A Wavelet Multigrid Preconditioner for Dirichlet Boundary-Value Problems in General Domains

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Abstract

We present a wavelet multigrid preconditioner for the conjugate gradient method which gives an efficient solver for the linear system arising from a wavelet-Galerkin discretization of a Dirichlet boundary-value problem via a penalty/fictitious domain formulation. The preconditioner is chosen to be a wavelet-based multigrid method for solving the same elliptic equation, however over the fictitious domain and with periodic boundary conditions. Numerical experiments described in the paper confirm the efficiency of this new iterative solver.

Key words: wavelets, penalty/fictitious domain formulation, Galerkin methods, multilevel methods, preconditioned cg-method
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1 Introduction

We shall provide a wavelet multigrid preconditioner for the conjugate gradient method applied to a class of linear systems arising by a wavelet-based discretization of the following elliptic model problem over a bounded domain ω in $\mathbf{R}^2$ (there is however no principal restriction to two dimensions),

$$-\alpha \triangle u + u = f, \qquad \text{in } \omega,$$

$$u = g, \qquad \text{on } \partial\omega,$$

where α is a positive constant.

First, we consider the above differential equation on a square with periodic boundary conditions (Section 3). We discretize this problem by a wavelet-Galerkin method similar to those discussed in [13], [14] and [15]. To solve the discrete problem, we have developed multilevel methods; the discretization step-size independent convergence rate of these methods can be proved by techniques closely related to those used for finite difference and finite element approximations (see, e.g. [8] and [17] for related references). A crucial tool for studying the multilevel solution of the wavelet-based approximate problems is the *Mallat transformation* described in e.g. [12]. Here, the Mallat transformation plays the fundamental role of *prolongation* and *restriction* between the consecutive levels.

Things become more complicated for boundary-value problems over generally shaped domains. In Section 4 we use the fictitious domain/penalty methods described in [6], [14] and [15] to reduce these problems to elliptic problems for closely related operators — in the embedding domain (box-shaped in general). The presence of the penalty term requires a special attention in order to achieve an efficient solver. Indeed, the condition number of the corresponding discrete system is dominated by the penalty term. A careful analysis shows how to overcome this difficulty. We end up with a modified linear system which is akin to that one we obtained by the discretization of the periodic problem over the square already studied in Section 3. Now, it is near at hand to use our periodic multigrid solver as a preconditioner for a conjugate gradient method applied to the modified system approximating the penalized boundary-value problem over the fictitous domain. The resulting iterative solver is highly efficient which is demonstrated by various numerical experiments.

We start our considerations by shortly recalling the necessary wavelet vocabulary.

2 Wavelet Analysis

2.1 Wavelet System

In this section, we will briefly recall from [3] various definitions and properties of the Daubechies wavelets. For a positive integer N, the Daubechies *scaling* function φ and *wavelet* function ψ of *order* N are defined as follows: There exist $2N$ real numbers a_i, $i = 0, 1, \ldots, 2N-1$, satisfying

$$\sum_k a_k = 2 \qquad \text{and} \qquad \sum_k a_{k+2l} a_k = 2\delta_{0,l} \quad \text{for all } l \in \mathbf{Z},$$

so that

$$\varphi(x) = \sum_k a_k \varphi(2x-k), \quad \text{for all } x \in \mathbf{R}, \tag{2.1}$$

and

$$\psi(x) = \sum_k b_k \varphi(2x-k), \quad \text{for all } x \in \mathbf{R}, \tag{2.2}$$

where $b_k = (-1)^k a_{2N-k-1}$, for $k = 2-2N, \ldots, 1$. The functions φ and ψ are compactly supported, with $\operatorname{supp}(\varphi) = \operatorname{supp}(\psi) = [0, 2N-1]$. For convenience, we define $a_k = 0$ for $k \notin [0, 2N-1]$.

Furthermore, φ and ψ are in $C^{\alpha(N)}$, the space of Hölder continuous functions with exponent $\alpha(N)$, where $\alpha(2) \approx 0.55, \alpha(3) \approx 1.09, \alpha(4) \approx 1.62$ and $\alpha(N) \approx 0.2075N$ for large N, see [5]. For $j, k \in \mathbf{Z}$, let us define

$$\varphi_k^j(x) := 2^{j/2}\varphi(2^j x - k), \quad \text{for all } x \in \mathbf{R}.$$

Set, for $j \in \mathbf{Z}$, $V_j = \text{closure}\ (\text{span}\{\varphi_k^j : k \in \mathbf{Z}\})$. Then, $\{\varphi_k^j :\ j, k \in \mathbf{Z}\}$ is an orthonormal basis for V_j. Also, $L^2(\mathbf{R}) = \text{closure}\ (\cup_j V_j)$, in the sense that for any function $f \in L^2(\mathbf{R})$, the orthogonal projection f_j of f onto V_j converges to f in $L^2(\mathbf{R})$ as $j \to +\infty$.

2.2 Mallat Transformations

The (periodic) *Mallat Transformations* h, g : $\mathbf{R}^n \to \mathbf{R}^{n/2}$, n even, of a vector $v \in \mathbf{R}^n$ are defined by

$$(h\,v)_k = \frac{1}{\sqrt{2}} \sum_{l=0}^{2N-1} a_l\, v_{l+2k}\,, \quad k = 0, \dots, n/2-1\,, \tag{2.3}$$

$$(g\,v)_k = \frac{1}{\sqrt{2}} \sum_{l=0}^{2N-1} b_l\, v_{l+2k}\,, \quad k = 0, \dots, n/2-1\,, \tag{2.4}$$

where we extend v periodically, i.e. $v_l = v_{l+n}$. The coefficients a_l in (2.3) and b_l in (2.4) are those in (2.1) and (2.2), respectively.

