, IL.* IEEE, Sept 1992.

[4] R. A. Gopinath and C. S. Burrus. On Cosine-Modulated Wavelet Orthonormal Bases. Technical Report CML TR92-06, Rice University, March 1992. IEEE Trans. in Image Processing, to appear.

[5] R. A. Gopinath and C. S. Burrus. Theory Of Modulated Filter Banks And Modulated Wavelet Tight Frames. Technical Report CML TR92-18, August 1992. Submitted to *Applied and Computational Harmonic Analysis: Wavelets and Signal Processing.*

[6] R. D. Koilpillai and P. P. Vaidyanathan. Cosine-Modulated Fir Filter Banks Satisfying Perfect Reconstruction. *IEEE Trans. in SP*, pages 770–783, April 1992.

[7] H. S. Malvar. Modulated QMF Filter Banks With Perfect Reconstruction. *Elect. Lett.*, 26:906–907, June 1990.

[8] J. Mau. Perfect Reconstruction Modulated Filter Banks. In *Proceedings of ICASSP, San Francisco, CA*, volume 4, pages IV–273. IEEE, 1992.

[9] T. Q. Nguyen. A Class of Generalized Cosine-Modulated Filter Banks. In *Proceedings of ISCAS, San Diego, CA*, pages 943–946. IEEE, 1992.

[10] L. Rabiner and D. Crochière. *Multirate Digital Signal Processing.* Prentice-Hall, 1983.

[11] P. P. Vaidyanathan. *Mutirate Systems and Filter Banks.* Prentice Hall, 1992.

[12] P. P. Vaidyanathan and Phuong-Quan Hoang. Lattice Structures for Optimal Design and Robust Implementation of Two-Channel Perfect Reconstruction Qmf banks. *IEEE Trans. in ASSP*, 36(1):81–93, 1988.

[13] M. Vetterli and C. Herley. Wavelets And Filter Banks: Theory And Design. *IEEE, Trans. in ASSP*, pages 2207–2232, Sept 1992.

$$N_p = \left\lfloor \frac{N}{2M} \right\rfloor J + (J - \hat{l})$$

Type 2 $N = 2Mk - 1$ & $N_p = Jk$ or,

$$N(\bmod 2M) = 2M - 1 - \hat{l} \in \{2M-1, \ldots, 2M-J\}$$

$$N_p = \left\lfloor \frac{N}{2M} \right\rfloor J + (J - \hat{l}).$$

The prototype filter in an FIR unitary MFB may be *linear-phase*. Type 1 unitary MFB has prototype symmetric about $\frac{2Mk-1}{2}$, and a Type 2 MFB is symmetric about $kM - 1$, iff

$$P_l(z) = \pm z^{-2k+1} P_{M-1-l}(z^{-1}).$$

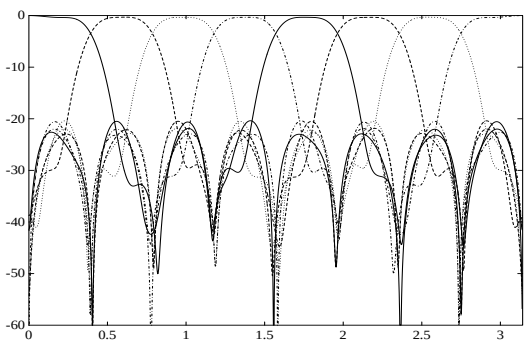


Figure 4. Type 2 MFB Design: $M = 8$, $N = 47$

A Type 1 FIR unitary MFB (as above) is said to be *canonical* if $N = 2Mk$, and a Type 2 FIR unitary MFB is said to be canonical if $N = 2Mk - 1$. Unitary MFBs considered in [6] and [8] can be considered canonical. Fig. 4 gives the frequency response of a Type 2 canonical MFB with $M = 8$ and $N = 47$ (designed numerically). The prototype filter is even symmetric and Table 1 gives $h(n)$ for $n \in \{0, \ldots, 23\}$. The rest of the coefficients are given by even symmetry.

