On the Robustness of the Damped V-cycle of the Wavelet Frequency Decomposition Multigrid Method

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Abstract

The V-cycle of the wavelet variation of the "Frequency decomposition multigrid method" of Hackbusch [*Numer. Math., 56, pp. 229-245, 1989*] is considered.

It is shown that its convergence speed is not affected by the presence of anisotropy provided that the corresponding coarse grid correction is damped sufficiently strong. Our analysis is based on properties of wavelet packets which are supplied and proved.

Numerical approximations to the speed of convergence illustrate the theoretical results.

Key words: wavelets, wavelet packets, robust multilevel methods, V-cycle
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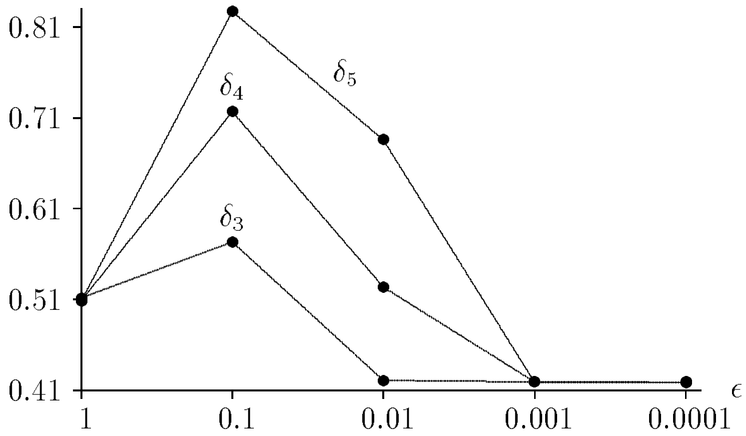

Figure 1: V-cycle convergence rates of the wavelet variation of the FDMGM for different discretization step sizes: $\delta_l = 2^{-l}$

1 Introduction

An iterative method for solving a linear system arising by the discretization of the *anisotropic* model problem

$$-\epsilon \frac{\partial^2}{\partial x^2} u(x,y) - \frac{\partial^2}{\partial y^2} u(x,y) + u(x,y) = f(x,y) \quad \text{in} \quad \Omega = (0,s)^2 , \tag{1.1}$$

$$u \quad \text{periodic}, \tag{1.2}$$

$0 < \epsilon \leq 1$, is said to be *robust* if its rate of convergence (i.e. spectral radius of the iteration matrix) is bounded smaller than 1 uniformly in ϵ and in the discretization step size.

A wavelet variation of the frequency decomposition mutigrid method (FDMGM) of Hackbusch [11] was presented in [15] and the robustness of the corresponding two-level method with respect to any intermediate level was verified.

However, due to numerical experiments the V-cycle implementation of the multilevel process seems not to be robust, see Figure 1. In this paper we are able to prove that a sufficiently damped version of the V-cycle is *almost robust* in the sense that the rate of convergence depends at most on the number of levels. Our result is of the same quality as the best estimates known for the convergence rates of multigrid solvers without regularity assumptions on the continuous problem, see [1], [3] and [16]. Indeed, the situation without regularity assumptions and the situation for the FDMGM are comparable in their lack of the *approximation property*. Hence, the standard proofs for the V-cycle convergence, see e.g. [10], are not applicable and adequate modifications are required which lead to level dependent convergence rates.

In the following section we shortly introduce the necessary vocabulary (wavelet packets, Mallat transformation, Wavelet-Galerkin discretization) to understand the considerations in Section 3. Here, we prove the convergence of the damped V-cycle. Basically, we use the same techniques as in Chapter 7.2 of [10]. For the ease of presentation two auxillary results which are crucial but rather technical are given in the appendix.