The Mallat Transformations satisfy (see [3], [12]),

$$\begin{aligned} h^t h \,+\, g^t g &= I\,, \\ h h^t \,=\, g g^t &= I\,, \\ g h^t \,=\, h g^t &= 0\,. \end{aligned}$$

We use I to denote the identity matrix of appropriate size throughout this paper.

3 Periodic Boundary-Value Problem

In this section, we are going to discuss a wavelet based multigrid method for the following simple but typical elliptic boundary-value problem,

$$-\,\alpha \triangle u + u = f, \qquad \text{in } \Omega, \tag{3.1}$$

$$u \text{ is periodic on the boundary of } \Omega, \tag{3.2}$$

where α is a positive constant and Ω is a square of side length s in $\mathbf{R}^2$ (for convenience, we assume that s is a positive integer and $s \geq 4N-3$).

3.1 Wavelet Based Discretization

We introduce the Sobolev space $H^1_p(\Omega)$, $\Omega = (0,s)^2$, with periodic boundary conditions,

$$H^1_p = H^1_p(\Omega) := \{v \in L^2(\Omega) \ : \ v_x,\, v_y \in L^2(\Omega), \\ v(0,y) = v(s,y),\, v(x,0) = v(x,s)\}.$$

The weak or variational formulation of the boundary-value problem (3.1), (3.2) becomes:

$$\text{find } u \in H^1_p : \ \mathcal{A}(u,v) = \int_\Omega fv\, dxdy, \qquad \text{for all } v \in H^1_p, \tag{3.3}$$

where $\mathcal{A}$ is the H^1_p-elliptic bilinear form

$$\mathcal{A}(u,v) = \int_\Omega (\alpha \nabla u \cdot \nabla v + uv)\, dxdy. \tag{3.4}$$

Due to the Lax-Milgram theorem [2] (3.3) has a unique solution u. For a Galerkin discretization of (3.3), we introduce the periodic wavelet-Galerkin spaces at level L, by

$$V^p_L = V^p_L(0,s) := \{v \in L^2(0,s) \ : \ v(x) = \sum_{k \in \mathbf{Z}} c_k \varphi^L_k(x),\, x \in [0,s], \\ \text{with } c_k = c_{k+2^L s}\}.$$

Obviously, V^p_L has the dimension $n_L = 2^L s$. The wavelet-Galerkin approximation $u^L \in V^p_L \otimes V^p_L$ to u is the unique solution of

$$\mathcal{A}(u^L, v^L) = \int_\Omega f v^L\, dxdy, \qquad \text{for all } v^L \in V^p_L \otimes V^p_L. \tag{3.5}$$

We expand u^L in

$$u^L(x,y) = \sum_{i,j \in \mathbf{Z}} u^L_{i,j} \varphi^L_i(x) \varphi^L_j(y),$$

where the expansion coefficients are periodic in each index with period n_L. Further, we define

$$f^L_{i,j} := \int_\Omega f(x,y) \varphi^L_i(x) \varphi^L_j(y)\, dxdy.$$

By introducing the following connection coefficients (see [1] and [11])

$$\Gamma_k = \int_{\mathbf{R}} \varphi'(x)\varphi'(x-k)\,dx, \quad k = 2-2N,\ldots,2N-2,$$

one can derive from (3.5) the following linear system for the unknowns $u_{k,l}^L$'s in compact form [15],

$$\alpha(c_L u^L + u^L c_L) + u^L = f^L, \tag{3.6}$$

where ($n = n_L$)

$$u^L = \begin{pmatrix} u_{11}^L & u_{12}^L & \cdots & u_{1n}^L \\ u_{21}^L & u_{22}^L & \cdots & u_{2n}^L \\ \vdots & \vdots & \vdots & \vdots \\ u_{n1}^L & u_{n2}^L & \cdots & u_{nn}^L \end{pmatrix}, \tag{3.7}$$

and f^L is arranged in the same way.

In (3.6) c_L is a $n \times n$ symmetric circulant matrix (see e.g. [4] for a discussion and algorithm concerning circulant matrices) with the first row

$$\delta_L^{-2}\,(\Gamma_0\,\Gamma_1\,\cdots\Gamma_{2N-2}\,0\cdots 0\,\Gamma_{2N-2}\,\cdots\Gamma_1), \tag{3.8}$$

where $\delta_L := 2^{-L}$ is the *discretization step-size.*

Lemma 3.1 *Let u be the solution of the variational problem* (3.3) *and let u^L be the approximate solution of* (3.5), *then*

$$||u - u^L||_0 \le C\delta_L^2||u||_2 \le C\delta_L^2||f||_0,$$

the constant C being independent of δ_L.

The norms $||\cdot||_s$ in the above lemma correspond to the Sobolev spaces $H^s(\Omega) = W^{s,2}(\Omega)$, see e.g. [2]. See [16] for the proof.

If we use U_L and F_L to denote the $n^2 \times 1$ vector resulting from u^L and f^L lexicographically, A_L to denote the corresponding $n^2 \times n^2$ coefficient matrix resulting from (3.6), then we have the desired linear system

$$A_L U_L = F_L. \tag{3.9}$$

3.2 A Multigrid Approach

Let A_k, $k = l, \cdots, L$, $0 \leq l < L$, denote the discretization matrix of (3.1) and (3.2) at level k. Thus, A_k has the dimension $n_k^2 \times n_k^2$ where $n_k = 2^k s$. Correspondingly, we have F_k's and the unknown U_k's, as in (3.9). In order to establish the multigrid process, we define the *basic iterative method* (BIM) on each level k by

$$U_k^{m+1} = U_k^m - L_k^{-1}(A_k U_k^m - F_k), \tag{3.10}$$

where $S_k = I - L_k^{-1} A_k$ is called the *iteration matrix* of (3.10) and where L_k is an "approximate" inverse of A_k. For instance, if $L_k = \beta^{-1} I$, then we have the *Richardson iteration*, and if $L_k = \beta^{-1} D_{A_k} := \beta^{-1}\mathrm{diag}(A_k)$, then we have the *damped Jacobi iteration*. Here β is a suitable damping factor. In our applications we will choose β sufficiently small such that the resulting Jacobi method is a symmetric iteration [9].

