The Spectral Theory of Multiresolution Operators and Applications

P. N. Heller R. O. Wells, Jr.

1 Introduction

In this article we explore the notion of the *multiresolution operator*, its spectral theory, and applications. This operator (also called the transition operator) has appeared as a fundamental tool in several aspects of wavelet theory, such as the Lawton-Cohen theorem on wavelet orthonormal bases [2, 18, 19], and the work of Eirola [8] and others [1, 3, 4, 5, 23] on the Sobolev smoothness of wavelet scaling functions. After introducing the multiresolution operator and observing its connection with the convolution and downsampling operations of multirate signal processing, we present a review of the work of Lawton and that of Eirola. Throughout the paper we work in the setting of rank m wavelet systems (m is the integer dilation factor, not necessarily 2). This involves some generalization of previous work, and yields initial results on the differentiability of wavelet scaling functions for rank $m > 2$. In particular, we find that the minimal support rank 3 scaling functions with N vanishing wavelet moments have Sobolev smoothness which grows only logarithmically with N.

2 Wavelets and Multiresolution Operators

We are considering here rank m wavelets of compact support [9, 13], defined by a *rank m wavelet matrix* $a = (a_{r,k})$ which is an $m \times mg$ matrix satisfying the orthogonality condition

$$\sum_k a_{r,k} a_{r',k+ml} = m\delta_{r,r'}\delta_{l,0} \tag{2.1}$$

and the sum condition

$$\sum_k a_{r,k} = m\delta_{r,0} \; . \tag{2.2}$$

If we look at the Fourier transforms (frequency responses in signal processing terminology)

$$A_r(\omega) = \frac{1}{m}\sum_k a_{r,k}e^{ik\omega}$$

of the wavelet sequences (filters), the conditions (2.1)-(2.2) can be restated as

$$\sum_{k,k'=0}^{m-1} A_r(\omega + 2\pi k/m)\,\overline{A_{r'}(\omega + 2\pi k'/m)} \equiv \frac{1}{m}\delta_{r,r'} \; , \tag{2.3}$$

and

$$A_r(0) = \delta_{r,0} \; . \tag{2.4}$$

Associated to the wavelet matrix is a system of scaling functions and wavelets; the rank m scaling function ϕ associated with the scaling sequence $a_{0,k}$ is the solution to the dilation equation

$$\phi(x) = \sum_k a_{0,k}\phi(mx - k) \; . \tag{2.5}$$

This dilation equation has a unique compactly supported solution in $L^2(\mathbf{R})\bigcap L^1(\mathbf{R})$; its Fourier transform $\hat{\phi}$ satisfies

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} A_0(\xi/m^j) \; .$$

The corresponding family of wavelets $\psi_{r,j,k}$ $(1 \leq r \leq m-1)$ is defined from ϕ by

$$\psi_r(x) = \sum_k a_{r,k}\phi(mx - k)$$

and

$$\psi_{r,j,k} = m^{-j/2}\psi_r(m^{-j}x - k) \; .$$

Lawton [17] proved that the collection $\{\psi_{r,j,k}(x)\}$ form a tight frame for $L^2(\mathbf{R})$.

We are now in a position to ask critical questions about the family of functions $\{\psi_{r,j,k}(x)\}$, such as "When is this tight frame an orthonormal basis?" and "How differentiable are the individual $\psi_{r,j,k}$?" Since the wavelets $\psi_{r,j,k}$

are linear combinations of dilates of the scaling function ϕ, these questions reduce to the study of ϕ: "When do the $\{\phi(x-k)\}$ form an orthonormal basis for the space $V_0 = \mathrm{Span}\,\{\phi(x-k)\}$?" and "How differentiable can we make $\phi(x)$?"

It is in answering these questions that one is led to the (rank m) *multiresolution operator* associated with the scaling sequence[1] $\{a_k\}$. This operator is defined in the frequency domain as the map taking a function u on $[0, 2\pi)$ to the function

$$Tu\,(\omega) = \left|A\left(\frac{\omega}{m}\right)\right|^2 u\left(\frac{\omega}{m}\right) + \left|A\left(\frac{\omega+2\pi}{m}\right)\right|^2 u\left(\frac{\omega+2\pi}{m}\right) + \ldots$$
$$\ldots + \left|A\left(\frac{\omega+2\pi(m-1)}{m}\right)\right|^2 u\left(\frac{\omega+2\pi(m-1)}{m}\right) ,$$

where we have extended the domain of definition of u by its natural periodization. The multiresolution operator[2] is defined in the time domain as the operator taking a sequence u_n to

$$(\check{T}u)_n = m \sum_{k,l} a_k \overline{a_l} u_{mn+k-l} \ . \tag{2.6}$$

If the function $u(\omega)$ has the Fourier series expansion $\sum_n u_n e^{in\omega}$ then (2.6) expresses the relationship between the Fourier coefficients u_n of u and those of $\check{T}u$; i.e. $(\check{T}u)_n$ is the n-th Fourier coefficient of Tu.

2.1 Multiresolution operators and multirate signal processing

The multiresolution operator has a natural meaning in the context of multirate signal processing: it represents convolution followed by downsampling – the two operations essential to moving between scales of a multiresolution analysis (hence the name multiresolution operator). In the time domain,

[1]Because of the special role of the scaling sequence $a_{0,k}$, we will often abbreviate our notation and refer to the scaling sequence as $\{a_k\}$ and its Fourier transform as $A(\omega)$.

[2]The operator T has been referred to variously as the *transition operator* and the *wavelet-Galerkin operator* in the literature; we take the liberty of introducing the name *multiresolution operator* because it more clearly illustrates the central role of this operator in understanding wavelet bases and multiresolution analysis.

we can interpret the multiresolution operator as performing convolution of a sequence with the autocorrelation $\overline{a} * a$, followed by downsampling by m (keeping every m-th sample). For further discussion of this from a signal processing perspective, see [6].