Table 1. Type 2 MFB Design : $M = 8$, $N = 47$

n	$h(n)$	n	$h(n)$	n	$h(n)$
0	-0.01039043657002	8	-0.03173360426080	16	0.23630041086624
1	-0.00892718132451	9	-0.01894585959823	17	0.27786900073705
2	-0.00493138246397	10	-0.00722810151190	18	0.31666554094133
3	0.00000000000000	11	0.00000000000000	19	0.35355339059327
4	-0.04911989181013	12	0.07199676091875	20	0.37689530886127
5	-0.05187043862195	13	0.11008290430172	21	0.39691483978902
6	-0.04950468889839	14	0.15119308952694	22	0.40955716141868
7	0.00000000000000	15	0.00000000000000	23	0.50000000000000

In general (by relaxing the assumption that $P_l(z)$ is a polynomial in z^{-1}) it follows that each of the J lattice has k_l parameters and N_p is as in Eqn. 9. However, the length N of the filters is not easy to determine. To see this notice the following: unitariness of the MFB is not perturbed if any polyphase component $P_l(z)$ is shifted by an arbitrary delay. While this does not change N_p it definitely does affect N, the length of the filters. For any given N, there exist M channel MFBs (Type 1 or Type 2) with N as the filter length [5].

For image processing applications one desires that the filters h_i (not the prototype) are linear phase. Despite the many advantages of MFBs they have the following disadvantage (see [5]):

Lemma 2 *Linear-phase FIR unitary MFBs do not exist.*

4. IMPLEMENTATION OF MFBs

If α is even, the analysis and synthesis banks can be implemented using Type III and Type II DCTs respectively, while if α is odd they can both be implemented using a Type IV DCT [5]. For example Fig. 6 shows the implementation of a Type 1 MFB analysis bank for M even.

5. MODULATED WAVELET TIGHT FRAMES

Modulated Wavelet Tight frames for the MFBs in [6] were constructed in [3]. We now extend this result to general FIR MFBs.

Theorem 3 *An FIR MFB forms a compactly supported modulated WTF iff for* $l \in \{0, 1, \ldots, J-1\}$, $\Theta_l = \frac{\pi}{4} + \frac{\pi}{2M}(\frac{\alpha}{2} - l)$ *where*

$$\Theta_l = \sum_{k=0}^{k_l - 1} \theta_{l,k}. \tag{10}$$

independent of whether the MFB is Type 1 or Type 2 and M is even or odd.

This shows that there exist modulated WTFs for all M and any given Type. Moreover, it gives a complete parameterization of all compactly supported modulated WTFs of both types. In fact if an MFB with filters of length N is parameterized by N_p parameters, then the corresponding WTF is parameterized by precisely, $N_p - J$ parameters. This follows directly from Eqn. 10 which puts an additional constraint on each of the J two-channel pairs. MWTFS associated with canonical MFBs are called canonical MWTFs. For example, the MWTF corresponding to a canonical Type 1 MFB with $N = 2M$ is uniquely given by ($i \in \{0, \ldots, M-1\}$, $n \in \{0, \ldots, N-1\}$)

$$h_i(n) = \sqrt{\frac{1}{2M}} \left[\sin(u_i) - \sin(v_i)\right], \tag{11}$$

where

$$u_i = \frac{\pi(i+1)(n+.5)}{M} - (2i+1)\frac{\pi}{4}$$

and

$$v_i = \frac{\pi i(n+.5)}{M} - (2i+1)\frac{\pi}{4}.$$

Similarly the MWTF corresponding to a canonical Type 2 MFB with $N = 2M - 1$ is also uniquely given by Eqn. 11 where in this case

$$u_i = \frac{\pi(i+1)(n+1)}{M} - (2i+1)\frac{\pi}{4}$$

and

$$v_i = \frac{\pi i(n+1)}{M} - (2i+1)\frac{\pi}{4}.$$

Fig. 5 shows the scaling function, wavelet and their Fourier transforms for the Type 1 canonical MWTF in Eqn. 11 with $M = 2$ and $N = 4$.