Now, we describe the recursive periodic *multigrid procedure* (MGP) for this periodic case. For this procedure we consider a range of levels with 0 being the coarsest and $L > 0$ being the finest level. We let h_k be the periodic Mallat transformation acting on data at level k as given in (2.4), where $n = n_k$, and we let H_k be the tensor product $h_k \otimes h_k$, where $0 \leq k \leq L$. We now have the varying quantities in our multigrid procedure:

k the actual level, $0 \leq k \leq L$,
w the approximate solution at level k, $w \in \mathbf{R}^{n_k^2}$,
b the righthand side and defect at level k, $b \in \mathbf{R}^{n_k^2}$.

Then we define $\mathbf{MGP}(k, w, b)$ as follows:

$\mathbf{MGP}(k, w, b)$

begin

if $k = 0$ **then** $w := A_l^{-1} b$ (*exact solution on the coarsest level*)
else

$$w := S_k^\nu w + \sum_{i=0}^{\nu-1} S_k^i L_k^{-1} b \quad (\nu \textit{ steps of BIM on level } k) \tag{3.11}$$

$$d := H_k (A_k w - b) \qquad (\textit{restriction of the defect to level } k-1)$$

$$v := 0$$

$$
\begin{array}{lll}
\mathbf{MGP}(k-1,v,d) & & (\textit{MGP starting on level } k-1 \\
 & & \quad \textit{with initial guess } v=0) \\
w \;:=\; w \,-\, H_k^t v & & (\textit{prolongation of the } k-1 \textit{ level approximation} \\
 & & \quad \textit{to the higher level } k, \textit{ coarse grid correction}\,)
\end{array}
\tag{3.12}
$$

end

One step of the *multigrid method* (MGM) is performed by

$$
\begin{aligned}
w &:= U_L^m, \\
&\mathbf{MGP}(L,w,F_L), \\
U_L^{m+1} &:= w.
\end{aligned}
\tag{3.13}
$$

Remark: a) The MGP describes one V-cycle with presmoothing only. To achieve a W-cycle, perform (3.12) twice. Applying (3.11) again after the coarse grid correction yields an MGM (3.13) with postsmoothing.

b) The linear system which has to be solved on the coarsest level has the relatively large dimension $n_0^2 = s^2 \geq (4N-3)^2$. However, it can be solved in an efficient manner if one considers its compact form (3.6) and uses the circulant structure of the matrix c_0.

3.3 Convergence Analysis

To prove the δ_L-independent convergence of MGM (3.13), we follow the theory of Hackbusch [8], and for this, we need the following notation. For simplicity we denote

$$
X_k := X_k(\Omega) := V_k^p \otimes V_k^p, \tag{3.14}
$$

and we will call X_k the (finite-dimensional) *periodic scaling space of level* k approximating $H_p^1(\Omega)$. For a finite dimensional space $V := \mathrm{span}\{e_1,\ldots,e_m\} \subset L^2(\Omega)$, we define the transformation $P : \mathbf{R}^m \to V$ as $P(\{x_k\}_{k=1}^m) := \sum_{k=1}^m x_k e_k$. We denote by R the adjoint operator of P with respect to the L^2 scalar product. We use P_k, R_k to denote such operators for the space X_k and we let $\|P_k\|$, $\|R_k\|$ denote the Euclidean norm of these finite-dimensional operators.

In order to be able to adapt the proofs in Section 6.3 in [8], we supply the following lemma. Its straightforward proof is omitted.

Lemma 3.2 *We have*

(i) $R_k P_k = I, \; \|P_k\| = \|R_k\| = 1,$

(ii) $H_k^t = R_k P_{k-1}$ *and* $H_k = R_{k-1} P_k$*,*

(iii) $A_{k-1} = H_k A_k H_k^t$.

By taking Lemma 3.1 into account, the *approximation property*

$$\|A_k^{-1} - H_k^t A_{k-1}^{-1} H_k\| \;\le\; C_A\, \delta_k^2, \quad 0 \le k \le L,$$

where C_A is a constant being independent of δ_k, follows readily from the standard proof for the finite element case, see Proposition 6.3.14 in [8]. Here and later $\|\cdot\|$ denotes the Euclidean norm. Consequently, we have the following theorem, cf. Section 7.2 of [8].

Theorem 3.3 *Let* $M_L^{\mathrm{mg}}(\nu_1, \nu_2)$ *denote the iteration matrix of an* MGM (3.13) *for a V-cycle with* ν_1 *presmoothing,* ν_2 *postsmoothing steps and with a* BIM *chosen to be a sufficiently strong damped Jacobi iteration. Then, if* $\nu_1 + \nu_2 > 0$*, the spectral radius of* $M_L^{\mathrm{mg}}(\nu_1, \nu_2)$ *satisfies*

$$\rho(M_L^{\mathrm{mg}}(\nu_1, \nu_2)) \le \frac{C}{\sqrt{C + \nu_1}\sqrt{C + \nu_2}} < 1,$$

where C *is a positive constant independent of* δ_L *and* ν_1*,* ν_2*.*

4 General Boundary-Value Problems

In this section, we will present a wavelet-based preconditioned conjugate gradient method (*cg*-method) [10] for solving the Dirichlet problem over a general shaped domain in higher dimensions (here we limit ourselves to two-dimensional problems, but the methodology and the algorithm can be carried over to any dimension in a straightforward manner).