Given an arbitrary sequence $\{h_n\}$, the frequency domain representation of the map $u_n \to (h * u)_{mn}$ (also written $(h * u) \downarrow m$) is

$$u(\omega) \to H\left(\frac{\omega}{m}\right) u\left(\frac{\omega}{m}\right) + H\left(\frac{\omega+2\pi}{m}\right) u\left(\frac{\omega+2\pi}{m}\right) + \\ \ldots + H\left(\frac{\omega+2\pi(m-1)}{m}\right) u\left(\frac{\omega+2\pi(m-1)}{m}\right) . \tag{2.7}$$

We use the notation T_H for this map and refer to H as the *symbol* of the operator T_H, in analogy with the theory of pseudodifferential operators. If H is a trigonometric polynomial (i.e. $\{h_n\}$) is a finite sequence), then T_H is a bounded linear operator on the Banach space $\mathcal{C}[0,\pi]$ of continuous functions on $[0,\pi]$ with the sup norm. Since the multiresolution operator takes the form (2.7) with $|A|^2$ playing the role of H, we can write $T_{|A|^2}$ for the multiresolution operator, and $\check{T}_{\overline{a}*a}$ for its time domain form.

2.2 Vanishing moments and the Daubechies coefficients

An early theorem on wavelets ([7], p.153) states that if a wavelet $\psi_r(x)$ is N times continuously differentiable and has sufficient decay to be in $L^1(\mathbf{R})$, then $\widehat{\psi_r}(\xi)$ must vanish to order N at $\xi = 0$. In order for this to happen, A_r must vanish to order N at $\omega = 0$, which leads us to the following definition:

Theorem 2.1 *A rank m wavelet system is said to have* degree N *if any of the following equivalent conditions hold:*

(i) $(d/d\xi)^n \widehat{\psi}_r(0) = 0,$ *for* $n = 0, 1, \ldots, N-1$ *and* $r = 1, 2, \ldots, m-1$.
(ii) $(d/d\omega)^n A_r(0) = 0,$ *for* $n = 0, 1, \ldots, N-1$ *and* $r = 1, 2, \ldots, m-1$.
(iii) $(d/d\omega)^n A_0\left(\frac{2\pi k}{m}\right) = 0,$ *for* $n = 0, 1, \ldots, N-1;$ $k = 1, 2, \ldots, m-1$.
(iv) $\sum_l (k+ml)^n a_{0,k+ml}$ *is independent of* k *for* $k = 0, 1, \ldots, m-1$ *and* $n = 0, 1, \ldots, N-1$.
(v) $A_0(\omega) = (1 + e^{i\omega} + \ldots + e^{i(m-1)\omega})^N Q(\omega)$ *for some trigonometric polynomial* $Q(\omega)$.

Further discussion of these issues can be found in [7], [10], and [22]. We note that $1+e^{i\omega}+\ldots+e^{i(m-1)\omega}$ is the Fourier transform of the Haar scaling sequence $\{1,1,\ldots,1\}$ of length m. This trigonometric polynomial and its square modulus play a critical role in the analysis of the multiresolution operator $T_{|A|^2}$, as we shall see below.

The main point is that we now have a computational condition to use in our attempt to find wavelet bases with a given degree of differentiability – since an N-times differentiable wavelet must have N vanishing moments, why not *impose* N vanishing moments and see how differentiable the resulting ϕ and ψ are? This was one of the ideas Daubechies used in the construction of orthonormal bases of compactly supported wavelets; she identified a minimal degree Q to define a wavelet system with N vanishing moments. This was generalized to rank m wavelets in [10, 21]. In fact, characterizations of these minimal length wavelet systems with N vanishing moments invariably describe the autocorrelations $|A_0(\omega)|^2$ or $|Q(\omega)|^2$ and then perform a spectral factorization [20] to obtain A_0 or Q. For a wavelet system with N vanishing moments, it is convenient to write

$$P(\omega)=|A_0(\omega)|^2=\left|\frac{1+e^{i\omega}+\ldots+e^{i(m-1)\omega}}{m}\right|^{2N}R(\omega) \tag{2.8}$$

where we have factored the square modulus $P(\omega)$ into two parts, N powers of the modulus squared Haar polynomial (*à la* definition (v) above) and a term $R(\omega)=|Q(\omega)|^2$ designed to restore orthogonality to the overall symbol. With this new notation, T_P is the same as $T_{|A|^2}$, and we henceforth use T_P to denote the multiresolution operator associated with the scaling sequence $\{a_k\}$.

Lawton [19] used the multiresolution operator to give a time-domain characterization of the Daubechies scaling coefficients, which we generalize now to the rank m case. The autocorrelation of the rank m minimal length scaling coefficients of degree N is the solution to

$$\sum_{n=-N}^{N-1}(mn+r)^p(\overline{a}*a)_{mn+r}=\frac{1}{m}\,\delta_{p,0}\,, \tag{2.9}$$
$$0\le p\le 2N-1\,,\ 1\le r\le m-1\,,$$

and

$$(\overline{a}*a)_{mn}=\delta_{n,0}\,; \tag{2.10}$$

i.e. it solves $m-1$ simultaneous Lagrange interpolation problems over the nonzero cosets of $\mathbf{Z}/m\mathbf{Z}$. The scaling sequence proper is then obtained from this autocorrelation by a spectral factorization. To connect these formulae with the multiresolution operator, let $u_k(n) = \delta_{n,k}$, for $k = 1, 2, \ldots, m-1$. Then $v_k(n) = \check{T}_{\overline{a}*a} u_k(n) = (\overline{a} * a)_{mn+r}$ and equations (2.9)–(2.10) follow from (v) in Theorem 2.1. These expressions have an explicit solution given in [11] by a formula for the symbol $P(\omega)$:

$$P(\omega) = \frac{1}{m} + \frac{2}{m^2} \sum_{r=1}^{m-1} \sum_{n=-N}^{N-1} \prod_{\substack{k=-N \\ k\neq n}}^{N-1} \frac{mk+r}{m(k-n)} e^{i(mn+r)\omega} . \tag{2.11}$$

