SOME THOUGHTS ON LEAST SQUARED ERROR OPTIMAL WINDOWS

*R.A.Gopinath**

Department of Electrical and Computer Engineering, Rice University, Houston, TX-77251

ABSTRACT

Windowing methods give simple designs of FIR filters. Window designs are generally not considered optimal in any meaningful sense. A typical desired FIR lowpass filter is specified by giving the passband, stopband and transition band responses of the filter. A particular class of transition responses viz., spline transition functions, give rise to windows that are optimal in a least squared sense [1]. Do all transition functions have associated windows that are least squared optimal? More importantly, given a window does there exist a (meaningful) transition function with respect to which the window is least squared optimal? In trying to answer the second question this note also characterizes all possible *lowpass extensions* of a given sequence and exhibits the unique minimum norm extension. This is used to show that a given finitely supported window $w_b(n)$, and a desired response with prescribed transition band edges (but no transition function) there exist infinitely many transition functions with respect to which the windowed FIR filter (obtained by windowing an ideal frequency response) is least squared optimal. The unique minimum norm lowpass extension of $w_b(n)$ is used to exhibit a particular transition function with an extremal property. Since it is desirable to have a monotone transition function we ask the question: can a given window be least squared optimal with respect to a monotone desired transition response? This question leads to a few open problems on non-negative definite lowpass extensions of sequences.

1. INTRODUCTION

Recently a simple and versatile method for the design of linear-phaser FIR filters with spline transition bands and optimal in a least-squared sense was introduced [1]. The optimal linear-phase filter of length N (wlog N may be assumed to be odd integer) with passband and stopband edges ω_p and ω_s respectively is given by

$$h(n) = \frac{\sin(\omega_o n)}{\pi n}\left[\frac{\sin\left(\frac{\Delta n}{2p}\right)}{\frac{\Delta n}{2p}}\right]^p = g(n)w(n), \qquad (1)$$

where $\omega_o = (\omega_s + \omega_p)/2$ and $\Delta = \omega_s - \omega_p$, and

$$g(n) = \frac{\sin(\omega_o n)}{\pi n}. \qquad (2)$$

This raises the following question: given an arbitrary window, say for example a Hamming window, does there exist a transition function (like the spline function above) such that the Hamming window is least-squares optimal? A related question is the following: given a transition function does there exist a window sequence $w(n)$ such that the least squared optimal FIR filter is given by $g(n)w(n)$?

This paper shows that all windows have associated transition functions that make them least-squared optimal. For every window there exists a a transition function which makes it *super-optimal*.

2. MAIN RESULTS

$D(\omega)$ is any *desired* frequency response prescribed by $D(\omega) = 1$, $|\omega| \le \omega_p$, $D(\omega) = 0, |\omega| \ge \omega_s$ and $D(\omega) = D(-\omega)$ (see Fig. 1 for example). Since one of the primary reasons for considering windows is to avoid the Gibbs' phenomenon (associated with discontinuities in $D(\omega)$) we will posit that $D(\omega)$ is continuous. The values of $D(\omega)$, $\omega_p \le |\omega| \le \omega_s$ will be called the transition function. A *window* $w(n)$ is any sequence such that $w(n) = w(-n)$. A window is finite if it is non-zero on a finite set. A function $f(t)$ of a real variable is said to be *absolutely continuous* in $[a,b]$ if given δ there exists a ϵ such that $\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \delta$ for all finite non-overlapping collections of intervals $\{(x_i', x_i)\}$ in $[a,b]$ with $\sum_{i=1}^{n} |x_i' - x_i| < \epsilon$. Absolutely continuous function are uniformly continuous and hence in particular continuous.