Let ω be a bounded domain in $\mathbf{R}^2$ with a Lipschitz-continuous boundary $\partial\omega$. We look for $u \in H^1(\omega)$, such that

$$-\alpha \triangle u + u = f, \qquad \text{in } \omega, \tag{4.1}$$

$$u = g, \qquad \text{on } \partial\omega, \tag{4.2}$$

where $f \in L^2(\omega)$, $g \in H^{\frac{1}{2}}(\partial\omega)$ and where α is a positive constant.

In order to avoid generating a complex grid matching the geometry of ω, we instead use the fictitious domain/penalty formulation following the idea in [15]. For $\epsilon > 0$, let Ω be a square containing ω. We seek a $u^\epsilon \in H_p^1(\Omega)$, such that

$$\int_\Omega (\alpha \nabla u^\epsilon \cdot \nabla v + u^\epsilon v) dx + \frac{1}{\epsilon} \int_{\partial\omega} u^\epsilon v \, ds = \int_\Omega \tilde{f} v \, dx + \frac{1}{\epsilon} \int_{\partial\omega} g v \, ds, \qquad (4.3)$$

for all $v \in H_p^1(\Omega)$, where, in (4.3), $\tilde{f}$ is an arbitrary L^2-extension of f in Ω. Using general results on penalty methods proved in, e.g. [7], Chapter 1, we can easily show that u^ϵ converges to $\tilde{u}$ in $H^1(\Omega)$, where $\tilde{u}$ is the $H^1(\Omega)$-extension of the solution of the following variational problem: $\tilde{u} \in H_p^1$, $\tilde{u} = g$ on $\partial\omega$,

$$\int_\Omega (\alpha \nabla \tilde{u} \, \nabla v + \tilde{u} \, v) dx \; = \; \int_\Omega \tilde{f} \, v \, dx$$

for all $v \in H_p^1$, such that $v = 0$ on $\partial\omega$.

4.1 The Wavelet-Galerkin Discretization

From the analysis in [15], we know that there is a wavelet expansion at level L for the numerical boundary measure $\mu^L \in V_L$, so that for any $g^L \in V_L$, one has

$$\int_{\partial\omega^L} g^L \, ds = \int_\Omega \mu^L g^L \, dx \longrightarrow \int_{\partial\omega} g \, ds$$

as $L \to \infty$. Therefore, by applying the Galerkin method to (4.3) with respect to the space X_L (3.14), we obtain the following linear system for $u^{\epsilon,L}$ written in compact form as follows

$$\alpha(c_L u^{\epsilon,L} + u^{\epsilon,L} c_L) + u^{\epsilon,L} + \frac{1}{\epsilon}\mu^L \times u^{\epsilon,L} = f^L + \frac{1}{\epsilon}\mu^L \times g^L, \qquad (4.4)$$

where $u^{\epsilon,L}$, f^L, g^L and μ^L are square matrices defined as in (3.7). The operation $A \times B$ is the multiplication of two matrices A, B of the same size obtained by multiplying corresponding entries.

Theoretically, the boundary measure μ is supported on $\partial\omega$. At level L, μ^L will have the same support as the gradient of χ_ω^L, the characteristic function of ω sampled at level L, see [15]. So geometrically, the support of μ^L is contained in a tubular neighborhood of $\partial\omega$ of width $2N/2^L$, where N is the order of the Daubechies wavelets. In our approximation, we set the entry of μ^L to be 1 where $\|\nabla\chi_\omega^L\|$ is not zero, that is, μ^L acts like the characteristic function of that tubular neighborhood, since we are going to choose ϵ very small. With this choice of μ^L, the approximate solution $u^{\epsilon,L}$ converges to the exact solution for the H^1-norm inside ω as $\epsilon \to 0$ and $L \to \infty$. On the boundary $\partial\omega$, we note that for all L, $u^{\epsilon,L} = g + C\epsilon$ where the constant C depends only on the norm of $||u^\epsilon||_{H^1}$ which is uniformly bounded in ϵ, see e.g. [7].

From (4.4) we can derive an $n_L^2 \times n_L^2$ linear system

$$A_L U_L^\epsilon + \frac{1}{\epsilon} M_L U_L^\epsilon = F_L + \frac{1}{\epsilon} M_L G_L, \tag{4.5}$$

obtained by rearranging all the expansion coefficients in lexicographical order. Note that M_L representing the numerical boundary measure μ^L is a diagonal matrix with diagonal elements either 0 or 1.

Remark: Because the entries of the circulant matrix c_L (3.8) increase with the factor δ_L^{-2} as L gets larger, the discrete penalty formulations (4.4) and (4.5) are meaningful only if the penalty parameter ϵ is much smaller than δ_L^2: $\epsilon \ll \delta_L^2$. From now on we will assume this natural condition for the penalty formulation.

4.2 A Preconditioned *cg*-Method

If one uses a standard multigrid method for solving (4.5) then the error explodes with the choice of small ϵ. To overcome this divergence one could try to apply a multigrid method with a block version of a BIM where one block is formed by the unkowns corresponding to the boundary $\partial\omega$, as suggested in [8]. However, the implementation of this approach depends strongly on the shape of the domain ω, a drawback we would like to avoid.

To derive an efficient solver (in terms of performance and convenient coding) we study the condition number κ of the matrix $A_{L,\epsilon} := A_L + \epsilon^{-1} M_L$ because $\kappa(A_{L,\epsilon})$ determines the convergence speed of the *cg*-method applied to (4.5), see e.g. [9]. We have that $\kappa(A_{L,\epsilon}) = O(\delta_L^{-2}\epsilon^{-1})$ and remembering that we have chosen $\epsilon \ll \delta_L^2$ above, ϵ^{-1} affects the condition number most.