An earlier approach to finding the rank m Daubechies coefficients with N vanishing moments appears in [10, 21]; there one obtained a formula for the minimal degree trigonometric polynomial $R(\omega)$:

$$R_N(x) = \sum_{n=0}^{N-1} \left\{ \frac{1}{n!} \left(\frac{d}{dx} \right)^n [H(x)]^{-N} \right\}_{x=1} (x-1)^n , \tag{2.12}$$

where $x = \cos\omega$ and

$$H(x) = \left| \frac{1 + e^{i\omega} + \ldots + e^{i(m-1)\omega}}{m} \right|^2$$

is the modulus squared of the Haar trigonometric polynomial.

2.3 An invariant finite-dimensional subspace for T_P

Let $\{a_k\}$ be a scaling sequence and T_P be the associated multiresolution operator, where $P = |A|^2$ as before. The infinite-dimensional operator T_P has interesting properties when restricted to certain finite-dimensional subspaces, which we now develop after Daubechies [7]. Let the trigonometric polynomial P be given as in (2.11), expressed in the form

$$P(e^{i\omega}) = \sum_{j=-J}^{J} p_j e^{ij\omega} . \tag{2.13}$$

Moreover define the $2J+1$-dimensional space of trigonometric polynomials

$$\mathcal{E}_J := \left\{ u(\omega) = \sum_{j=-J}^{J} u_j e^{ij\omega} \right\} ,$$

as well as the distinguished subspace

$$\mathcal{F}_{J,N} := \left\{ f \in \mathcal{E}_J : f(\omega) = (1-\cos\omega)^N g(\omega) \right\} .$$

We now consider the action of the multiresolution operator T_P and the *reduced multiresolution operator* T_R defined by (2.7) with symbol R, the remainder in the factorization $P = H^N R$.

Lemma 2.2 *Suppose that the wavelet system associated with the scaling sequence $\{a_k\}$ has degree N (the first N wavelet moments vanish). Then*
(i) *The subspace $\mathcal{E}_J$ of $\mathcal{C}[0,\pi]$ is an invariant subspace for the operator T_P, i.e.*

$$T_P : \mathcal{E}_J \to \mathcal{E}_J .$$

The operator T_P acting on $\mathcal{E}_J$ has the matrix representation $t_{j,k} = p_{mj-k}$, or

$$T = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ & \ddots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ p_{J-m+2} & p_{J-m+3} & \cdots & p_J & 0 & \dots & 0 \\ p_{J-2m+2} & p_{J-2m+3} & \cdots & p_{J-m} & \cdots & p_J & 0 \\ & & & \ddots & & & \\ 0 & \dots & 0 & p_{J-m+2} & \cdots & p_{J-1} & p_J \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ & & & & & \ddots & \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

with respect to the basis $\left\{ e^{-iJ\omega}, \dots, 1, \dots, e^{iJ\omega} \right\}$.

(ii) $T_P|_{\mathcal{E}_J}$ *has eigenvalues m^{-n}, $n = 0, 1, \dots, 2N-1$.*

(iii) *Suppose $f \in \mathcal{F}_{J,N}$, $f(\omega) = (1-\cos\omega)^N g(\omega)$; then*

$$T_P f(\omega) = \frac{1}{m^{2N}} (1-\cos\omega)^N T_R\, g(\omega) . \tag{2.14}$$

Hence $\mathcal{F}_{J,N}$ is an invariant subspace for T_P, and moreover if f is an eigenvector for T_P in $\mathcal{F}_{J,N}$ with eigenvalue μ, then

$$T_R g(\omega) = m^{2N} \mu g(\omega) \,,$$

so that g is an eigenfunction for T_R with eigenvalue $m^{2N}\mu$.

Remark: Part (iii) establishes a one-to-one correspondence between eigenfunctions of $T_P|_{\mathcal{F}_{J,N}}$ and eigenfunctions of $T_R|_{\mathcal{E}_{J-N}}$. This will enable us to reduce the study of the spectrum of T_P to that of T_R in the work to follow.

Proof: To prove (i), given $u \in \mathcal{E}_J$ and P as in (2.13), extend $T_P u$ to be an even function on $[-\pi, \pi]$, and expand it in a Fourier series on this interval. The sum over m terms in the definition of T_P offsets the dilation by $\frac{1}{m}$ so that no fractional powers of sine or cosine appear. The orthogonality of $\{\sin k\omega, \cos k\omega\}$ ensures that all but a finite number of coefficients in the Fourier expansion of $T_P u$ vanish, and we find that $T_P u \in \mathcal{E}_J$.

(ii) is strongly related to Theorem 2.1. As in that theorem, the factorization

$$P(\omega) = \left| \frac{1 - e^{im\omega}}{m(1 - e^{i\omega})} \right|^{2N} R(\omega) \tag{2.15}$$

implies that the partial moments

$$M_{k,n} = \sum_l (k + ml)^n p_{k+ml}$$

are equal to a number M_n independent of k, $k = 0, 1, \ldots, m-1$ for $n = 0, 1, \ldots, 2N-1$. Following [7, 12] if we consider the polynomial sequence $e_n = (j^n)_{j=-J,\ldots,J} \in \mathcal{E}_J$, we find that

$$e_n T = m^{-(n+1)} \sum_{k=0}^{n} \tbinom{n}{k} M_{n-k} e_k \,.$$

Since $M_0 = 1$ with our normalizations, we see that each element e_n of $\mathcal{E}_J$ is a generalized left eigenvector of T_P with eigenvalue m^{-n} for $n = 0, 1, \ldots, 2N-1$.