2 Wavelet Analysis

2.1 Wavelet Packet System

We will briefly recall various definitions and properties of wavelet packets [5] generated by the Daubechies scaling functions constructed in [6]. For a positive integer N, the Daubechies wavelet packets $\{\psi^l\}_{l\in\mathbf{N}_0}$ of order N are defined as follows: There exist $2N$ real numbers a_i, $i = 0, 1, \ldots, 2N-1$, satisfying

$$\sum_k a_k = 2 \qquad \text{and} \qquad \sum_k a_{k+2m}\, a_k = 2\delta_{0,m} \quad \text{for all } m \in \mathbf{Z}, \tag{2.1}$$

so that

$$\begin{aligned} \psi^{2l}(x) &= \sum_{k=0}^{2N-1} a_k\, \psi^l(2x-k)\,, && \qquad (2.2)\\ & && l = 0, 1, 2, \ldots \\ \psi^{2l+1}(x) &= \sum_{k=0}^{2N-1} b_k\, \psi^l(2x-k)\,, && \qquad (2.3) \end{aligned}$$

where $b_k = (-1)^k a_{2N-k-1}$. The function ψ^0 is called *scaling function* and ψ^1 is called *wavelet*. All wavelet packets are compactly supported, with $\mathrm{supp}(\psi^l) = [0, 2N-1]$. For convenience, we define $a_k = 0$ for $k \notin [0, 2N-1]$.

The wavelet packets are in $C^{\alpha(N)}$, the space of Hölder continuous functions with exponent $\alpha(N)$, where $\alpha(2) \approx .55$, $\alpha(3) \approx 1.09$ and $\alpha(N) \approx 0.2075 \cdot N$

for large N [9]. We refer to [14] for comprehensive introduction to wavelet packets.

For the multilevel process defined in Section 3 we will need the second order *connection coefficients* ([2], [12])

$$\Gamma_k^l := \int_{\mathbf{R}} (\psi^l)'(x-k)\,(\psi^l)'\,dx\,, \quad 2-2N \leq k \leq 2N-2\,, \tag{2.4}$$

of wavelet packets with $N \geq 3$. Due to the recursive definition of wavelet packets, their connection coefficients can be easily computed with $O(\ln(l+1)\,N^2)$ operations from the connection coefficients Γ_k^0 of the scaling function ψ^0,

$$\Gamma_k^{2l} = 4 \sum_{r,s} a_r\, a_{s-r+2k}\, \Gamma_s^l\,, \tag{2.5}$$

$$\Gamma_k^{2l+1} = 4 \sum_{r,s} b_r\, b_{s-r+2k}\, \Gamma_s^l\,. \tag{2.6}$$

2.2 Mallat Transformations

The (periodic) *Mallat transformations* $h, g : \mathbf{R}^n \to \mathbf{R}^{n/2}$, n even, of a vector $v \in \mathbf{R}^n$ are defined by

$$(h\,v)_k = \frac{1}{\sqrt{2}} \sum_{l=0}^{2N-1} a_l\, v_{l+2k}\,, \tag{2.7}$$

$$k = 0, 1, \ldots, n/2-1\,,$$

$$(g\,v)_k = \frac{1}{\sqrt{2}} \sum_{l=0}^{2N-1} b_l\, v_{l+2k}\,, \tag{2.8}$$

where we extend v periodically, i.e. $v_l = v_{l+n}$. The coefficients a_l in (2.7) and b_l in (2.8) are those in (2.2) and (2.3), respectively. The Mallat transformations satisfy (see [6], [13]),

$$\begin{aligned} h^t h + g^t g &= I\,, \\ hh^t = gg^t &= I\,, \\ gh^t = hg^t &= 0\,. \end{aligned}$$

We use I to denote the identity matrix of appropriate size throughout this paper.

In later sections we will need the tensor products of the Mallat operators h and g. Therefore, we define first the tensor product of vectors and then of linear operators. Suppose $v \in \mathbf{R}^n$ and $w \in \mathbf{R}^m$, then the tensor product

$$v \otimes w := (v_0 w_0, v_0 w_1, \ldots, v_0 w_{m-1}, v_1 w_0, \ldots, v_1 w_{m-1}, \ldots\ldots, v_{n-1} w_0, \ldots, v_{n-1} w_{m-1})^t$$

is in $\mathbf{R}^{nm}$. The tensor product $A \otimes B : \mathbf{R}^{nm} \to \mathbf{R}^{nm}$ of two linear operators $A : \mathbf{R}^n \to \mathbf{R}^n$ and $B : \mathbf{R}^m \to \mathbf{R}^m$ is defined by

$$(A \otimes B)(v \otimes w) := Av \otimes Bw,$$

for all $v \in \mathbf{R}^n$ and all $w \in \mathbf{R}^m$.

