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could be fractal (since $T_0^n f(t)$ converges only in $L^2$). *One way* to prevent this is to impose $K$-regularity - one forces $H_0(z)$ to be of the form

$$H_0(z) = \left( \frac{1 + z^{-1} + \ldots + z^{-(M-1)}}{M} \right)^K Q(z). \qquad (11)$$

for maximal possible $K$ (and $Q(z)$ a polynomial in the FIR case). The autocorrelation $(H_0(z)H_0(z^{-1}))$ of such scaling filters that also satisfy Eqn. 3 can be explicitly obtained [12, 27, 53, 19]. In this case, one obtains all wavelet filters from Eqn. 6. Also one can solve numerically for the $v_i$'s and $V_0$ that satisfy Eqn. 11 to obtain $K$-regular WTFs. Regularity as defined here is a property of $h_0$, not a statement that $\psi_0(t)$ (and hence the wavelets) are in $C^{K-1}$. Practically it means that $H_0(\omega)$ is $(K-1)^{th}$ order flat at $\omega = 0$, or equivalently the scaling function and its translates represent polynomials of degree $(K-1)$ exactly.

WTFs have a 1-1 correspondence to unitary FBs with $H_0(\omega) = \sqrt{M}$. This suggests that one can impose this linear constraint on other types of unitary FBs to get corresponding WTFs. For unitary MFBs this demands that $\sum_k \theta_{l,k} = \frac{\pi}{4} + \frac{\pi}{2M}(\frac{\alpha}{2} - l)$, $\{\theta_{l,k}\}$ is the set of lattice angle parameters of the $l^{th}$ two-channel unitary FIR FB [23, 22]. WTFs corresponding to unitary FBs with symmetry restrictions (like linear-phase etc.) can also be obtained [52, 24].

## 4. FB TREES AND WAVELET PACKETS

$M$-band WTFs give a flexible tiling of the TF plane. They are associated with a particular tree-structured FB, where the lowpass channel at any depth is split into $M$ bands. Clearly, an arbitrary tree-structured filter bank, where all channels are split into sub-channels (using FBs with potentially different number of bands), would give a very flexible signal decomposition. The wavelet analog of this is known as the wavelet packet decomposition (WPD) [10]. For a given signal (or class of signals), one can (for a fixed set of filters) obtain the best (in some sense) FB tree-topology. For a binary tree an efficient scheme using entropy as the criterion has been developed - the best wavelet packet basis algorithm [10].

## 5. TIME-VARYING FBs AND LOCAL BASES

FB trees (wavelet packets) afford flexible decompositions of the TF plane. However, all TF tiles at a particular frequency level have the same time-resolution. The frequency-axis in the TF plane is split arbitrarily (by the choice of tree-topology) while the time-axis split is governed by the sampling rate in each channel (which is constant in a particular channel). Time-varying FBs (wavelet analogs being local sine/cosine bases [1]) are useful when the time-axis is to be split arbitrarily (the frequency-axis split being constant bandwidth). Here the time-axis is split into bins, and in each bin a certain FB is used. By combining the two dual concepts of FB trees, and time-varying FBs one can obtain arbitrary tiling of the TF plane [28, 29, 30]. One can use Eqn. 5 to design and implement time-varying filter bank trees [21].

We now give a list of FB structures and their TF analogs. A single FB (many channels) is like the DSTFT, a lowpass recursive FB tree gives the DWT, an arbitrary FB tree gives the WPD, time-varying FBs give local bases and arbitrary time-varying FB trees give arbitrary TF decompositions. Orthonormality of all these decompositions can be obtained by allowing only unitary FBs.

## 6. GENERALIZATIONS

Just as unitary FBs give ON wavelet bases, so also general PR filter banks give rise to wavelet biorthogonal bases [9, 58, 20]. Multi-d unitary FBs give ON wavelet bases on $\mathbb{R}^n$ [34].

## 7. CONCLUSION

Unitary FBs with an additional linear constraint give rise to ON wavelet bases. Unitary FB trees give rise to wavelet packet bases. Time-Varying FBs give rise to local bases. Time-Varying FB trees give arbitrary tiling of the TF plane. The advantage of using these techniques for TF decompositions is that unitary FBs are easy to design and implement. Other techniques to tile the TF plane also do exist: for example, see the matching pursuits algorithm in [37].