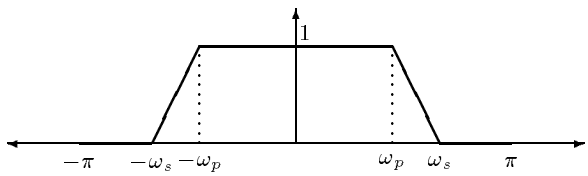

Figure 1. A Desired Frequency Response $D(\omega)$

Fact 1 *The optimal FIR approximation of $D(\omega)$ is "windowed" (i.e., $d(n)$ is of form the $g(n)w(n)$) if and only if $D(\omega)$ is absolutely continuous in the transition region and $D(\omega_0 - \omega) + D(\omega_0 + \omega) = 1$ for $|\omega| < \frac{\Delta}{2}$.*

Proof: Let the transition function be absolutely continuous. Then there exists $W(\omega)$ supported in $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$ such that for $|\omega| \le \frac{\Delta}{2}$ (see Royden [3] pp. 110 Theorem 14)

$$D(-\omega_o + \omega) = \int_{-\frac{\Delta}{2}}^{\omega} W(\lambda)d\lambda \qquad (3)$$

Since $D(\omega) = D(-\omega)$ it follows that $W(\omega) = W(-\omega)$. Also notice that $1 = D(\omega_p) = D(-\omega_p) = \int W(\omega)d\omega$. Now $w(n)$, the inverse Fourier transform of $W(\omega)$ is an real even-symmetric function. Then $d(n) = g(n)w(n)$ is verified to be the inverse Fourier transform of $D(\omega)$. Conversely let $d(n) = g(n)w(n)$ and let $W(\omega)$ be the Fourier transform of an even-symmetric $w(n)$. Then

$$D(\omega) = \int_{-\omega_o}^{\omega_o} W(\omega - \lambda)d\lambda = \int_{\omega-\omega_o}^{\omega+\omega_o} W(\lambda)d\lambda \qquad (4)$$

and hence (see Royden [3] pp. 88 Proposition 14) $D(\omega)$ is absolutely continuous for all ω and hence in particular in the transition region. Moreover, since $W(\omega) = W(-\omega)$ it follows that $D(\omega_0 - \omega) + D(\omega_0 + \omega) = 1$ for $|\omega| < \frac{\Delta}{2}$. □

Remark: Continuity of the transition response is not sufficient for the existence of an associated window. For example if $D(-\omega_s + \omega)$ is the Cantor ternary function (see Royden pp. 50 Problem 48) then $D(\omega)$ is continuous and monotone in the transition region and satisfies $D(\omega_0 - \omega) + D(\omega_0 + \omega) = 1$ for $|\omega| < \frac{\Delta}{2}$. However, since $D(\omega)$ is not absolutely continuous (see Royden pp. 111 Problem 15) there is no window associated with this choice of $D(\omega)$. Since Fact 1 requires $D(\omega_0 - \omega) + D(\omega_0 + \omega) = 1$ for $|\omega| < \frac{\Delta}{2}$, we will require this of the desired response $D(\omega)$ in the rest of this note.

Given a fixed $\Delta > 0$ and positive integer K, a finite window $w_b(n)$, with $w_b(n) = 0$ $|n| \ge K$ is said to have a *lowpass extension* if there exists $w_e(n)$, even-symmetric with $w_e(n) = 0, |n| < K$ such that $W_b(\omega) + W_e(\omega) = 0$, for $|\omega| > \frac{\Delta}{2}$. The sequence

*This work was supported in part by Texas Instruments and in part by Bell Northern Research

$w(n) = w_b(n) + w_e(n)$ will be referred to as a lowpass extension of $w_b(n)$.

Fact 2 *For a finite window $w_b(n)$ there exists a desired $D(\omega)$ if and only if $w_b(n)$ has a lowpass extension.*

Proof: Let $w_b(n)$ have a lowpass extension $w(n) = w_b(n) + w_e(n)$ and let $d(n) = \alpha g(n)w(n)$ where $\frac{1}{\alpha} = \int W(\omega)d\omega$. Then $D(\omega)$ is given by

$$D(\omega) = \alpha \int_{-\pi}^{\pi} W(\omega - \lambda)d\lambda = \alpha \int_{\omega-\omega_o}^{\omega+\omega_o} W(\lambda)d\lambda \qquad (5)$$

and is seen to be a desired response. Conversely let $w_b(n)$ not have a lowpass extension and let $D(\omega)$ be an associated desired response. Let $w(n) = w_b(n) + w_e(n)$ (necessarily not lowpass) be the corresponding extension. Then $supp(W(\omega)) = [-A, A] \supset [-\frac{\Delta}{2}, \frac{\Delta}{2}]$. Therefore there exists an open set in $[\omega_o - A, \omega_o + A] - [\omega_o - \frac{\Delta}{2}, \omega_o + \frac{\Delta}{2}]$ in which $D(\omega)$ does not vanish and hence $D(\omega)$ cannot be a desired response. The contradiction implies $w_b(n)$ has a lowpass extension. □