In a first step we therefore try to eliminate the influence of ϵ. Since $\epsilon \ll \delta_L^2$ we consider the limit of the family $\{U_L^\epsilon\}_{\epsilon>0}$ of solutions of (4.5) as ϵ tends to zero. We will use the maximum norm $\|\cdot\|_\infty$.

Lemma 4.1 *Let $U_L^\epsilon = A_{L,\epsilon}^{-1}(F_L + \epsilon^{-1} M_L G_L)$ be the solution of* (4.5), *then there exists a $U_L \in \mathbf{R}^{n_L^2}$ such that*

$$\|U_L^\epsilon - U_L\|_\infty = O(\epsilon).$$

Moreover, U_L is uniquely determined by

$$(I - M_L)A_L U_L = (I - M_L)F_L \qquad \textit{and} \qquad M_L U_L = M_L G_L\,.$$

Proof: Without loss of generality we may assume that $M_L = \operatorname{diag}(m_i : 1 \leq i \leq n)$ with $m_i = 1$ for $1 \leq i \leq k < n$ and $m_i = 0$ otherwise. Here, we set $n = n_L^2$. Using Cramer's rule the i-th component of U_L^ϵ can be expressed by

$$(U_L^\epsilon)_i = \frac{1}{\det \epsilon A_{L,\epsilon}} \underbrace{\det\left(\epsilon a_{L,\epsilon}^1 \cdots \epsilon a_{L,\epsilon}^{i-1}\ \epsilon F_L + M_L G_L\ \epsilon a_{L,\epsilon}^{i+1} \cdots \epsilon a_{L,\epsilon}^n\right)}_{=:\ \Delta_i(\epsilon)},$$

where $a_{L,\epsilon}^i$ is the i-th column of $A_{L,\epsilon}$. Denoting the identity matrix on $\mathbf{R}^k$ by I_k we write

$$\epsilon A_{L,\epsilon} = \begin{pmatrix} I_k & 0 \\ 0 & \epsilon I_{n-k} \end{pmatrix} \begin{pmatrix} \epsilon A_{L,k} + I_k & \epsilon B \\ B^t & A'_{L,k} \end{pmatrix}$$

with $A_{L,k} = \{(A_L)_{ij} : 1 \leq i, j \leq k\}$, $A'_{L,k} = \{(A_L)_{ij} : k+1 \leq i, j \leq n\}$ and $B = \{(A_L)_{ij} : 1 \leq i \leq k,\ k+1 \leq j \leq n\}$. Hence, $\det \epsilon A_{L,\epsilon} = \epsilon^{n-k} P(\epsilon)$ and $P(0) = \det A'_{L,k} > 0$. With the same argument we can show that $\Delta_i(\epsilon)$ can be written as $\Delta_i(\epsilon) = \epsilon^{n-k} \widetilde{\Delta}_i(\epsilon)$. Therefore the limit $(U_L)_i := \lim_{\epsilon \to 0} (U_L^\epsilon)_i = \widetilde{\Delta}_i(0)/P(0)$ exists. Both, $P(\epsilon)$ and $\widetilde{\Delta}_i(\epsilon)$, are polynomials in ϵ of degree 1 at least. Thus, $P(\epsilon) = P(0) + O(\epsilon)$ and $\widetilde{\Delta}_i(\epsilon) = \widetilde{\Delta}_i(0) + O(\epsilon)$. This implies $|(U_L^\epsilon)_i - (U_L)_i| = O(\epsilon)$.

We multiply (4.5) from the left with $I - M_L$ and with ϵM_L to get $(I - M_L)A_L U_L^\epsilon = (I - M_L)F_L$ and $\epsilon M_L A_L U_L^\epsilon + M_L U_L^\epsilon = \epsilon M_L F_L + M_L G_L$, respectively. Taking the limit as ϵ tends to zero gives the statement. $\square$

Instead of the ill-conditioned system (4.5) we now choose to solve

$$(I - M_L)A_L(I - M_L)\xi_L = (I - M_L)(F_L - A_L M_L G_L) \tag{4.6}$$

on the range $\mathrm{R}(I - M_L)$ of $I - M_L$. The limit U_L is then given by

$$U_L \;=\; \xi_L^\star \;+\; M_L G_L\,,$$

where $\xi_L^\star$ is the unique solution of (4.6) in $\mathrm{R}(I - M_L)$.

Remark: The implementation of the *cg*-method for solving (4.6) is straightforward. Indeed, the *cg*-iteration has only to be restricted to the subspace $\mathrm{R}(I-M_L)$ which can be done easily. Moreover, we do not need to reorder the unkowns. This is a crucial fact because the system (4.6) can also be written in compact form, cf. (4.4), which is well suited for an efficient coding.

So far we have gotten rid of the influence of the penalty parameter ϵ. Still, the condition number of $\widetilde{A}_L = (I - M_L)A_L(I - M_L)$ on $\mathrm{R}(I - M_L)$ increases like $O(\delta_L^{-2})$. So, it makes sense to consider using the periodic multigrid method presented in Section 3.2 for solving (3.9) as a preconditioner for the *cg*-method acting on (4.6). With the same number ν of pre- and postsmoothing steps the iteration matrix of (3.13) becomes $M_L^{\mathrm{mg}}(\nu,\nu) = I - W_L^{-1}A_L$, where W_L is symmetric and positive-definite. The transformed matrix $(I - M_L)W_L^{-1}(I - M_L)$ is also positive-definite on $\mathrm{R}(I - M_L)$ and we may write $\widetilde{W}_L := ((I - M_L)W_L^{-1}(I - M_L))^{-1}$ on $\mathrm{R}(I - M_L)$. Now, we solve the symmetric system (4.7) which is equivalent to (4.6),

$$\widetilde{W}_L^{-1/2}\,\widetilde{A}_L\,\widetilde{W}_L^{-1/2}\,\widetilde{\xi}_L \;=\; \widetilde{F}_L\,, \quad \widetilde{\xi}_L \in \mathrm{R}(I - M_L), \tag{4.7}$$

with $\widetilde{F}_L = \widetilde{W}_L^{-1/2}(I - M_L)(F_L - A_L M_L G_L)$. In using the *cg*-method to solve (4.7) only the action of W_L^{-1} on a vector v has to be computed, see e.g. [9], which can be realized by one step of the multigrid iteration (3.13) with starting guess zero and right hand side v.