In many applications it is desirable to design K-regular WTFs ([2]). K-regular MWTFs can be designed numerically. In signal processing applications, "smoothness" of the scaling functions (and hence wavelets) may be more crucial than the precise degree of regularity. With an appropriate concept of smoothness, such regular WTFs and MWTFs can be designed ([4, 5]).

6. CONCLUSION

This paper gives a complete theory of MFBs and modulated WTFs of compact support. MFBs can be classified into Type 1 and Type 2 MFBs. Depending on the type and the parity of, M, the number of channels, an MFB reduces to a set of $J(\approx M/2)$ two-channel PR FBs. A complete parameterization of all unitary FIR MFBs can be obtained. There do not exist unitary FIR linear-phase MFBs. All MFBs can be implemented efficiently using the DCT. Modulated filter banks are associated with modulated wavelet tight frames. All compactly supported multiplicity M MWTFs (of Type 1 and Type 2) are parameterized explicitly.

where $\alpha \in \mathbf{Z}$, is called the *the modulation phase*. Notice that ϵ_i and γ_i are independent of the filter length. In Eqn. 1, by substituting for $h_i(n)$ and $g_i(n)$ (from Eqn. 4) PR conditions for the general MFB can be obtained (see [5] for details). Every MFB, it turns out, decomposes into roughly $M/2$ two channel PR FBs. For $l \in \mathbf{Z}$, let $p_l(n) = h(Mn + l)$, $q_l(n) = h(Mn - l)$, and for $j \in \{0, 1\}$, $p_{l,j}(n) = h(2Mn + Mj + l)$, $q_{l,j}(n) = g(2Mn + Mj - l)$. Let $P_l(z)$, $Q_l(z)$, $P_{l,j}(z)$ and $Q_{l,j}(z)$ be the corresponding $\mathcal{Z}$-transforms. Then we have the following characterization of PR for MFBs (FIR or otherwise):

Theorem 1 (MFB PR Theorem) *An MFB is PR iff* **when α is an odd integer** *$P_l(z)$, $P_{\alpha-l}(-z)$, $Q_l(z)$ and $Q_{\alpha-l}(-z)$ form a two-channel PR FB (scaled by $\sqrt{\frac{M}{2}}$), and* **when α is an even integer** *if $l + \frac{\alpha}{2} (\mathrm{mod}\, M) \neq 0$, $P_l(z)$, $P_{\alpha-l}(z)$, $Q_l(z)$ and $Q_{-\alpha-l}(z)$ form a two-channel PR FB (scaled by $\sqrt{\frac{M}{2}}$) and $P_{\frac{\alpha}{2},0}(z) Q_{\frac{\alpha}{2},0}(z) = \frac{1}{M}$.*

Thus PR conditions on an MFB reduce to two-channel PR conditions on appropriate pairs of polyphase components of $h(n)$ and $g(n)$ (i.e $p_l(n)$ and $q_l(n)$). It turns out that the unitariness of the MFB is equivalent to the unitariness of the two-channel PR FBs. Moreover, when α is even, unitariness requires $P_{\frac{\alpha}{2},0}(z)$ to be allpass. The statement of Theorem 1 is redundant since $P_{l+Mk}(z) = z^k P_l(z)$ and $Q_{l+Mk}(z) = z^{-k} Q_l(z)$. One only needs to know PR conditions corresponding to $P_l(z)$ and $Q_l(z)$ for $l \in \{0, \ldots, M-1\}$. Moreover, the PR conditions depend only on the parity of α leading to the definition of two types of MFBs.