Does every finite window have a lowpass extension? Is it unique? More generally, does every finite sequence (not necessarily a window) have a lowpass extension. This paper shows that all finite sequences have infinitely many lowpass extensions (or more generally bandpass/highpass extensions all defined naturally). Also even-symmetric sequences have even-symmetric extensions and odd symmetric sequences have odd-symmetric extensions (implying that lowpass extensions exist for finite windows). The space of all extensions (for arbitrary finite sequences) is linear. One can obtain uniqueness of the extension by postulating that the extension has minimum energy (or norm).

The uniqueness question is readily settled as follows: if $h_b(n)$ is a finite sequence with a unique extension, by removing the boundary values of $h_b(n)$ one gets a sequence that also has a unique extension which must anticipate the values removed. Hence lowpass extensions (for sequences/windows) are nonunique. Contrast this with the continuous time case where bandlimited extensions of any finite signal is unique; a fact that follows from the analyticity of bandlimited signals.

A sequence $h(n)$ is said to be *gapped* if it vanishes in a finite set. The largest K such that $h(n) = 0$, $|n| \leq K$ will be called the *gap*. The nonuniqueness of lowpass extensions implies the existence of (nontrivial) gapped lowpass sequences.

Fact 3 *For every integer K there exist lowpass sequences with gap K.*

Proof: Let $h_b(n)$ be a sequence that vanishes for $|n| < K$. If $h(n) = h_b(n) + h_e(n)$ and $h'(n) = h_b(n) + h'_e(n)$ are distinct extensions then $h'_e - h_e = h' - h$ is lowpass with gap at least K. □

Gapped functions form a linear-space, G_K, and gapped lowpass functions $G_{K,\Delta}$ form a subspace whose orthogonal complement is given by G_{K,Δ^c} (defined to be gapped highpass functions). Moreover, $G_K = G_K^s \oplus G_K^a$, where G_K^s and G_K^a are spaces of even and odd-symmetric gapped functions. Similarly, $G_{K,\Delta} = G_{K,\Delta}^s \oplus G_{K,\Delta}^a$. In particular, for a finite window $w_b(n)$ we are looking for $w_e(n) \in G_K^s$ such that $w_b(n) + w_e(n)$ is lowpass. As a corollary of the previous lemma we have the following result:

Fact 4 *Lowpass extensions are unique modulo gapped functions.*

The results above would not make sense if lowpass extensions do no exist! First consider the lowpass window extension problem. It suffices to find a basis for $\{w_b \mid w_b(n) = 0, |n| \geq K\}$ (a linear space of dimension K) and exhibit an extension for each basis element. Let F be the invertible operator that maps even-symmetric sequences into one-sided sequences. For an even symmetric sequence $h(n)$, let $\hat{h}(n)$ denote Fh. That is, $\hat{h}(n) = h(n)$, $n \geq 0$. Let L be the lowpass projection operator corresponding to a passband of $[-\omega_o, \omega_o]$. That is Lh is the output of an ideal lowpass filter driven by h. L induces a corresponding operator $\hat{L}$ on one-sided sequences: $\hat{L}\hat{h} = FLF^{-1}\hat{h}$. Let T_K denote the symmetric truncation operator on sequences.

$$(T_K h)(n) = \begin{cases} h(n) & \text{for } |n| < K \\ 0 & \text{otherwise} \end{cases} \qquad (6)$$