(iii) follows by the factorization (2.15) of P:

$$\begin{aligned} T_P f(\omega) &= T_P \left|1 - e^{i\omega}\right|^{2N} g(\omega) \\ &= \left|\frac{1 - e^{i\omega}}{m(1 - e^{i\omega/m})}\right|^{2N} R(\frac{\omega}{m}) \left|1 - e^{i\omega/m}\right|^{2N} g(\frac{\omega}{m}) + \dots \\ &= \frac{1}{m^{2N}} \left|1 - e^{i\omega}\right|^{2N} T_R\, g(\omega) \, . \end{aligned}$$

□

3 The Lawton-Cohen theorem on wavelet orthonormal bases

With the definition and elementary properties of the multiresolution operator in hand, we can now begin to apply it to problems of wavelet theory. The first question we asked was "When do the integer translates of ϕ, $\{\phi(x-k)\}$, form an orthonormal basis for the space V_0 defined as their span?" In other words, when does the sequence h_k defined by

$$h_k := \int \phi(x)\phi(x-k)dx$$

satisfy

$$h_k = \delta_{k,0} \; ?$$

By applying the scaling relation (2.5) to the definition of h_k, we find

$$\begin{aligned} h_k &= \sum_{j,l} a_j \overline{a}_l \int \phi(mx-j)\overline{\phi(mx-l-mk)}dx \\ &= \sum_{j,l} a_j \overline{a}_l h_{2k+l-j} \end{aligned}$$

Simply put,

$$h = \check{T}_{\overline{a}*a} h \; .$$

However, since the scaling coefficients a_k satisfy the orthogonality relation

$$\sum_k a_k a_{k+ml} = \delta_{0,l} \, ,$$

then

$$\check{T}_{\overline{a}*a}\delta_{0,l} = \delta_{0,l} * (a * \overline{a}) \downarrow m = (a * \overline{a}) \downarrow m = \delta_{0,l}\,,$$

or

$$\delta_{0,l} = \check{T}_{\overline{a}*a}\delta_{0,l}\,.$$

Thus both the (as yet undetermined) sequence $\{h_k\}$ and the Kronecker delta $\delta_{0,l}$ are eigensequences for $\check{T}_{\overline{a}*a}$ with eigenvalue 1. The fact that the scaling sequence $\{a_k\}$ satisfies the quadratic wavelet matrix condition (2.1) is *equivalent* to $\delta_{0,l}$ being an eigenvector of the multiresolution operator $\check{T}_{\overline{a}*a}$ with eigenvalue 1. This observation was made by Lawton [17], proving that the wavelet system $\{\psi_{r,j,k}\}$ is a tight frame if $\delta_{0,l}$ is such an eigenvector. Furthermore, Lawton showed that if $\delta_{0,l}$ is the only eigenvector for eigenvalue 1, then $\{\psi_{r,j,k}\}$ is an orthonormal wavelet basis. It is the content of a later theorem [2, 18, 21] that this is also a necessary condition for orthonormality:

Theorem 3.1 *(Lawton-Cohen-Gopinath) The set $\{\phi(x-k)\}$ form an orthonormal basis for V_0 (and, as a result, the set $\{\psi_{r,j,k}\}$ form an orthonormal basis for $L^2(\mathbf{R})$) if and only if 1 is a nondegenerate eigenvalue for the multiresolution operator $\check{T}_{\overline{a}*a}$ (and T_P as well).*

The lack of redundancy in the eigenvectors for eigenvalue 1 implies that there is no redundancy in the tight frame $\{\psi_{r,j,k}\}$. Observe that in the time domain we wish the unique eigensequence for $\check{T}_{\overline{a}*a}$ to be $\delta_{0,l}$, while in the frequency domain the unique eigenfunction of T_P to be the constant $u_0(\omega) \equiv 1$; in either case the eigenvalue is 1.

This theorem is not vacuous, as can be seen by the "degenerate Haar" scaling sequence

$$a^{dH} = \{1, 0, \ldots, 0, 1, 1, \ldots, 1\}$$

where m zeros follow the first 1 and then $m-1$ ones follow the string of zeros. The scaling function ϕ associated with a^{dH} is the characteristic function of the interval $\left[0, 2 + \frac{1}{m-1}\right]$, and $\phi(x)$ and $\phi(x-1)$ are clearly not orthogonal, although the corresponding wavelet system $\{\psi_{r,j,k}\}$ is a tight frame.

Cohen [2] has developed additional equivalent conditions for this theorem which involve "sets congruent to $[-\pi, \pi] \bmod 2\pi$" and invariant cycles under the map $\omega \rightarrow m\omega \bmod 2\pi$. We do not elaborate on this here, other than to note that he also uses multiresolution operators in the Fourier domain in his characterization of orthonormality.

4 Differentiability estimates via the spectral theory of the multiresolution operator

The multiresolution operator T_P arises in a second area of wavelet theory, the study of the differentiability of solutions to the scaling equation (2.5). In this section we present an overview of the work of Eirola [8], generalized to the rank m setting, and give our preliminary results for wavelet systems of rank $m = 3$. Detailed proofs and extensions of this work appear in [14]. Work on related problems appears in [1, 3, 4, 5, 23].