Definition 1 *An MFB is Type 1 if $\alpha = M - 1$ and Type 2 if $\alpha = M - 2$.*

Depending on M and and the type of the MFB, α could be even or odd, and therefore, without loss of generality, Type 1 and Type 2 MFBs are the only MFBs that have to be studied. Henceforth we assume the modulation phase $\alpha \in \{M-1, M-2\}$. There are four possibilites for MFBs depending on the type of the MFB and the parity of M. In each case, the MFB PR Theorem can be reinterpreted. Let

$$J = \begin{cases} \frac{M}{2} & \text{for Type 1 MFB, } M \text{ even} \\ \frac{M-1}{2} & \text{for Type 1 MFB, } M \text{ odd} \\ \frac{M-2}{2} & \text{for Type 2 MFB, } M \text{ even} \\ \frac{M-1}{2} & \text{for Type 2 MFB, } M \text{ odd.} \end{cases} \quad (6)$$

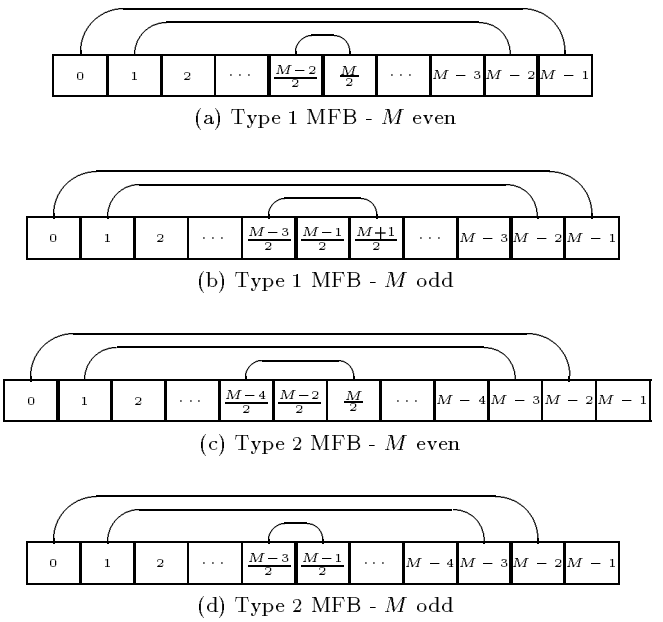
Theorem 2 *Every PR MFB has J two-channel PR pairs made of polyphase components as shown in Fig. 3. Also for Type 2 MFBs*

$$\begin{bmatrix} Q_{M-1,0} \\ Q_{M-1,1} \end{bmatrix} = \frac{1}{P_{M-1,0}^2 - z^{-1} P_{M-1,1}^2} \begin{bmatrix} P_{M-1,0} \\ z^{-1} P_{M-1,1} \end{bmatrix}. \quad (7)$$

Furthermore, for Type 1 MFBs with M odd, and Type 2 MFBs with M even,

$$P_{J,0}(z) Q_{J,0}(z) = \frac{1}{M}, \quad (8)$$

Recall that each of the J PR pairs are given by $P_l(z)$, $Q_l(z)$, $P_{\alpha-l}(-z)$, and $Q_{\alpha-l}(-z)$ for $l \in \{0, \ldots, J-1\}$. When the filters are FIR, Eqn. 7 implies $P_{M-1}(z) = A\sqrt{\frac{2}{M}} z^n$ and $Q_{M-1}(z) = \frac{1}{A}\sqrt{\frac{2}{M}} z^{-n}$ for some integer n and non-zero A. Similarly Eqn. 8 is equivalent to $P_{J,0}(z) = A\sqrt{\frac{1}{M}} z^n$ and $Q_{J,0}(z) = \frac{1}{A}\sqrt{\frac{1}{M}} z^{-n}$ for some integer n and non-zero A.