This induces the following operator, $\hat{T}_K$, on one-sided sequences: $\hat{T}_K = FT_KF^{-1}$. Let E denote the *invertible* operator that acts on sequences that are zero for $|n| \geq K$ giving a vector of size e $2K-1$ indexed in $\{-(K-1), \ldots, K-1\}$ and let $\mathbf{h}$ denote Eh. That is, $\mathbf{h}(i) = (Eh)(i) = h(i)$ Similarly let $\hat{E}$ be the *invertible* operator on one-sided sequences that vanish for $n \geq K$ and giving a vector of size K indexed in $\{0, \ldots, K-1\}$ and let $\hat{\mathbf{h}}$ denote $\hat{E}\hat{h}$. That is, $\hat{\mathbf{h}}(i) = (\hat{E}\hat{h})(i) = \hat{h}(i)$. Then corresponding to L acting on sequences there are matrices $\mathbf{L}$ and $\hat{\mathbf{L}}$ acting on vectors of size $2K-1$ and K respectively and defined by $\mathbf{Lh} = ET_KLE^{-1}\mathbf{h}$ and $\hat{\mathbf{L}}\hat{\mathbf{h}} = \hat{E}\hat{T}_K\hat{L}\hat{E}^{-1}\hat{\mathbf{h}}$. $\mathbf{L}$ corresponds to considering a vector as a sequence, lowpass filtering it and truncating it to a vector of the same size. $\hat{\mathbf{L}}$ corresponds to considering the vector as an even symmetric sequence (by reflection), lowpass filtering it, and truncating the resulting even symmetric sequence to get a vector of the original length. The entries of $\mathbf{L}$ and $\hat{\mathbf{L}}$ are given respectively by

$$(\mathbf{L})_{n,k} = \frac{\sin\left(\pi\frac{\Delta}{2}(n-k)\right)}{\pi(n-k)}, \qquad (7)$$

and

$$(\hat{\mathbf{L}})_{n,k} = \begin{cases} \frac{\sin\left(\pi\frac{\Delta}{2}\right)n}{\pi n} & \text{for } k = 0 \\ \frac{\sin((n+k)\frac{\Delta}{2}\pi)}{\pi(n+k)} + \frac{\sin\left((n-k)\frac{\Delta}{2}\pi\right)}{\pi(n-k)} & \text{for } k = 1, \ldots, K-1 \end{cases} \qquad (8)$$

Now consider the lowpass extension problem for finite windows. Assume that $\hat{\mathbf{L}}$ has a complete set of eigenvectors (notice that this would have been immediate if $\hat{\mathbf{L}}$ is symmetric - which it is not!); for $i = 0, \ldots, K-1$, $\hat{\mathbf{L}}\hat{\mathbf{v}}_i = \lambda_i\hat{\mathbf{v}}_i$. Then every finite window can be represented as a linear combination as follows:

$$w_b = \sum_{i=0}^{K-1} a_i F^{-1}\hat{E}^{-1}\hat{\mathbf{v}}_i = \sum_{i=0}^{K-1} a_i v_i \qquad (9)$$

Now define $v_{i,e}(n) = (\frac{L}{\lambda_i} - I)v_i(n)$. Notice that $v_{i,e}(n) \in G_{K,\Delta}^s$. Moreover, $v_i(n) + v_{i,e}(n) = \frac{L}{\lambda_i}v_i(n)$ is lowpass and hence is a extension of $v_i(n)$. Thus for an arbitrary finite window (even-symmetric sequence) $w_b(n) = \sum_{i=0}^{K-1} a_i v_i(n)$ one has an explicit lowpass extension given by

$$\sum_i \frac{a_i}{\lambda_i} L v_i(n). \qquad (10)$$

Notice that in Eqn. 10 one would have a problem if any eigenvalue is zero. The existence of a complete set of eigenvectors with positive eigenvalues does not seem to be readily provable for $\hat{\mathbf{L}}$. However, using a more general result for arbitrary finite sequences (i.e., for the matrix $\mathbf{L}$) this can be proved. In the general case the proof is based on discrete-prolate-spheroidal sequences [4]. The essential fact is that $\mathbf{L}$ is the matrix that represents the quadratic form corresponding to the fraction of lowpass energy of a finite sequence. Since finite sequences have energy in all bands this is a positive definite matrix. Moreover, from the symmetric and Toeplitz nature of $\mathbf{L}$ it follows that the eigenvectors are symmetric or antisymmetric. In fact $\mathbf{L}$ has the following three properties [4]:

- It is symmetric (unlike $\hat{L}$) and positive definite.