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Every rank $n$ projection matrix $P$ is (non-uniquely) of the form $w_1 w_1^T + \ldots + w_n w_n^T$, where $w_i$ are unit norm vectors. Therefore if $L = \sum_{i=1}^{K-1} \delta_i$, one has the (non-unique) Householder factorization ([54])

$$H_p(z) = \prod_{i=L-1}^{1} \left[ I - v_i v_i^T + z^{-1} v_i v_i^T \right] V_0. \tag{6}$$

The unitary FIR FB design problem becomes: choose $\{v_i\}$ and $V_0$ to minimize filter approximation error. Implementation of Eqn. 6 makes unitary FBs computationally more efficient than general FBs. When $M = 2$, a lattice parameterization of $H_p(z)$ is also sometimes useful [56] (corresponding factorization for $M > 2$ is not as useful as Eqn. 6 [15]).

For large $N$ and $M$, unitary FB design is still impractical. The solution is to use modulated FBs (MFBs) where (see Fig. 2) the filters are modulates of prototypes filters $h(n)$ and $g(n)$ [48, 40, 41, 32, 39, 25]:

$$h_i(n) = h(n) \cos\left( \frac{\pi}{2M} (2i+1)(n - \frac{\alpha}{2}) \right), \tag{7}$$

$$g_i(n) = g(n) \cos\left( \frac{\pi}{2M} (2i+1)(n + \frac{\alpha}{2}) \right), \tag{8}$$

where $\alpha$ is an integral modulation phase. An MFB is PR iff appropriate pairs of polyphase components (roughly $M/2$ pairs) of $h$ and $g$ form two-channel PR FBs [23]. Moreover, an MFB is unitary iff $g(n) = h(-n)$ and in this case the PR pairs form two-channel unitary FBs. Therefore, the Givens' (or Householder) parameterization of two-channel unitary FIR FBs parameterizes all unitary FIR MFBs. FIR MFBs can be efficiently implemented using the two-channel PR pairs and a DCT (for analysis and synthesis).

Sometimes (e.g., in image processing applications) one desires the filters to be linear-phase. Unfortunately, unitary FIR MFBs cannot have linear-phase filters. Unitary FBs with linear-phase can be parameterized [52, 46]. Moreover, for many other types of symmetry restrictions on the filters also the factorization in Eqn. 5 can be specialized [24]. This parameter reduction is important for design and implementation.

Tree-structured FBs are obtained by further decomposing the subband signals using FBs. Tree-structured (two-dimensional) FBs have been used in image coding [5]. Nonuniform FBs have also been designed by several authors [45, 33]. For a treatment of the multi-d FB problem see [6, 7, 18]. One difficulty in multiple dimensions is that unitary $H_p(z)$ cannot be factorized as in Eqn. 6. However, when $M = 2$, a complete Redheffer product based factorization has been reported recently [2].

## 2. NON-STATIONARY SIGNAL ANALYSIS

Non-stationary signals are best described in the TF plane. For "exploratory signal analysis" TF information can be obtained by a number of methods: Wigner-Ville distribution [60], the CWD [8] etc. Many applications require the decomposition of a signal into TF *atoms* centered at particular points in the TF plane. The "optimal" decomposition depends on the signal (or class of signals) studied. All classical TF decompositions (like the Discrete Short-Time Fourier Transform (DSTFT)), however, are signal independent.

### 2.1. DSTFT

The DSTFT uses basis functions (atoms) of the form $w_{j,k}(t) = w(t - k\tau_0) e^{-\imath j \omega_0 t}$ where $w(t)$ is a window function [17]. If these functions form an orthonormal (ON) basis $x(t) = \sum_{j,k} \langle x, w_{j,k} \rangle \, w_{j,k}(t)$. The DSTFT coefficients, $\langle x, w_{j,k} \rangle$, estimate the presence of a signal component centered at $(k\tau_0, j\omega_0)$ in the TF plane, i.e., the DSTFT gives a uniform tiling of the TF plane with the atoms $\{w_{j,k}(t)\}$. If $\Delta_t$ and $\Delta_\omega$ are time and frequency resolutions respectively of $w(t)$, then the uncertainly principle demands that $\Delta_t \Delta_\omega \leq 1/2$ [13]. Moreover, if the basis is ON, the Balian-Low theorem implies either $\Delta_t$ or $\Delta_\omega$ is infinite [13]. Both $\Delta_t$ and $\Delta_\omega$ can be controlled by the choice of $w(t)$, but for any particular choice, there will be signals for which either the time or frequency resolution is not adequate.