The usual way to obtain an analytic estimate for the condition number of $\widetilde{W}_L^{-1/2}\,\widetilde{A}_L\,\widetilde{W}_L^{-1/2}$ is to establish an estimate of the type

$$\gamma\,\widetilde{W}_L \;\le\; \widetilde{A}_L \;\le\; \Gamma\,\widetilde{W}_L \quad \text{on } \mathrm{R}(I - M_L) \tag{4.8}$$

with numbers $0 < \gamma \le \Gamma$. The notation $A \le B$ signifies that $B - A$ is positive semi definite.

At this time we do not know how to prove (4.8) with meaningful bounds. Two principal difficulties are: a) it is not clear whether the underlying continuous expression of (4.8) can be used, b) it is not clear whether the relation $(1-\rho_L)W_L \leq A_L \leq W_L$, where ρ_L is the spectral radius of $M_L^{\mathrm{mg}}(\nu,\nu)$, may be of any help. Nevertheless, the numerical experiments for the *cg*-method acting on (4.7) described in the next section are impressive.

4.3 Numerical Experiments

We consider the boundary-value problem (4.1), (4.2), with respect to two different geometric domains, the disk (Example I)

$$\omega_1 \;=\; \{(x,y)\in \mathbf{R}^2 \,:\, x^2+y^2 < 1/16\}$$

and the disk with re-entrant corner (Example II)

$$\omega_2 = \{(x,y)\in \mathbf{R}^2 \,:\, x^2+y^2 < 1/16,\; y < |x|\}.$$

In both examples the right hand side f is chosen to be $f \equiv 1$ and the boundary function g is chosen to be $g \equiv 0$. One of the numerical difficulties with this setting is the appearance of boundary layers if α is small compared to 1.

For the domain ω_1 the exact solution u is known to have the following representation

$$u(x,y) \;=\; 1 \;-\; \frac{\mathrm{J}_0(i\sqrt{x^2+y^2}/\sqrt{\alpha})}{\mathrm{J}_0(i/\sqrt{\alpha}/4)}, \qquad i=\sqrt{-1}, \tag{4.9}$$

where J_0 is the Bessel function of the first kind of order 0.

The boundary-value problem (4.1), (4.2), with the domain ω_2 has less than full elliptic regularity. In this example we study the dependence of the preconditioned *cg*-method on the regularity of the underlying boundar-value problem.

We let the fictitious domain Ω be the square defined by

$$\Omega \;=\; \{(x,y)\in \mathbf{R}^2 \,:\, |x|,|y| < 1/2\}.$$

Remark: To minimize the number of discretization points in $\Omega - \omega_k$, $k=1,2$, one could use $\widetilde{\Omega} = \{(x,y)\in \mathbf{R}^2 \,:\, |x|,|y| < 1/4\}$ as smallest possible box-shaped fictitious domain. We chosed the larger domain Ω for our computations to visualize more clearly the action of the numerical boundary measure μ^L as well as the periodicity of the solution on Ω.

Table 1: Example I: necessary numbers of iterations to yield an Euclidean norm of the residue smaller than 0.01. The discretization step-size is 1/256.

	$\alpha = 1$	$\alpha = 0.01$	$\alpha = 0.0001$
CG^ϵ	1538	1219	136
CG	781	608	68
PCG	79	99	17

In the sequel we will use the following abbreviations: CG^ϵ denotes the *cg*-method applied to (4.5) with $\epsilon = 10^{-8}$, CG denotes the *cg*-method acting on (4.6) and PCG stands for the *cg*-method applied to the preconditioned system (4.7) where W_L originates from the multigrid iteration (3.13) with one pre- and one postsmoothing step. The computational costs of one iteration step of CG^ϵ and CG are almost identical, whereas one step of PCG is more expensive. However, all three methods coincide in the order of their computational costs which is $O(n_L^2)$. We realized the three methods in the MATLAB computer system on a Sun Sparc 2 workstation and we found that the cpu time for one step of PCG was about four times the cpu time of one step of CG. In our experiments each iteration is started with starting guess 0 and the underlying Daubechies scaling function always has order $N = 3$.

For the disk ω_1 (Example I), Table 1 shows the number of iteration steps needed by the three methods to yield an Euclidean norm (not weighted with the discretization step-size) of the residue smaller than 0.01. Table 2 contains the same numbers with respect to ω_2 (Example II). The discretization step-size belonging to both tables is $\delta_8 = 1/256$. As expected from our theoretical considerations, CG outperforms CG^ϵ and PCG outperforms CG. Moreover, in the case $\alpha = 1$, PCG is less affected by the lack of full elliptic regularity than the other two methods. If α is sufficiently small then CG and PCG are comparable because A_L is close to the identity matrix.

Figures 1, 2 and 3 display cross sections through the origin of the approximate solution (solid line) as well as through the exact solution (4.9) (dashed line) for Example I with different choices of α. The approximations are obtained by terminating PCG after the Euclidean norm of the residue was smaller than 0.01. In addition to the region of interest $[-0.25, 0.25]$ the approximate solution is also plotted over the whole cross section of the fictitious domain. The support of the numerical boundary measure μ^L of

Table 2: Example II: necessary numbers of iterations to yield an Euclidean norm of the residue smaller than 0.01. The discretization step-size is 1/256.