- It has a complete set of $(2K-1)$ eigenvectors $\mathbf{v}_i$ with positive eigenvalues λ_i.
- The eigenvectors can be ordered so that $\mathbf{v}_i$, $i = 0, \ldots, K-1$ are even-symmetric eigenvectors and $\mathbf{v}_i$, $i = K, \ldots, 2K-2$ are odd-symmetric eigenvectors.

The existence of the K even-symmetric eigenvectors $\mathbf{v}_i$ imply the existence of the corresponding eigenvectors $\hat{\mathbf{v}}_i$ of $\hat{\mathbf{L}}$ and with the same eigenvalue.

We have essentially shown that finite sequences, even-symmetric, odd-symmetric or otherwise have infinitely many lowpass extensions that preserve the symmetry property. Clearly the lowpass band, $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$, could be replaced by *any* band and hence one has the corresponding results for highpass and bandpass situations.

Since there are infinitely many lowpass extensions a natural question is whether the one in Eqn. 10 has an extremal property. Indeed one can show that it is the unique minimum energy (or norm) lowpass extension. To see this consider the operator $A = (I - T_K)L$ acting on G_K. This is a symmetric, non-negative definite (because it is the kernel of the quadratic form representing the lowpass energy of any sequence in G_K) Toeplitz operator. Therefore, A has a complete eigenbasis. Clearly any gapped sequence in $G_{K,\Delta}$ is an eigenvector with eigenvalue 1:

$$(I - T_K)Lh = (I - T_K)h = h - T_K h = h - 0 = h. \tag{11}$$

Any highpass gapped sequence in G_{K,Δ^c} is also an eigenvector with eigenvalue zero:

$$(I - T_K)Lh = (I - T_K)0 = 0. \tag{12}$$

In fact from the Toeplitz, symmetric nature of A, it is clear that $G^s_{K,\Delta}$ and $G^a_{K,\Delta}$ are distinct (mutually orthogonal) eigenspaces with eigenvalue 1, while G^s_{K,Δ^c} and G^s_{K,Δ^c} are distinct (mutually orthogonal) eigenspaces with eigenvalue 0. Are there any more eigenvectors? Indeed for all $i \in \{0, \ldots, 2K-1\}$ $v_{i,e}(n)$ is an eigenvector with eigenvalue $(1 - \lambda_i)$ since

$$\begin{aligned}(I - T_K)Lv_{i,e}(n) &= (I - T_K)L(v(n) - v_i(n)) \\ &= (I - T_K)v(n) - \lambda_i(I - T_K)v(n) \\ &= (1 - \lambda_i)v_{i,e}(n). \end{aligned} \tag{13}$$

There cannot be any more eigenvectors; for if there is one with eigenvalue α (α has be in $(0,1)$ because A corresponds to measuring the lowpass fraction of a gapped signal's energy), this would imply the existence of another eigenvector for the matrix $\mathbf{L}$. But $\mathbf{L}$ can only have $2K-1$ eigenvectors! Moreover since the corresponding eigenvalues are different from 0 and 1, these vectors are orthogonal to $G^s_{K,\Delta}$, G^s_{K,Δ^c}, $G^a_{K,\Delta}$ and G^a_{K,Δ^c}. This gives a complete description of gapped functions G_K as a direct sum of orthogonal spaces

$$G_K = G^s_{K,\Delta} \oplus G^s_{K,\Delta^c} \oplus G^a_{K,\Delta} \oplus G^a_{K,\Delta^c} \oplus V^s_{K,\Delta} \oplus V^a_{K,\Delta} \oplus V_{K,\Delta^c} \oplus V^a_{K,\Delta^c}. \tag{14}$$