2.3 Wavelet-Galerkin Discretization of the Model Problem

We introduce the Sobolev space $H_p^1(\Omega)$, $\Omega = (0, s)^2$, with periodic boundary conditions,

$$\begin{aligned} H_p^1 = H_p^1(\Omega) := \{ v \in L^2(\Omega) \ : \ & v_x,\, v_y \in L^2(\Omega), \\ & v(0, y) = v(s, y),\, v(x, 0) = v(x, s) \}. \end{aligned}$$

The weak or variational formulation of the model problem (1.1), (1.2), reads:

$$\text{find } u \in H_p^1 : \ \mathcal{A}(u, v) = \int_\Omega f v \, dx dy, \qquad \text{for all } v \in H_p^1, \tag{2.9}$$

where $\mathcal{A}$ is the H_p^1-elliptic bilinear form

$$\mathcal{A}(u, v) = \int_\Omega (\epsilon\, u_x v_x + u_y v_y + uv) \, dx dy.$$

Due to the Lax-Milgram theorem [4] (2.9) has a unique solution u. For a Wavelet-Galerkin discretization of (2.9), we assume that $N \geq 3$ and that s in (1.1) is an integer greater than $4N - 3$. Further, we set $\psi_{l,k}^0(x) = 2^{l/2} \psi^0(2^l x - k)$ and introduce the Wavelet-Galerkin spaces

$$\begin{aligned} V_l = V_l(0, s) := \{ v \in L^2(0, s) \ : \ & v(x) = \sum_{k \in \mathbf{Z}} c_k \psi_{l,k}^0(x),\, x \in [0, s], \\ & \text{and } c_k = c_{k + 2^l s} \}. \end{aligned}$$

Obviously, V_l has the dimension $n_l = 2^l s$. The Wavelet-Galerkin approximation $u_l \in V_l \otimes V_l \subset H_p^1$ to u is the unique solution of

$$\mathcal{A}(u_l, v_l) = \int_\Omega f v_l \, dxdy, \qquad \text{for all } v_l \in V_l \otimes V_l.$$

A convergence proof of $u_l \to u$ is given in [17]. For u_l we have the following expansion $u_l(x,y) = \sum_{i,j} u_{i,j}^l \psi_{l,i}^0(x)\psi_{l,j}^0(y)$ where the expansion coefficients $u_{i,j}^l$ are periodic with period n_l in each index. We now define $f_{i,j}^l = \int_\Omega f(x,y)\psi_{l,i}^0(x)\psi_{l,j}^0(y)\, dxdy$. If we order the $u_{i,j}^l$'s and $f_{i,j}^l$'s, $0 \le i,j \le n_l - 1$, lexicographically and denote the resulting vectors U_l and F_l, respectively, then we have the following linear system for the n_l^2 unknowns U_l,

$$A_{l,0}\, U_l = F_l, \tag{2.10}$$

where

$$A_{l,0} \;=\; \epsilon\, c_l^0 \otimes I + I \otimes c_l^0 + I \otimes I \tag{2.11}$$

is the Galerkin approximation (stiffness matrix) of $\mathcal{A}$ in $V_l \otimes V_l$. In (2.11) c_l^0 is a symmetric and circulant $n_l \times n_l$ matrix [8], which is completely determined by the connection coefficients Γ_k^0 (2.4),