(a) Type 1 MFB - M even

(b) Type 1 MFB - M odd

(c) Type 2 MFB - M even

(d) Type 2 MFB - M odd

Figure 3. Two Channel PR Pairs in a PR MFB

3. UNITARY FIR MFBs

Theorem 2 notwithstanding, design of arbitrary MFBs is difficult as there is no simple parameterization of two channel PR FBs. However, this is not the case for unitary FIR MFBs. Unitariness of an MFB is easily characterized (see [5]):

Lemma 1 *An MFB is unitary iff $g(n) = h(-n)$ and moreover this implies that the J two channel PR FBs associated with a unitary MFB are themselves unitary.*

A unitary MFB is determined by one prototype filter. The J two-channel PR pairs are also unitary. In the FIR case, the $P_l(z)$ and $Q_l(z)$ not involved in the J unitary PR pairs have to be pure delays (Eqns. 7-8). Each of the J pairs can then be parameterized by the well-known lattice parameterization of two-channel unitary FBs ([12]). Every two channel unitary FB has filters of length $2k$ parameterized by k unknowns. For simplicity assume that for $l \in \{0, 1, \ldots, M-1\}$, $P_l(z)$ has only positive powers of z^{-1}. Then for a Type 1 MFB, we have for $l \in \{0, 1, \ldots, J-1\}$, $\begin{bmatrix} P_{l,0}(z) & P_{M-1-l,0}(z) \\ P_{l,1}(z) & -P_{M-1-l,1}(z) \end{bmatrix}$ is given by

$$\prod_{k=k_l-1}^{1} \begin{bmatrix} \cos\theta_{l,k} & z^{-1}\sin\theta_{l,k} \\ \sin\theta_{l,k} & -z^{-1}\cos\theta_{l,k} \end{bmatrix} \begin{bmatrix} \cos(\theta_{l,0}) & \sin\theta_{l,0} \\ \sin\theta_{l,0} & -\cos\theta_{l,0} \end{bmatrix}$$

where $\{\theta_{l,k}\}$ is a set of k_l parameters determining the l^{th} lattice. The total number of parameters determining the MFB is therefore

$$N_p = \sum_{l=0}^{J-1} k_l. \quad (9)$$

Since $P_l(z) = P_{l,0}(z^2) + z^{-1} P_{l,1}(z^2)$, $P_l(z)$ is FIR of length $2k_l$. This imposes constraints on the lengths of possible unitary FIR MFBs (that satisfy $P_l(z)$ is a polynomial in z^{-1}). In fact, N and N_p for the four possibilites of MFBs (with this assumption) can be obtained easily ([5]).

Type 1: $N = 2Mk$ & $N_p = Jk$ or,

$N(\mathrm{mod}\, 2M) = 2M - \hat{l} \in \{2M-1, 2M-2, \ldots, 2M-J+1\}$

THEORY OF MODULATED FILTER BANKS AND MODULATED WAVELET TIGHT FRAMES

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ABSTRACT

Perfect reconstruction (PR), FIR, unitary modulated filter banks (MFBs) have been constructed by several authors [7, 6, 8, 9]. The form of modulation used in these papers are not the same. This paper exhibits the most general form of modulation (that depends on an integral parameter, *the modulation phase*,) and characterizes PR for this general MFB problem (no FIR assumptions on the filters are made as in [7, 6, 9, 8]). An M channel PR MFB reduces to $J(\approx M/2)$ two-channel PR FBs. Unitariness of an MFB is equivalent to the unitariness of each of the two-channel FBs. Type 1 and Type 2 MFBs are introduced. All FIR unitary MFBs are parameterized. Unitary MFBs are associated with modulated wavelet tight frames (MWTFs). All compactly supported MWTFs are parameterized. MFBs can be implemented efficiently using Type III and Type II DCT (when the modulation phase is even) and Type IV DCT (when the modulation phase is odd).