where $V_{K,\Delta} = Span\,\{v_{i,e}\}_{i=0}^{K-1}$, $V^a_{K,\Delta} = Span\,\{v_{i,e}\}_{i=K}^{2K-2}$, $V_{K,\Delta^c} = Span\,\{u_{i,e}\}_{i=0}^{K-1}$, $V^a_{K,\Delta^c} = Span\,\{u_{i,e}\}_{i=K}^{2K-2}$, the $u_{i,e}$ is the highpass extension of w_i, one of the eigenvectors of the finite matrix $\mathbf{H}$ that can be defined similar to $\mathbf{L}$ (replacing lowpass by highpass everywhere).

Now consider a finite window $w_b(n)$ with the unique extension $w_e(n) \in V_{\Delta,K}$. Any other extension is of the form $w_e(n) + p(n)$, $p(n) \in G^s_{K,\Delta}$. Therefore

$$\|w_e + p\|^2 = \|w_e\|^2 + \|p\|^2 + 2\langle w_e, p\rangle = \|w_e\|^2 + \|p\|^2 \geq \|w_e\|^2, \tag{15}$$

and hence we have proved the following result:

Fact 5 *Every finite window $w_b(n)$ has a unique minimum norm lowpass extension given by Eqn. 10.*

Putting it all together we have the answer to our original question:

Theorem 1 *Every finite window is squared-error optimal.*

It would be desirable to have $D(\omega)$ absolutely continuous and monotone in the transition region. This could place restrictions on the choice of finite windows $w(n)$ that may be considered optimal (indeed it does).

Fact 6 *$D(\omega)$ is monotone and absolutely continuous if and only if $w(n)$ has a lowpass extension that is non-negative definite.*

Proof: The monotonicity of the transition region implies that in Eqn. 4 $W(\omega)$ has to be non-negative and this in turn implies that $w(n)$ is non-negative definite. The converse argument is also immediate. □

Open Problems:

1. Characterize finite even-symmetric sequences with non-negative definite lowpass extensions.
2. Are such extensions unique? If not, what is the cone of all such extensions? Is this convex cone infinite dimensional?
3. Does an alternating projections algorithm converge to a a non-negative definite lowpass extension? That is does the following recursion with $w_b = w^(0)$ converge:

$$w^{(i)} = w_b + (I - T_K)PLw^{(i-1)},$$

where P is an operator that zeros out the Fourier transform at all frequencies where it is negative.

The set $w_e + G_{K,\Delta}$ gives all lowpass extensions of w_b, where $w^\star_e$ is the minimum norm extension. Each extension gives a transition function $D(\omega)$ with respect to which the window w_b is least-squared optimal. Now w_b is said to be *super-optimal* with respect to a given transition function if this transition function has the least squared error; it solves the problem

$$\inf_{p \in G_{K,\Delta}} \sum_{|n|\geq K} |w_e(n)g(n)|^2 \sum_{|n|\geq K} |(w^\star_e(n) + p(n))g(n)|^2.$$

The notion of super-optimality suggests that if a window is super-optimal with respect to a given transition function, then the frequency response of the filter $h(n) = w_b(n)g(n)$, will resemble this response more than any other. It is easy to find bounds for the ℓ^1 and ℓ^2 norms of the error which suggest that the minimum norm extension gives a transtion function that leads to super-optimality. Indeed,

$$\|w_e g\|_1 = \sum_{|n|\geq K} |w_e(n)g(n)| \leq \|w_e\|_2 \|g\|_2,$$

and

$$\|w_e g\|_2 = \left(\sum_{|n|\geq K} |w_e(n)g(n)|^2\right)^{1/2} \leq \|w_e\|_2 \sup_{|n|\geq K} |g(n)|,$$

and the bounds are minimized by $w^\star_e$. Hence the following conjecture.