$$c_l^0 \;=\; \delta_l^{-2} \begin{pmatrix} \Gamma_0^0 & \Gamma_1^0 & \cdots & \Gamma_p^0 & 0 & 0 & \cdots & 0 & \Gamma_p^0 & \cdots & \Gamma_2^0 & \Gamma_1^0 \\ \Gamma_1^0 & \Gamma_0^0 & \cdots & \cdots & \Gamma_p^0 & 0 & \cdots & 0 & 0 & \Gamma_p^0 & \cdots & \Gamma_2^0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Gamma_1^0 & \cdots & \Gamma_p^0 & 0 & 0 & 0 & \cdots & \Gamma_p^0 & \cdots & \cdots & \Gamma_1^0 & \Gamma_0^0 \end{pmatrix},$$

where $p = 2N - 2$ and $\delta_l = 2^{-l}$ is the *discretization step size*. We denote a circulant matrix by its first row,

$$c_h^0 \;=\; \delta_l^{-2}\, \mathrm{Cir}_{n_l}(\Gamma_0^0\, \Gamma_1^0\, \cdots\, \Gamma_p^0\, 0\, \cdots\, 0\, \Gamma_p^0\, \cdots\, \Gamma_1^0).$$

3 Multilevel Scheme

3.1 Definitions and Notation

For the definition of the FDMGM we supply some notation. A more detailed discussion of the wavelet variation of the FDMGM can be found in [15], see also the original paper [11]. First, we define the matrices $A_{l,m} \in \mathbf{R}^{n_l^2 \times n_l^2}$ by

$$A_{l,m} \;:=\; \epsilon\, c_l^m \otimes I \,+\, I \otimes c_l^0 \,+\, I \otimes I\,, \tag{3.1}$$

where

$$c_l^m = \delta_l^{-2}\,\mathrm{Cir}_{n_l}(\Gamma_0^m \;\cdots\; \Gamma_{2N-2}^m\; 0 \;\cdots\; 0\; \Gamma_{2N-2}^m \;\cdots\; \Gamma_1^m) \tag{3.2}$$

with the connection coefficients Γ_k^m (2.4) of the wavelet packet ψ^m (2.2) resp. (2.3). We have that

$$A_{l-1,2m} = (h \otimes h)A_{l,m}(h^t \otimes h^t) \quad \text{and} \quad A_{l-1,2m+1} = (g \otimes h)A_{l,m}(g^t \otimes h^t),$$

where h and g are the Mallat transformations (2.7) and (2.8) of appropriate dimension. All of the matrices $A_{l,m}$ are positive definite.

The iteration matrix of the *basic iterative method* (BIM) (also called *smoothing iteration* [10]) with respect to $A_{l,m}$ will be denoted by

$$S_{l,m} = I - W_{l,m}^{-1}A_{l,m}\,. \tag{3.3}$$

The matrix $W_{l,m}$ characterizes the special BIM, for instance the choice $W_{l,m} = \beta^{-1}\mathrm{diag}(A_{l,m})$, $\beta \in \mathbf{R}$, gives the damped Jacobi iteration.

Now, we are able to formulate the recursive multilevel procedure. We suppose that all needed connection coefficients are precomputed via (2.5) resp. (2.6) and stored. We let $L_e \geq 0$ be the coarsest level, that is the level where the corresponding linear systems are solved exactly, and we let L be the level number with respect to the finest discretization. We have four variable quantities in our procedure:

- l is the varying level, $L_e \leq l \leq L$,
- m is a non-negative integer on level l, $0 \leq m \leq 2^{L-l} - 1$,
- $w \in \mathbf{R}^{n_l^2}$ is a starting guess,
- $b \in \mathbf{R}^{n_l^2}$ is a right hand side or a defect.