	$\alpha = 1$	$\alpha = 0.01$	$\alpha = 0.0001$
CG^{ϵ}	2081	1217	138
CG	1051	610	69
PCG	99	104	17

Table 3: Example I ($\alpha = 1$): necessary numbers of iterations to yield an Euclidean norm of the residue smaller than $r_L = 0.01 \cdot 2^{L-8}$, $L = 6, 7, 8, 9$.

	$\delta_6 = 1/64$	$\delta_7 = 1/128$	$\delta_8 = 1/256$	$\delta_9 = 1/512$
CG	168	371	781	1599
PCG	36	54	79	118

width $2N\delta_8$ can be seen clearly. Without any sophisticated discretization techniques wavelet-Galerkin methods provide stable approximations to the exact boundary layers, cf. Figure 3, an observation already made in e.g. [13]. No Gibbs phenomenon occurs.

The approximate solution for $\alpha = 0.01$ of Example II (disk with re-entrant corner) is plotted in Figure 4 restricted to the domain of interest ω_2.

At the end we compare CG and PCG for $\alpha = 1$ and varying discretization step-sizes $\delta_L = 2^{-L}$, $L = 6, 7, 8, 9$. To get a meaningful result we have to adapt the stop criterion to the discretization step-size. We stop CG and PCG if the Euclidean norm of their residue is smaller than $r_L = 0.01 \cdot 2^{L-8}$. The needed numbers of iteration steps are presented in the Tables 3 and 4. If the discretization step-size is divided by 2 then the necessary iterations for CG are more than doubled, in contrast to this, the necessary iterations for PCG increase only with about the factor 1.5.

Table 4: Example II ($\alpha = 1$): necessary numbers of iterations to yield an Euclidean norm of the residue smaller than $r_L = 0.01 \cdot 2^{L-8}$, $L = 6, 7, 8, 9$.

	$\delta_6 = 1/64$	$\delta_7 = 1/128$	$\delta_8 = 1/256$	$\delta_9 = 1/512$
CG	219	501	1051	2256
PCG	46	66	96	145

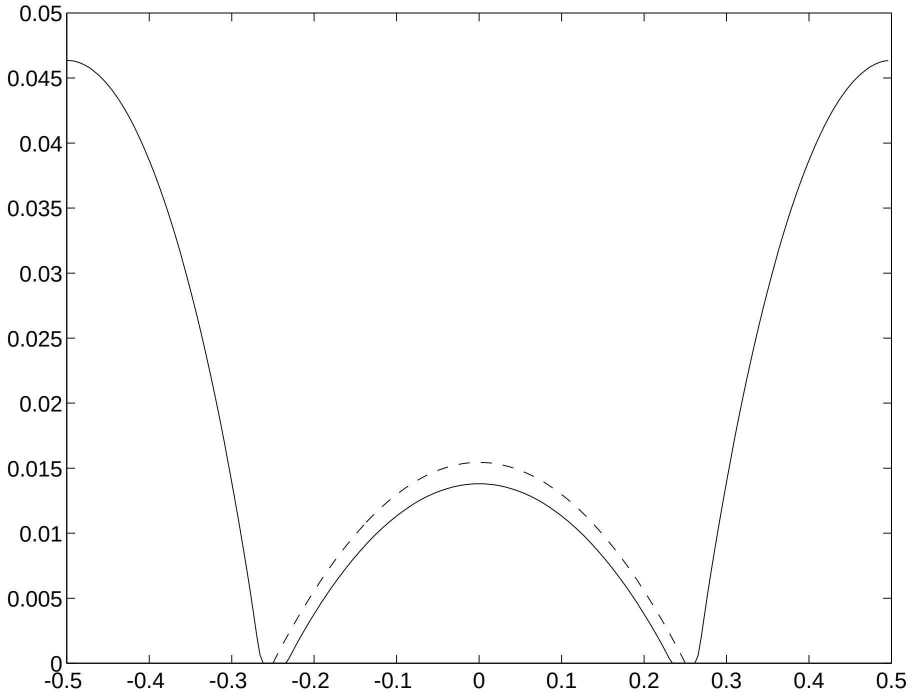

Figure 1: Example I: cross section through the origin of the approximate solution after 79 iterations of PCG (solid line) and through the exact solution (dashed line) for $\alpha = 1$ and $\delta_8 = 1/256$.

Conclusion: In this paper we introduced a wavelet-based multigrid method for an elliptic model problem over a square with periodic boundary conditions. Further, we showed how this multigrid iteration can be used as a preconditioner for a *cg*-method applied to a linear system originating from a wavelet-Galerkin discretization of a Dirichlet boundary-value problem via a penalty/fictitious domain formulation.

Our presented (and not presented) experiments indicated the efficiency of our PCG method compared to CG. Nevertheless, for an analytic statement an estimate like (4.8) is needed.