### 2.2. DWT

The Discrete Wavelet Transform (DWT) is another signal independent tiling of the TF plane suited for signals where high frequency signal components have shorter duration than low frequency signal components. TF atoms for the DWT, $\{\psi_{j,k}(t)\} = \left\{ 2^{j/2} \psi(2^j t - k) \right\}$, are obtained by translates and dilates of the wavelet function $\psi(t)$. One shrinks/stretches the wavelet to capture high/low frequency components of the signal. If these TF atoms form an ON basis $x(t) = \sum_{j,k} \langle x, \psi_{j,k} \rangle \, \psi_{j,k}(t)$. The DWT coefficients, $\langle x, \psi_{j,k} \rangle$, estimate the presence of a signal component centered at $(2^{-j}k, 2^j)$ in the TF plane, giving yet another tiling of the TF plane. It turns out that the DWT (for compactly supported wavelets) can be efficiently computed using two-channel unitary FIR FBs [12].

Notice that for non-stationary signal analysis, in general one desires methods for arbitrary tiling of the TF plane, not just the two special case above (their importance notwithstanding). A general way to obtain ON wavelets $\psi(t)$ is using unitary FIR FB theory.

## 3. ON WAVELETS AND UNITARY FBs

Several ON wavelet bases were constructed [42, 3] before the multiresolution analysis (MRA) approach to ON wavelet bases was discovered [36]. Mallat noticed the connection between MRA and two-channel FBs[36]. Daubechies exploited this connection to construct ON wavelet bases of compact support [12]. A comprehensive treatment of wavelets associated with two-channel FBs (and more) is given in [14, 49]. We describe the precise theoretical and computational relationship between $M$-band wavelets and $M$-channel FBs [20, 61]. Consider a unitary FIR FB such that $\sum_n h_0(n) = \sqrt{M}$. Equivalently (in Eqn. 6) the first row of $V_0$ is $\left[ \begin{array}{ccc} 1/\sqrt{M} & \ldots & 1/\sqrt{M} \end{array} \right]$. Define the operators $T_i$:

$$T_i f(t) = \sqrt{M} \sum_k h_i(k) f(Mt - k). \tag{9}$$

Given $f(t)$, $\int f(t) dt = 1$, $T_0^n f$ converges to the unique *scaling function* $\psi_0(t)$. Clearly $T_0 \psi_0(t) = \psi_0(t)$. For $i \in \{1, \ldots, M-1\}$ define the wavelets $\psi_i(t) = T_i \psi_0(t)$. The translates and dilates of the wavelet, it can be shown, forms a wavelet tight frame (WTF) [20]. That is, if $\psi_{i,j,k}(t) = M^{j/2} \psi_i(M^j t - k)$, then

$$x(t) = \sum_{i=1}^{M-1} \sum_{j,k} \langle x, \psi_{i,j,k} \rangle \, \psi_{i,j,k}(t). \tag{10}$$

In wavelet terminology $h_0$ is the scaling filter and the rest are wavelet filters. The tile corresponding to the atom $\psi_{i,j,k}(t)$ is centered at $(M^{-j}k, iM^j)$ in the TF plane.

If we are given $\langle x, \psi_{0,J,k} \rangle$, then the output of a unitary FB with analysis filters $\{h_i(-n)\}$ gives $\langle x, \psi_{i,J-1,k} \rangle$, $i \in \{0, \ldots, M-1\}$ [20]. Therefore, the DWT coefficients for all scales $j < J$, can be computed by using this FB in a tree-structure where the lowpass channel is recursively split into $M$-channels.

Let $a(n) = \int \psi_0(t) \psi_0(t+n) \, dt$. A compactly supported WTF is an ON basis iff $[\downarrow M][H_0(z) H_0(z^{-1}) A(z)] = A(z)$ [35, 14, 19]. Even if one has an ON basis, the scaling function

# A TUTORIAL OVERVIEW OF FILTER BANKS, WAVELETS AND INTERRELATIONS

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## ABSTRACT

Two channel filter banks (FBs) were developed for subband speech coding [16]. Since then FB theory has grown to include FBs with more than two channels, tree-structured FBs etc. A rich theory for the design and implementation of FBs exists. Wavelets were originally developed as an alternative to the STFT for non-stationary signal analysis [26]. Since then it has been realized that this is one of several techniques for decomposition of non-stationary signals into time-frequency (TF) atoms. FBs and these TF decompositions (one discrete-time and the other continuous-time) are closely related theoretically and practically. This paper explores this relationship.

## 1. PR FBs AND TMs

FBs decompose *discrete* signals into subband signals which are encoded efficiently based on perceptual criteria [11]. FBs (known as QMF banks) have been studied since 1977 [16, 31, 47, 50]. In these FBs, the input signals could not be recovered exactly from subband signals - i.e., the FBs were not PR. Yet FBs were attractive partly because of efficient implementations using polyphase representations [4]. Nontrivial PR FBs were not constructed until 1984 [51, 43]. PR results have since been extended to $M$ channel FBs. In 1986 the FB problem and the transmultiplexer (TM) problem (TMs are used for TDM/FDM conversion) were shown to be duals [59].