We then define the following recursive multilevel procedure MLP:

MLP$(l,\, m,\, w,\, b)$
begin
if $l = L_e$ **then** $w := A_{l,m}^{-1}b$ **else**

$w := S_{l,m}^{\nu_1}w + \sum_{j=0}^{\nu_1-1} S_{l,m}^j W_{l,m}^{-1}b$ (ν_1 steps of the BIM)

$d := A_{l,m} w - b,\ v_0 = 0,\ v_1 = 0$
MLP$(l-1,\ 2m,\ v_0,\ (h \otimes h) d)$
MLP$(l-1,\ 2m+1,\ v_1,\ (g \otimes h) d)$
$w := w\ -\ \beta_m\,(h^t \otimes h^t) v_0\ -\ \beta_m\,(g^t \otimes h^t) v_1$
$w := S_{l,m}^{\nu_2} w\ +\ \sum_{j=0}^{\nu_2-1} S_{l,m}^{j} W_{l,m}^{-1} b \quad (\nu_2 \text{ steps of the BIM})$
end

The above procedure describes a damped V-cycle with *damping parameters* $\beta_m \in \mathbf{R}$. One step of the multilevel method for solving (2.10) on level $L \geq 1$ is performed by

$$\begin{array}{rcl} w & := & U_L^\mu, \\ & \mathrm{MLP}(L,\ 0,\ w,\ F_L), & \\ U_L^{\mu+1} & := & w. \end{array} \tag{3.4}$$

The convergence rates of the above iteration without damping ($\beta_m \equiv 1$) for $L \in \{3, 4, 5\}$ with $L_e = 0$ and for the Daubechies wavelet packets $N = 3$ are plotted in Figure 1 on page 2. We used the damped Jacobi iteration as BIM with $\nu_1 = 2$ and $\nu_2 = 0$.

Next, we consider the iteration matrix $M_{L,0}(\nu_1, \nu_2)$ of the iteration (3.4). Let $M_{l,m}(\nu_1, \nu_2)$ denote the iteration matrix of the iteration

$$\begin{array}{rcl} w & := & U_l^\mu, \\ & \mathrm{MLP}(l,\ m,\ w,\ F_l), & \\ U_l^{\mu+1} & := & w, \end{array} \tag{3.5}$$

that is, we start our multigrid procedure to solve $A_{l,m} U_l = F_l$ on level l.

Lemma 3.1 *The above defined matrices $M_{l,m}(\nu_1, \nu_2)$, $L_e < l \leq L$, for the iteration* (3.5) *satisfy the recursion*

$$\begin{aligned} M_{l,m}(\nu_1, \nu_2)\ =& \\ & S_{l,m}^{\nu_2} \Big(I\ -\ \beta_m\,(h^t \otimes h^t) \{ I - M_{l-1,2m}(\nu_1, \nu_2) \}\, A_{l-1,2m}^{-1}\,(h \otimes h) A_{l,m} \\ & \qquad -\ \beta_m\,(g^t \otimes h^t) \{ I - M_{l-1,2m+1}(\nu_1, \nu_2) \}\, A_{l-1,2m+1}^{-1}\,(g \otimes h) A_{l,m} \Big)\, S_{l,m}^{\nu_1} \end{aligned} \tag{3.6}$$

with $M_{L_e,m}(\nu_1, \nu_2) = 0$.

Proof: The recursion (3.6) follows by transfering the proof of Lemma 7.1.4 in [10] to our situation.

3.2 Robustness of the damped V-cycle

For our considerations we will need the Euclidean (spectral) norm $\|w\| = \langle w,w\rangle^{1/2}$ as well as the energy-norms relative to the $A_{l,m}$'s, $\|w\|_{l,m} = \langle w,w\rangle_{l,m}^{1/2}$ $:= \langle\ A_{l,m}w,w\rangle^{1/2}$. The associated matrix norms are denoted by the same symbols. With $\varrho(B)$ we abbreviate the spectral radius of B which is the largest absolut value of the eigenvalues of B.

The following proposition has been proved in [15].

Proposition 3.2 *There exist positive numbers $\sigma_m < 1$ being independent of ϵ and δ_l such that*

$$\varrho(\ A_{l-1,2m}^{-1}\, B_{l,m}\, A_{l-1,2m+1}^{-1}\, B_{l,m}^t) \;\le\; \sigma_m \tag{3.7}$$

with $B_{l,m} = (h\otimes h)A_{l,m}(g^t\otimes h^t) = \epsilon\, hc_l^m g^t \otimes I$. Further, the strong Cauchy inequality

$$|\langle w,v\rangle_{l,m}| \;\le\; \sqrt{\sigma_m}\,\|w\|_{l,m}\,\|v\|_{l,m}$$

holds true for all $w,v\in \mathbf{R}^{n_l^2}$ satisfying $w = (h^th\otimes h^th)w$ and $v = (g^tg\otimes h^th)v$, respectively.