4.1 Sobolev Estimates for the scaling function ϕ

We defined the scaling function ϕ as the unique solution in $L^2(\mathbf{R}) \bigcap L^1(\mathbf{R})$ of the dilation equation (2.5). Seeking to create smooth scaling functions, we identified a distinguished set of scaling sequences $\{a_k\}$, those of minimal length having degree N (i.e. those whose wavelets have N vanishing moments). We now ask the question "Given such a scaling sequence, how differentiable is the corresponding ϕ?" We answer this by finding the maximal Sobolev space that ϕ belongs to. Given a real number s, the Sobolev space $\mathcal{H}^s$ is defined by

$$\mathcal{H}^s := \left\{ f \,:\, \|f\|^2_{\mathcal{H}^s} = \int_{\mathbf{R}} |\hat{f}(\xi)|^2 \left(1 + |\xi|^2\right)^s d\xi < \infty \right\} .$$

Recall that for $f : \mathbf{R} \to \mathbf{C}$, $f \in \mathcal{H}^s \Rightarrow f \in \mathcal{C}^\alpha$, for $\alpha < s - \frac{1}{2}$. That is, if $s > n + \frac{1}{2}$, then f has n continuous derivatives. We define the *Sobolev smoothness* of a function $\phi \in L^2(\mathbf{R})$ to be

$$s(\phi) := \sup \{ s \,:\, \phi \in \mathcal{H}^s \} .$$

Observe that $s(f) = s_0$ does not imply that $f \in \mathcal{H}^{s_0}$, only that $f \in \mathcal{H}^{s_0 - \epsilon}$ for all $\epsilon > 0$. Our goal in this section is, given a minimal length scaling sequence $\{a_k\}$, of degree N, to determine a formula relating the Sobolev smoothness $s(\phi)$ to the spectrum of the multiresolution operator T_P.

We obtain a solution to the scaling equation (2.5) by taking an initial guess $\phi_0(x) := \frac{sin(\pi x)}{\pi x}$ and then iterating:

$$\phi_{n+1} := \sum_k a_k \phi_n(mx - k) .$$

In the Fourier transform domain this equation becomes

$$\hat{\phi}_{n+1}(\xi) = A(\xi/m)\hat{\phi}_n(\xi/m) \; .$$

Letting χ_S be the characteristic function of a set S, we have $\frac{\widehat{sin(\pi x)}}{\pi x} \equiv \chi_{[-\pi,\pi]}$, and moreover

$$\hat{\phi}_{n+1}(\xi) = \prod_{j=1}^{n} A(\xi/m^j) \; \chi_{[-m^n\pi, m^n\pi]} \; .$$

Since the ϕ_n converge pointwise and in $L^2(\mathbf{R})$ to ϕ (proof in [7]), they will converge in $\mathcal{H}^s$ if there is a uniform bound on $\|\phi_n\|^2_{\mathcal{H}^s}$. The s-th Sobolev norm of ϕ_n is bounded by

$$\|\phi_n\|^2_{\mathcal{H}^s} \leq C \left(\|\hat{\phi}_n\|^2_{L^2} + \int_{\mathbf{R}} |\hat{\phi}_n(\xi)|^2 |\xi|^{2s} d\xi \right) \; , \tag{4.16}$$

and so we concentrate on bounding the right-hand term in (4.16).

4.1.1 A change of variable

We now perform a change of variable which relates the multiresolution operator T_P to the integral $\int_{\mathbf{R}} |\hat{\phi}_n(\xi)|^2 |\xi|^{2s} d\xi$ appearing in the right-hand side of (4.16). Fixing the Sobolev exponent s, write

$$e_n(\xi) := |\hat{\phi}_n(\xi) - \hat{\phi}_{n-1}(\xi)|^2 |\xi|^{2s} \; .$$

If we can show that

$$\int_{\mathbf{R}} e_n(\xi) d\xi \leq C\epsilon^n \text{ for some } \epsilon < 1$$

then

$$\int_{\mathbf{R}} |\hat{\phi}_n(\xi)|^2 |\xi|^{2s} d\xi \leq C \sum_{k=1}^{n} \epsilon^k < C\frac{1}{1-\epsilon} < \infty$$

uniformly, and so $\phi(x) \in \mathcal{H}^s$ by (4.16). We can estimate the integrals $\int_{\mathbf{R}} e_n(\xi) d\xi$ using the multiresolution operator T_P. Observe that $e_n(-\xi) = e_n(\xi)$ and

$$e_{n+1}(\xi) = \left| A\left(\frac{\xi}{m}\right)\right|^2 \left| \hat{\phi}_n\left(\frac{\xi}{m}\right) - \hat{\phi}_{n-1}\left(\frac{\xi}{m}\right)\right| |\xi|^{2s} = m^{2s} P\left(\frac{\xi}{m}\right) e_n\left(\frac{\xi}{m}\right) \; .$$

Since $\hat{\phi}_n(\xi) = 0$ for $|\xi| > m^{n-1}\pi$,

$$\int_{\mathbf{R}} e_n(\xi)d\xi = \int_{|\xi|\le m^{n-1}\pi} e_n(\xi)d\xi \ .$$

Now we use T_P to collapse this integral over a compact interval (that grows with n) into an integral over the fixed interval $[0,\pi)$. Define

$$\begin{aligned}\kappa_1(\omega) = 2[\, & e_1(\omega) + e_1(\omega-2\pi) + e_1(\omega+2\pi) + e_1(\omega-4\pi) \ + \\ & \ldots + e_1(\omega + \frac{m-2}{2}2\pi) + e_1(\omega - \frac{m}{2}2\pi)\,] \quad \text{, for } m \text{ even ,}\end{aligned}$$

and

$$\begin{aligned}\kappa_1(\omega) = 2[\, & e_1(\omega) + e_1(\omega-2\pi) + e_1(\omega+2\pi) + e_1(\omega-4\pi) \ + \\ & \ldots + e_1(\omega + \frac{m-3}{2}2\pi)\,] \quad \text{, for } m \text{ odd .}\end{aligned}$$