An $M$-channel FB is shown in Fig. 1. Transposition of the analysis/synthesis banks gives the dual TM. The FB problem is to choose $\{h_i\}$ and $\{g_i\}$ so that they approximate ideal bandpass filters (see Fig. 2) and $y(n) = x(n)$ (PR condition).

A number of variations are possible: the number of channels, $L$, is not equal to $M$, the filters are linear-phase or have symmetry restrictions, the problem is in multiple dimensions, the filters are non-uniform band, the signals are periodic etc.

The FB/TM problem is a constrained minimization problem - to minimize the approximation error subject to the PR constraint. This brute force approach to FB design is sometimes quite powerful [44, 45]. For moderate $M$ and filter length $N$, however, the method becomes impractical.

The algebraic PR conditions are independent of filter lengths, causality of filters, dimensionality etc. If $H_p(z)$ and $G_p(z)$ are the polyphase component matrices of the analysis and synthesis filters [55, 20], then an FB is PR iff $G_p^T(z)H_p(z) = I$ and a TM is PR iff $H_p(z)G_p^T(z) = I$. The FB and TM PR conditions are identical iff $M = L$. In a PR FB, if the input is $g_j(n)$, so is the output. But an output of $g_j(n)$ is obtained by letting $d_i(n) = \delta(n)\delta(i-j)$. The FB PR property ($G_p^T(z)H_p(z)$) captures precisely this fact [20]:

$$\sum_n h_i(n)g_j(Ml-n) = \delta(l)\delta(i-j). \qquad (1)$$

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A similar analysis shows that a TM is PR iff

$$\sum_i \sum_n h_i(Mn+n_1)g_i(-Mn-n_2) = \delta(n_1-n_2). \qquad (2)$$

(Para)unitary FBs/TMs are a class of PR FBs/TMs where

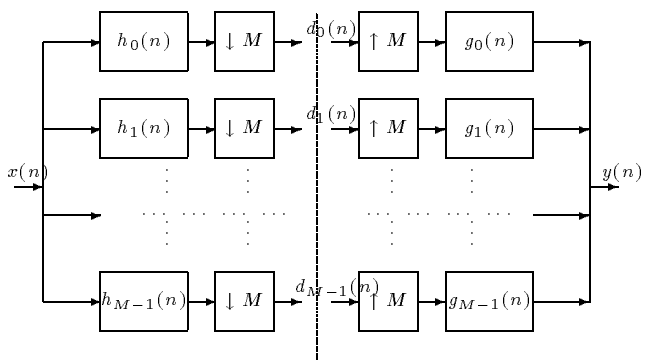


Figure 1. An $M$ Channel Filter Bank

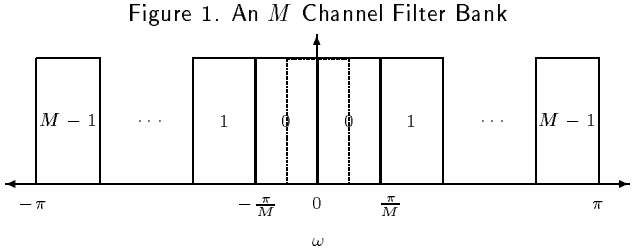


Figure 2. Ideal Responses of Filters

$g_i(n) = h_i(-n)$. Such FBs/TMs can be parameterized converting the design problem to an unconstrained minimization problem [54]. For unitary FBs, $G_p(z) = H_p(z^{-1})$ and PR equivalent to $H_p(z)$ being unitary on the unit circle ($H_p^T(z^{-1})H_p(z) = I$). Eqn. 1 and Eqn. 2 become respectively

$$\sum_n h_i(n)h_j(Ml+n) = \delta(l)\delta(i-j), \qquad (3)$$

$$\sum_i \sum_n h_i(Mn+n_1)h_i(Mn+n_2) = \delta(n_1-n_2). \qquad (4)$$

Filters satisfying Eqn. 3 are known as LOTs [38, 57].

If $MK - M < N \leq MK$, $H_p(z)$ is a degree $(K-1)$ polynomial matrix. If $H_p(z)$ is unitary, then there exist $(K-1)$ projection matrices $P_i$ of rank $\delta_i$ and an orthogonal matrix $V_0$ such that [21]

$$H_p(z) = \prod_{i=K-1}^{1} \left[I - P_i + z^{-1}P_i\right] V_0. \qquad (5)$$

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