The numbers σ_m will play a critical rôle in our convergence analysis of the V-cycle presented in this section. They are uniformly bounded smaller than 1, i.e.

$$\sigma \;:=\; \sup\{\sigma_m\,|\,m\in\mathbf{N}_0\} \;<\; 1, \tag{3.8}$$

which will be proved in Lemma A.2 (appendix).

From now on we require a symmetry condition for our BIM, that is the matrix $W_{l,m}$ in (3.3) satisfies

$$W_{l,m}^t \;=\; W_{l,m} \;\ge\; A_{l,m} \quad \text{and} \quad \|W_{l,m}\| \;\le\; C_W\,\delta_l^{-2}\,. \tag{3.9}$$

The notation $A\le B$ signifies that A and B are symmetric matrices and that $B-A$ is positive semi definite.

Remark: The (sufficiently) damped Jacobi iteration as well as the symmetric Gauß-Seidel iteration fulfill (3.9), see e.g. [10].

We introduce the abbreviations $p_0 = h \otimes h$, $p_1 = g \otimes h$ and the transformed matrices

$$\widehat{M}_{l,m}(\nu_1,\nu_2) \;=\; A_{l,m}^{1/2}\, M_{l,m}(\nu_1,\nu_2)\, A_{l,m}^{-1/2}\,, \quad \widehat{p}_i \;=\; A_{l-1,2m+i}^{-1/2}\, p_i\, A_{l,m}^{1/2}\,, \quad i = 0,1\,,$$

$$\widehat{S}_{l,m} \;=\; A_{l,m}^{1/2}\, S_{l,m}\, A_{l,m}^{-1/2} \;=\; I - X_{l,m} \;\text{ with }\; X_{l,m} \;=\; A_{l,m}^{1/2}\, W_{l,m}^{-1} A_{l,m}^{1/2}\,.$$

The recursion (3.6) implies

$$\widehat{M}_{l,m}(\nu_1,\nu_2) \;=\; \widehat{S}_{l,m}^{\nu_2} \left(I \;-\; \beta_m \sum_{i=0}^{1} \widehat{p}_i^{\,t} \left\{ I \;-\; \widehat{M}_{l-1,2m+i}(\nu_1,\nu_2) \right\} \widehat{p}_i \right) \widehat{S}_{l,m}^{\nu_1}\,.$$

Before we formulate the robustness result for the damped V-cycle in Theorem 3.8 we first supply some preparatory lemmas and corollaries.

Lemma 3.3 *If* $\beta_m \in [0, (1+\sqrt{\sigma_m})^{-1}]$ *then*

$$0 \;\le\; Q_{l,m} \;:=\; I \;-\; \beta_m\, (\widehat{p}_0^{\,t}\, \widehat{p}_0 \;+ \widehat{p}_1^{\,t}\, \widehat{p}_1) \;\le\; I\,. \tag{3.10}$$

Proof: The relation (3.10) is equivalent to $0 \;\le\; \beta_m\, (\widehat{p}_0^{\,t}\, \widehat{p}_0 + \widehat{p}_1^{\,t}\, \widehat{p}_1) \;\le\; I$ which is satisfied if $\beta_m \in [0, \varrho(\widehat{p}_0^{\,t}\, \widehat{p}_0 + \widehat{p}_1^{\,t}\, \widehat{p}_1)^{-1}]$ because $\widehat{p}_0^{\,t}\, \widehat{p}_0 + \widehat{p}_1^{\,t}\, \widehat{p}_1$ is positive semi definite. We have that $\widehat{p}_0^{\,t}\, \widehat{p}_0 + \widehat{p}_1^{\,t}\, \widehat{p}_1 \;=\; A_{l,m}^{1/2}\, U^t\, D^{-1}\, U\, A_{l,m}^{1/2}$ with the 2×2-block diagonal matrix $D = \mathrm{diag}(A_{l-1,2m},\, A_{l-1,2m+1})$ and with the mapping $U :\; \mathbf{R}^{n_l^2} \to \mathbf{R}^{n_l^2/2}$ defined by

$$U \;:=\; \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \;=\; \begin{pmatrix} h \otimes h \\ g \otimes h \end{pmatrix}. \tag{3.11}$$