Setting $\kappa_{n+1} := m^{2s} T_P \, \kappa_n$; we have

Lemma 4.1

$$\int_{|\xi|\le m^n\pi} e_{n+1}(\xi)d\xi = \int_0^\pi \kappa_{n+1}(\omega)d\omega = \int_0^\pi m^{2sn} T_P^n \kappa_1(\omega)d\omega \ . \tag{4.17}$$

The proof can be found in [14]. We see from Lemma 4.1 that the Sobolev smoothness of a scaling function ϕ defined by a scaling sequence $\{a_k\}$ is given by

$$s(\phi) = \sup\left\{ s \ : \ \int_0^\pi m^{2sn} T_P^n \kappa_1(\omega)d\omega \le C\epsilon^n \text{ for some } \epsilon < 1 \right\} . \tag{4.18}$$

4.1.2 The study of the positive operator T_R

The Sobolev regularity of ϕ now depends solely on the family of integrals

$$\int_0^\pi T_P^n \kappa_1(\omega)d\omega \ .$$

As a first step in bounding these integrals, we reduce the study of iterates of the multiresolution operator T_P to the study of iterates of the *reduced* multiresolution operator T_R. It can be checked that $\kappa_1(\omega)$ vanishes to order $2N$ at $\omega = 0$, and

$$\kappa_1(\omega) = (1 - \cos\omega)^N v_1(\omega) \ ,$$

for some $v_1 \in \mathcal{C}[0,\pi]$. If $s > \frac{1}{2}$, then $v_1 \in \mathcal{C}^1_{[0,\pi]}$. Using the relation (2.14),

$$\int_0^\pi T_P^n \kappa_1(\omega) d\omega = \int_0^\pi m^{2Nn}(1-\cos\omega)^N T_R^n\, v_1(\omega)\, d\omega \le c_N\, m^{2Nn} \int_0^\pi T_R^n v_1(\omega) d\omega \ .$$

This is advantageous because $R(\omega)$ is a purely positive trigonometric polynomial, and so T_R is a positive operator on $\mathcal{C}[0,\pi]$. We will be able to bound the integral

$$\int_0^\pi T_R^n v_1(\omega) d\omega \tag{4.19}$$

using the spectral radius ρ of the operator T_R on $\mathcal{C}[0,\pi]$.

We now examine this spectral radius, developing infinite-dimensional analogues of the Perron-Frobenius theorem. Recall that the spectral radius ρ of a linear operator T is defined as

$$\rho(T) = \lim_{n\to\infty} \left(\sup_{\|u\|=1} \|T^n u\|^{1/n} \right) \ .$$

We aim to compute $\rho(T_R)$ in terms of the spectral radius of the finite-dimensional operator $\widetilde{T}_R = T_R|_{\mathcal{E}_J}$. We will use the theory of positive operators as developed in [15]. A *cone* in $\mathcal{C}[0,\pi]$ is a closed subset which does not contain a one-dimensional subspace, and which contains halflines emanating from the origin. The subset K of $\mathcal{C}[0,\pi]$ consisting of nonnegative continuous functions on $[0,\pi]$ is such a cone. We also introduce the subset K^0 of K consisting of all elements $u \in K$ such that $u(\omega) > 0$ for all $\omega \in [0,\pi]$. With these preliminaries, the following theorem is proven in [8, 14]:

Theorem 4.2 *Let $R \in \mathcal{C}[0,\pi]$ be such that $R \in K^0 \cap \mathcal{E}_J$, then there exists a function $\tilde{u} \in K^0 \cap \mathcal{E}_J$ and a real number $\mu > 0$ such that:*
i) $\widetilde{T}_R\,\tilde{u} = \mu\tilde{u}$ ($\tilde{u}$ is an eigenvector of $\widetilde{T}_R$ with eigenvalue μ), and $\rho(\widetilde{T}_R) = \mu$.
ii) $T_R\,\tilde{u} = \mu\tilde{u}$ ($\tilde{u}$ is an eigenvector of T_R with eigenvalue μ), and $\rho(T_R) = \mu$.
iii) If $0 \neq u_0 \in K \cap \mathcal{C}^1_{[0,\pi]}$, then there exists an $n_0 > 0$ such that $T_R^{n_0} u_0 \in K^0$, and

$$\lim_{n\to\infty} \frac{T_R^n u_0}{\|T_R^n u_0\|} = \tilde{u} \tag{4.20}$$

The limit in (4.20) is uniform for $\omega \in [0,\pi]$, i.e., is a strong limit in the Banach space $\mathcal{C}[0,\pi]$.

Applying this theorem with $R(\omega)$ as defined in (2.12), so that $J = N-1$, we obtain two critical corollaries:

Corollary 4.3 *If μ is the largest eigenvalue of $T_R|_{\mathcal{E}_{N-1}}$ and u_0 is an arbitrary nonnegative nonzero function $\in \mathcal{C}^1_{[0,\pi]}$, then given $\epsilon > 0$, there exist c_ϵ, C_ϵ such that*

$$c_\epsilon(\mu-\epsilon)^n \leq \int_0^\pi T_R^n u_0(\omega) d\omega \leq C_\epsilon(\mu-\epsilon)^n \, .$$

Applying this to the integral (4.19), with $u_0 = v_1$, we find

$$s(\phi) = \sup\left\{ s \, : \, m^{2s-2N}\mu < 1 \right\} \, ,$$

yielding

Corollary 4.4 *If μ is the largest eigenvalue of $T_R|_{\mathcal{E}_{N-1}}$ and $\mu < \frac{1}{m}$,*

$$s(\phi) = N - \frac{\log\mu}{2\log m} \, . \tag{4.21}$$

Thus we have obtained an explicit formula for the Sobolev smoothness of the scaling function ϕ in terms of the maximal eigenvalue of the associated reduced multiresolution operator T_R acting on the finite-dimensional space $\mathcal{E}_{N-1}$.