Conjecture 1 *Every window is super-optimal with respect to $D(\omega)$ associated with its the minimum norm lowpass extension.*

3. TRANSITION FUNCTIONS FOR TRADITIONAL WINDOWS

In this section we construct the unique transition function (resulting from a minimum norm lowpass extension) associated with classical windows like the Hamming window, Kaiser window, Hanning window etc. Once cannot implement Eqn. 10 directly because the positive numbers λ_i can be arbitrarily small. However, one can easily show that the following discrete-version of the Papoulis-Gerschberg iteration ([2]) converges to the unique minimum norm extension:

$$w^{(i)} = w_b + (I - T_K)Lw^{(i-1)}.$$

In continuous time, since lowpass extensions are unique, the Papoulis-Gerschberg algorithm always converges to the unique extension. In the discrete-case, since lowpass extensions are non-unique, it is conceivable that the algorithm may converege to an arbitrary lowpass extension. However, it does converge to the unique minimum norm extension as can be seen from the fact that the limit

$$w = (I - (I - T_K)L)^{-1} w_b,$$

is zero if $w_b = 0$.

Figs. 2-6 show the desired frequency responses for which the rectangular, Hamming, Kaiser and Hanning windows of length $N = 51$, with $\omega_p = 2\pi/5$ and $\omega_s = 3\pi/5$ are least-squares optimal. These frequency responses correspond to the minimum norm lowpass extension. Notice that in all cases except the Kaiser window, the transition function is not monotone; the lowpass extension of the window is *not* positive definite. The behavior of the Hanning and Hamming windows, both being examples of generalized cosine windows are similar. For the rectangular window the desired response has ripples that in a sense "anticipates" the Gibbs phenomenon. Recall that the rectangular window in the figure is also least squares optimal with respect to the ideal lowpass filter with cutoff at $\frac{\pi}{2}$. However, the particular transition function obtained in this paper explicitly brings out that the rectangular window approximates a desired response with ripples; hence it may be an undesirable window. If one increases N, the number of ripples increase, the height remaining the same, again indicative of the Gibb's phenomenon (see Fig. 6).

REFERENCES

[1] C. S. Burrus, A. W. Soewito, and R. A. Gopinath. Least squared error fir filter design with transition bands. *IEEE Trans. ASSP*, ASSP-40(6):1327–1340, June 1992.

[2] A. Papoulis. A new algorithm in spectral analysis and band-limited extrapolation. *IEEE Trans. on CAS*, CAS-22:735–742, 1975.

[3] H. L. Royden. *Real Analysis.* McMillan Publishing Company, 3 edition, 1989.

[4] D. Slepian. Prolate spheroidal wave functions, fourier analysis, and uncertainty - v: The discrete case. *Bell Syst. Tech. J.*, 57:1371–1430, May-June 1978.