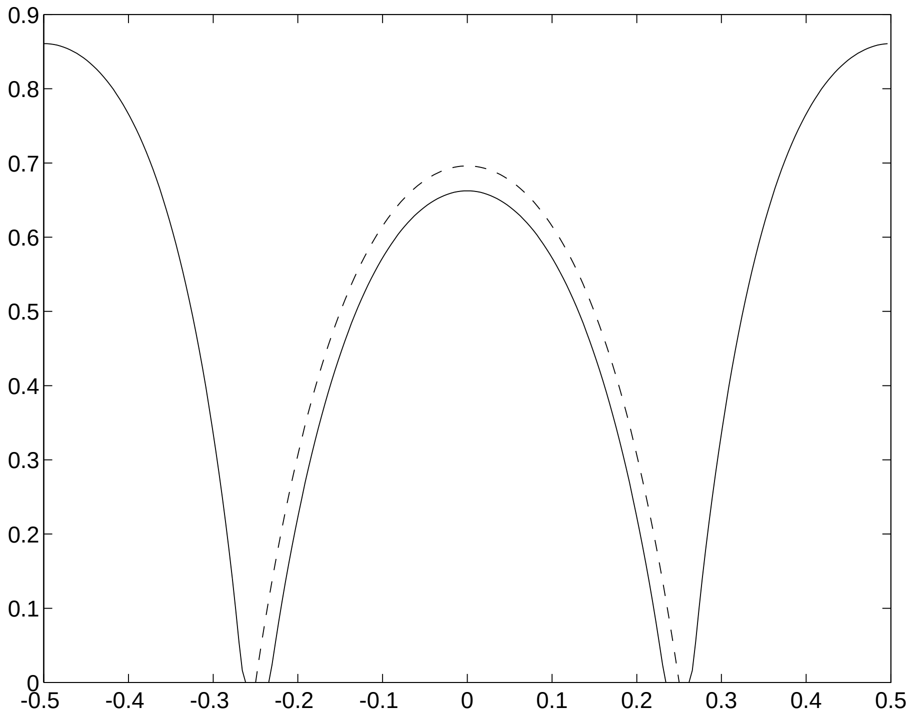

Figure 2: Example I: cross section through the origin of the approximate solution after 99 iterations of PCG (solid line) and the exact solution (dashed line) for $\alpha = 10^{-2}$ and $\delta_8 = 1/256$.

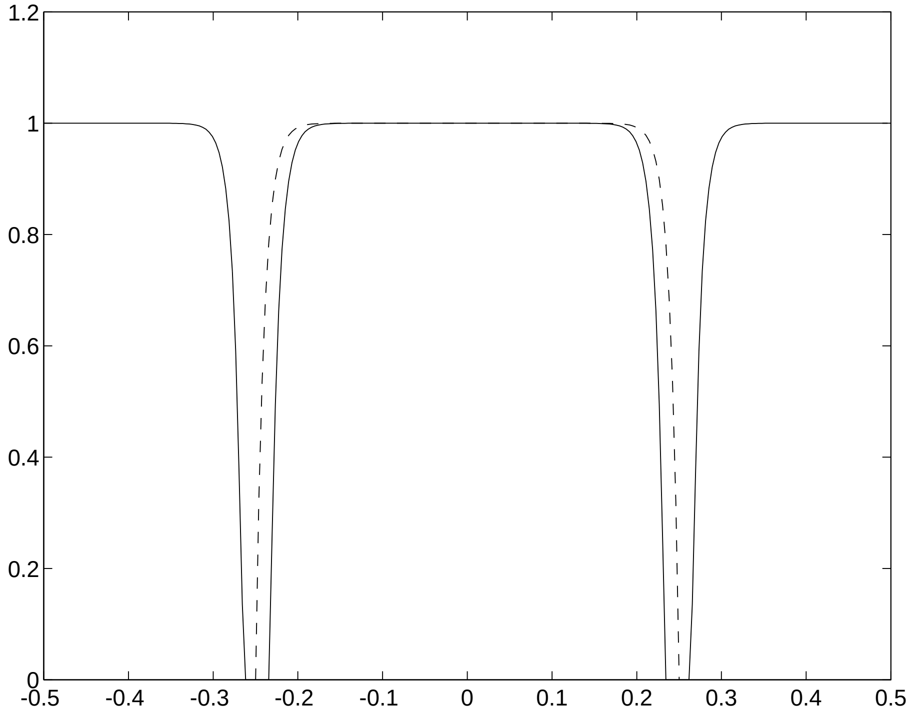

Figure 3: Example I: cross section through the origin of the approximate solution after 17 iterations of PCG (solid line) and the exact solution (dashed line) for $\alpha = 10^{-4}$ and $\delta_8 = 1/256$.

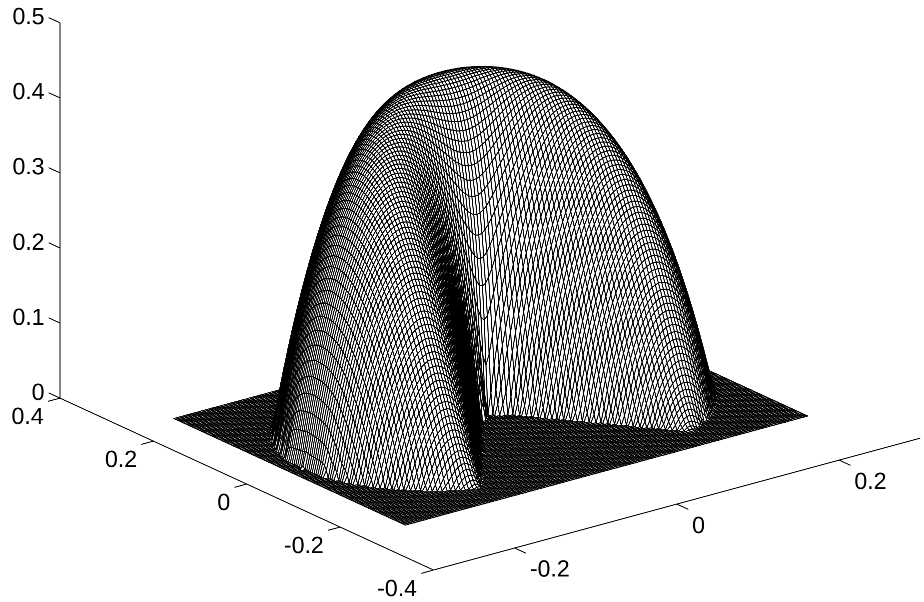

Figure 4: Example II: the approximate solution for $\alpha = 0.01$ and $\delta_8 = 1/256$ after 104 iterations of PCG. The Euclidean norm of the residue is smaller than 0.01

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