4.2 Results for $m = 2, 3$

In the cases $m = 2$ and $m = 3$, we are able to apply this machinery in two ways. First, given explicit formulas for the trigonometric polynomial $R(\omega)$, one may compute its eigenvalues on the subspace $\mathcal{E}_{N-1}$ and obtain the maximal Sobolev exponent exponent from equation (4.21). Using the formula

$$R(\omega) = \sum_{n=0}^{N-1} \frac{(N-1+n)!}{(N-1)!\, n!} (1-\cos\omega)^n$$

in the $m = 2$ case (see [7] for a derivation) and the formula

$$R(\omega) = \sum_{n=0}^{N-1} \frac{(2N-1+n)!}{(2N-1)!\, n!} \left(\frac{2}{3}\right)^n (1-\cos\omega)^n \tag{4.22}$$

when $m = 3$ (obtained in [10]), we compute the Sobolev smoothness of minimal support scaling functions[3] ϕ_N of degree N. The results are tabulated for $N = 1, 2, \ldots, 10$ in Table 4.1 and graphed for $N = 1, 2, \ldots, 25$ in Figure 4.1.

N	m=2 Sobolev exponent	m=3 Sobolev exponent
1	0.5	0.5
2	1.0	0.9087
3	1.4150	1.1599
4	1.7756	1.2950
5	2.0968	1.3665
6	2.3884	1.4133
7	2.6587	1.4499
8	2.9147	1.4809
9	3.1617	1.5081
10	3.4027	1.5323

Table 4.1: Sobolev smoothness for minimal support scaling functions with N vanishing moments

However, one can go much further. Eirola [8] used the notion of T_R as a positive operator mapping a cone K to itself to develop asymptotic estimates for the spectral radius of T_R as $N \to \infty$. For m odd, define the cone

$$K^J = \left\{ u \in K^0 \cap \mathcal{E}_J \; : \; u(\omega) = \sum_{j=0}^{J} u_j (1 - \cos\omega)^j \, , \; u_j \geq 0 \right\}.$$

Eirola defines an analogous cone for m even. In [14] we prove

Lemma 4.5 *Given $R \in K^J$,*

$$T_R : K^J \to K^J \; .$$

[3] Warning! We have just switched notation and now use ϕ_N to refer to the degree N scaling function of minimal support, *not* the N-th iterative approximation to the scaling function ϕ.

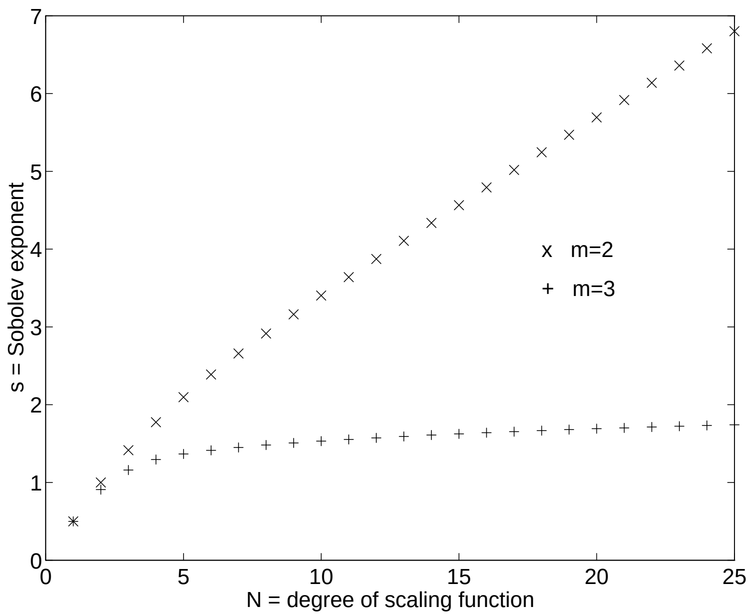

Figure 4.1: Sobolev exponents for minimal support degree N scaling functions, $m = 2$ and $m = 3$, $N = 1, 2, \ldots, 25$.

Now define the dual cone K^{J^*} and adjoint operator T_R^*, and consider the evaluation functionals η_θ defined by

$$\langle \eta_\theta, u \rangle = u(\theta) \; .$$

One can prove:

Lemma 4.6 *If $R \in K^J$, and there exists $\lambda > 0$ and $\theta \in [0, \pi]$ such that*

$$T_R^* \eta_\theta - \lambda \eta_\theta \in K^{J^*} \; ,$$

then $\mu \geq \lambda$, and conversely, if

$$\lambda \eta_\theta - T_R^* \eta_\theta \in K^{J^*} \; ,$$

then $\mu \leq \lambda$.

In the case $m = 2$, Eirola applied this technique with $J = N - 1\,,\ \theta = \frac{2\pi}{3}$:

$$T_R^* \eta_{\frac{2\pi}{3}} = R\left(\frac{2\pi}{3}\right) \eta_{\frac{2\pi}{3}} + R\left(\frac{\pi}{3}\right) \eta_{\frac{\pi}{3}} \ .$$

Since $R\left(\frac{2\pi}{3}\right) >> R\left(\frac{\pi}{3}\right)$, this says that $\eta_{\frac{2\pi}{3}}$ is "almost" an eigenvector for T_R. Applying Lemma 4.6, it can be shown [8] that

$$R\left(\frac{2\pi}{3}\right) \leq \mu \leq R\left(\frac{2\pi}{3}\right)(1 + o(N)) \ . \tag{4.23}$$

Since we know

$$R\left(\frac{2\pi}{3}\right) = \sum_{n=0}^{N-1} \frac{(N-1+n)!}{(N-1)!\,n!} \left(\frac{3}{2}\right)^n \ ,$$

equation (4.23) yields an asymptotic value for the Sobolev exponent s:

$$s(\phi_N) \approx N - \frac{N \log 3}{2 \log 2} \approx .2075N \ .$$

This result has also been obtained by Cohen and Conze [3] and Volkmer [24], using different methods.