1. INTRODUCTION

An M-channel filter bank (see Fig. 1.) is PR if $y(n) = x(n)$. PR is equivalent to ([10])

$$\sum_{i=0}^{M-1} \sum_{n} h_i(Mn + n_1) g_i(-Mn - n_2) = \delta(n_1 - n_2). \quad (1)$$

A PR FB is said to be *unitary* if $g_i(n) = h_i(-n)$. Unitary (FIR) FBs are practically important since they can be completely parameterized (important for their design) and efficiently implemented [11]. Moreover, they give rise to orthonormal wavelet bases for $L^2(\mathbb{R})$. Every unitary FIR FB that satisfies $\sum_n h_0(n) = \sqrt{M}$ is associated with a wavelet tight frame (WTF) and conversely. That is, if the *scaling* function, $\psi_0(t)$, and the $(M-1)$ *wavelets*, $\psi_i(t)$, are defined by

$$\psi_i(t) = \sqrt{M} \sum_k h_i(k) \psi_0(Mt - k), \quad (2)$$

$i \in \{0, \ldots, M-1\}$, then $\{\psi_{i,j,k}\} \stackrel{\text{def}}{=} \left\{M^{j/2} \psi_i(M^j t - k)\right\}$, $i \in \{1, \ldots, M-1\}$, forms a mutiplicity M compactly supported WTF for $L^2(\mathbb{R})$ ([2, 13]). For every function $f(t) \in L^2(\mathbb{R})$ one has

$$f(t) = \sum_{i=1}^{M-1} \sum_{j,k} \langle f, \psi_{i,j,k}(t) \rangle \, \psi_{i,j,k}(t). \quad (3)$$

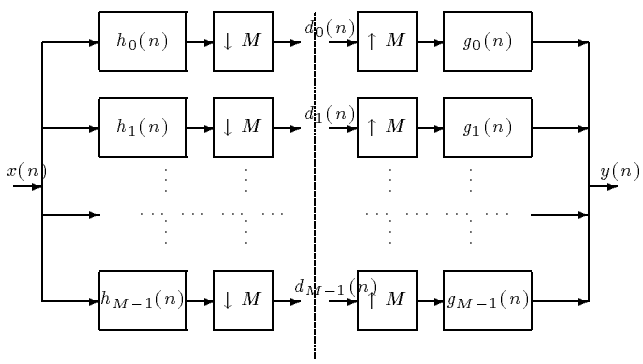

Figure 1. An M Channel Filter Bank

Modulated FBs (MFBs) are a special class of FBs where the analysis and synthesis filters are obtained by modulation of prototype filters. They are based on the assumption that ideally one desires the $h_i(n)$ and $g_i(n)$ to correspond to bandpass filters occupying adjacent bands (see Fig. 2).

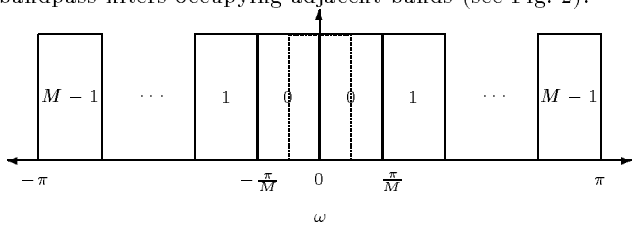

Figure 2. Ideal Responses of Filters

$$h_i(n) = h(n) \cos\left(\frac{\pi}{2M}(2i+1)n + \epsilon_i\right) \quad (4a)$$

and

$$g_i(n) = g(n) \cos\left(\frac{\pi}{2M}(2i+1)n + \gamma_i\right), \quad (4b)$$

where ϵ_i and γ_i are phase factors. The filters $h(n)$ and $g(n)$ are analysis and synthesis prototypes respectively. Modulated filter banks are easier to design since only the prototype filters need to be designed. Moreover, they can be efficiently implemented using the DCT (see Section 4.).

2. PR MODULATED FILTER BANKS

If an MFB is to be PR, then without loss of generality it can be shown that ϵ_i and γ_i are, assuming they are linear in i, *necessarily* of the form (see [1, 5] for a proof)

$$\epsilon_i = -\frac{\pi}{2M}(2i+1)\frac{\alpha}{2} \;\; ; \;\; \gamma_i = \frac{\pi}{2M}(2i+1)\frac{\alpha}{2}, \quad (5)$$

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