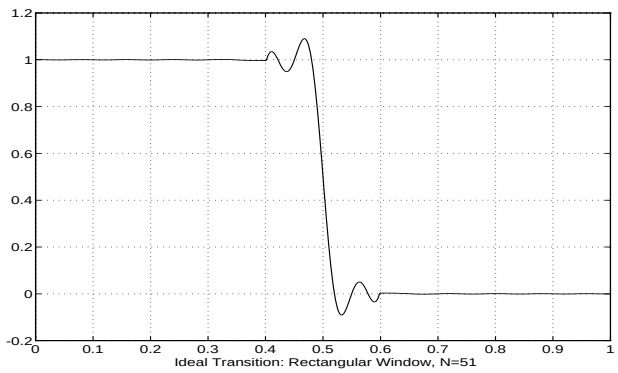

Figure 2. Rectangular Window

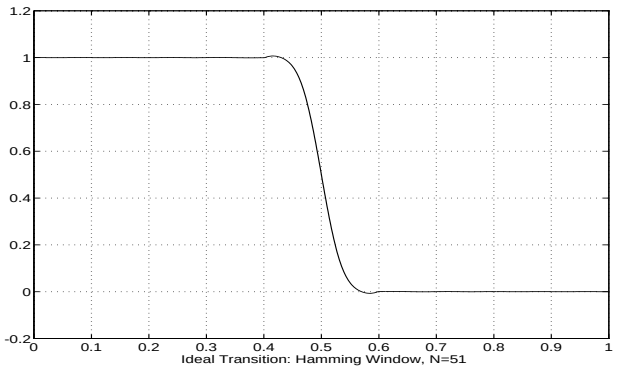

Figure 3. Hamming Window

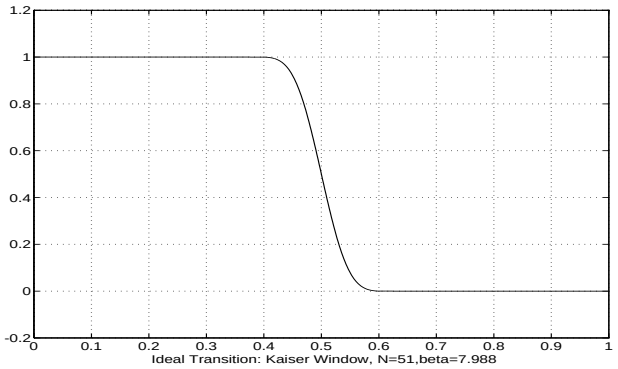

Figure 4. Kaiser Window

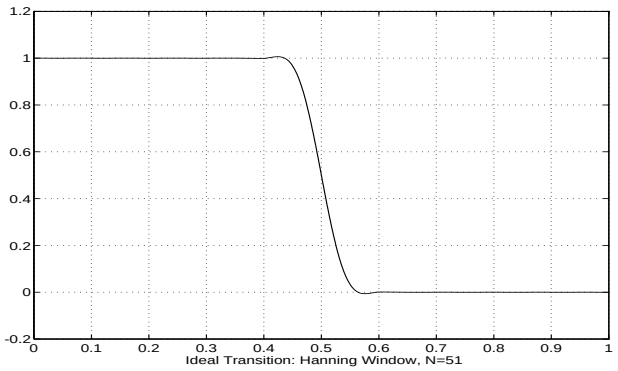

Figure 5. Hanning Window

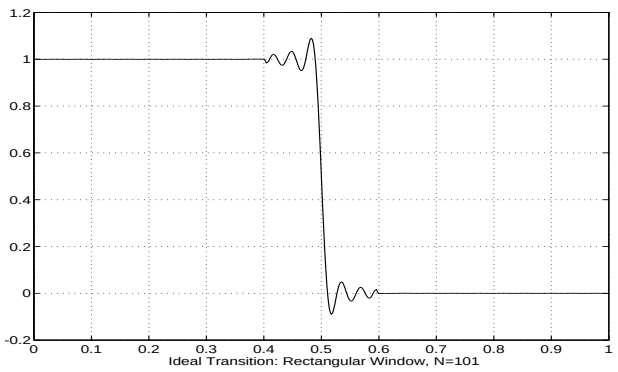

Figure 6. Rectangular Window: $N = 101$

SOME THOUGHTS ON LEAST SQUARED ERROR OPTIMAL WINDOWS

R.A.Gopinath[1]

Department of Electrical and Computer Engineering, Rice University, Houston, TX-77251

Windowing methods give simple designs of FIR filters. Window designs are generally not considered optimal in any meaningful sense. A typical desired FIR lowpass filter is specified by giving the passband, stopband and transition band responses of the filter. A particular class of transition responses viz., spline transition functions, give rise to windows that are optimal in a least squared sense [1]. Do all transition functions have associated windows that are least squared optimal? More importantly, given a window does there exist a (meaningful) transition function with respect to which the window is least squared optimal? In trying to answer the second question this note also characterizes all possible *lowpass extensions* of a given sequence and exhibits the unique minimum norm extension. This is used to show that a given finitely supported window $w_b(n)$, and a desired response with prescribed transition band edges (but no transition function) there exist infinitely many transition functions with respect to which the windowed FIR filter (obtained by windowing an ideal frequency response) is least squared optimal. The unique minimum norm lowpass extension of $w_b(n)$ is used to exhibit a particular transition function with an extremal property. Since it is desirable to have a monotone transition function we ask the question: can a given window be least squared optimal with respect to a monotone desired transition response? This question leads to a few open problems on non-negative definite lowpass extensions of sequences.