In the $m = 3$ case, we have applied the same cone-based estimates to obtain bounds on the maximal eigenvalue of T_R. This time we use η_π as the evaluation functional in Lemma 4.6, and find:

$$R(\pi) \leq \mu \leq R(\pi)\ (1 + o(N)) \ .$$

Computing

$$R(\pi) = \sum_{n=0}^{N-1} \frac{(2N-1+n)!}{(2N-1)!\,n!} \left(\frac{4}{3}\right)^n \ ,$$

the Sobolev differentiability of the $m = 3$ wavelets grows much more slowly with N:

$$s(\phi_N) \approx \frac{\log N}{4 \log 3} \ .$$

These results are developed in [14].

To show how differently the differentiability of these functions grows, in Figure 4.2 we juxtapose the graphs of ϕ_N for $m = 2, 3$ and $N = 2, 10$. While the two functions with $N = 2$ don't look that different, the $m = 2$, $N = 10$ function is clearly smoother than the example with $m = 3$, $N = 10$.

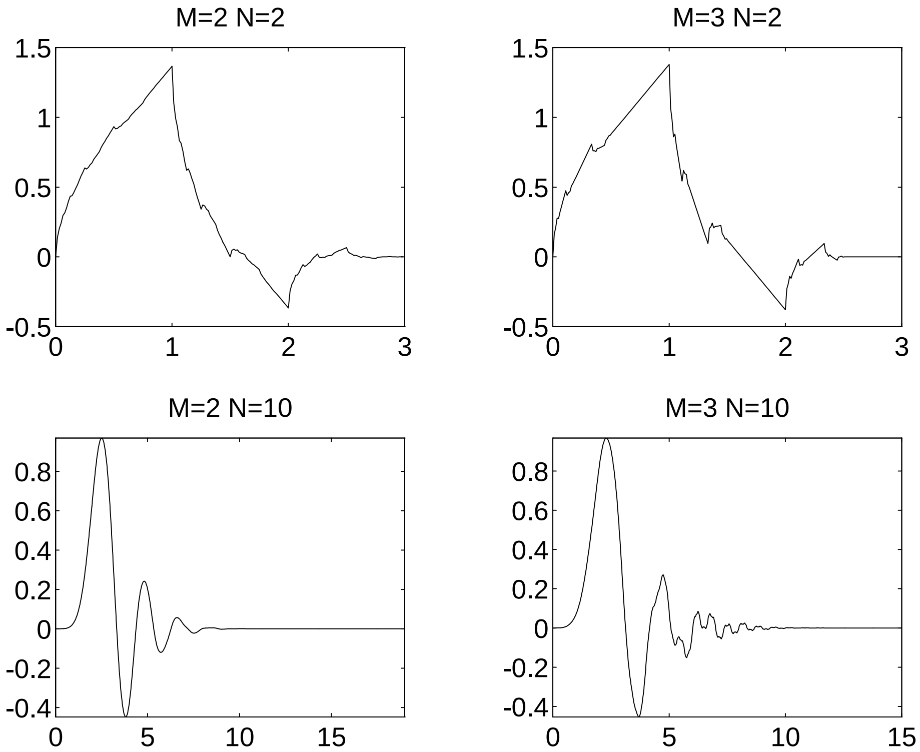

Figure 4.2: Scaling functions for $m = 2$, $N = 2$, $N = 10$ (top row) and $m = 3$, $N = 2$, $N = 10$ (bottom row).

5 Conclusion

We have introduced the multiresolution operator T_P associated with a wavelet scaling sequence $\{a_k\}$, and found the spectral theory of this operator to be critical for two important applications: deciding when a wavelet tight frame is actually an orthonormal basis, and in determining the Sobolev smoothness of a rank m scaling function of degree N (one whose first N wavelet moments vanish). Cohen [2] and Lawton [17, 18] have shown that a scaling sequence $\{a_k\}$ leads to a wavelet orthonormal basis if and only if 1 is a nondegenerate eigenvalue of the associated multiresolution operator T_P.

In section 4 we described the work of Eirola [8] and its generalization to the rank m case. We indicated that the minimal support degree N scaling function ϕ_N has Sobolev smoothness $s(\phi_N)$ which is linear in N for rank $m = 2$ and logarithmic in N for rank $m = 3$. It appears that in general $s(\phi_N)$ for these scaling functions will be linear in N for m even and logarithmic in N for m odd; this is still under investigation.

Are there families of rank m wavelet scaling functions ϕ_L defined by scaling sequences of length L which have Sobolev smoothness $s(\phi_L)$ which is linear in L for odd m? This question was raised by Daubechies [7] and has not been resolved, to our knowledge. Experimental results indicate the existence of a family of scaling sequences for rank $m = 3$ for which $s(\phi_L)$ depends algebraically on L (e.g. $s(\phi_L) \approx L^{5/9}$), but these results are very preliminary. Further development of these ideas is reported in [14].

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Peter Niels Heller
Aware, Inc.
One Memorial Drive
Cambridge, MA 02142
heller@aware.com

Raymond O. Wells, Jr.
Computational Mathematics Laboratory
Rice University
Houston, TX 77251
wells@math.rice.edu