Now, let us estimate the spectral radius $\varrho(A_{l,m}^{1/2}\, U^t\, D^{-1}\, U\, A_{l,m}^{1/2})$:

$$\begin{aligned} \varrho(A_{l,m}^{1/2}\, U^t\, D^{-1}\, U\, A_{l,m}^{1/2}) \;&=\; \varrho(D^{-1}\, U\, A_{l,m} U^t) \;=\; \varrho(I \;-\; (I - D^{-1}\, U\, A_{l,m} U^t)) \\ &\le\; 1 \;+\; \varrho(I - D^{-1}\, U\, A_{l,m} U^t)\,. \end{aligned}$$

The estimate (3.7) in Proposition 3.2 implies that

$$r_m(\epsilon, \delta_l) \;:=\; \varrho(I - D^{-1}\, U\, A_{l,m} U^t) \;\le\; \sqrt{\sigma_m}\,. \tag{3.12}$$

Lemma 3.4 *Suppose* (3.9) *and let* $\tilde{Q}_{l,m} := I - A_{l,m}^{1/2} U^t (U A_{l,m} U^t)^{-1} U A_{l,m}^{1/2}$. *Then,*

$$0 \le \tilde{Q}_{l,m} \le C\, X_{l,m}\,, \tag{3.13}$$

where C is a constant being independent of ϵ, δ_l and m.

Proof: We follow the proof of Lemma 6.4.6 in [10]. The statement of Lemma B.1 (appendix) implies $\tilde{Q}_{l,m} \le C_A \delta_l^2 A_{l,m}$ and (3.9) can be rewritten as $I \le C_W \delta_l^{-2} W_{l,m}^{-1}$ which gives $A_{l,m} \le C_W \delta_l^{-2} X_{l,m}$. Hence, (3.13) holds with $C = C_A C_W$.

Corollary 3.5 *Suppose* (3.9) *and let $\beta_m \in [0, (1+\sqrt{\sigma_m})^{-1}]$. Then,*

$$0 \le Q_{l,m} \le C\, X_{l,m} + d_m\, I\,, \tag{3.14}$$

where $d_m = 1 - \beta_m(1-\sqrt{\sigma_m})$ and where C is the constant from (3.13).

Proof: We have that $Q_{l,m} - \tilde{Q}_{l,m} = A_{l,m}^{1/2} U^t ((U A_{l,m} U^t)^{-1} - \beta_m D^{-1}) U A_{l,m}^{1/2}$ with U and D as in the proof of Lemma 3.3. Further,

$$\begin{aligned}
\varrho(Q_{l,m} - \tilde{Q}_{l,m}) &= \varrho(I - \beta_m D^{-1} U A_{l,m} U^t) \\
&= \varrho(\,(1-\beta_m) I + \beta_m (I - D^{-1} U A_{l,m} U^t)\,) \\
&\le 1 - \beta_m + \beta_m\, \varrho(I - D^{-1} U\, A_{l,m} U^t) \\
&\le 1 - \beta_m + \beta_m \sqrt{\sigma_m} = d_m\,,
\end{aligned}$$

where we used (3.12). From (3.13) follows that

$$\begin{aligned}
0 \le Q_{l,m} &\le \tilde{Q}_{l,m} + \varrho(Q_{l,m} - \tilde{Q}_{l,m})\, I \\
&\le C\, X_{l,m} + d_m\, I\,.
\end{aligned}$$

□

We set $\widehat{M}_{l,m} := \widehat{M}_{l,m}(\nu/2, \nu/2)$ which formally makes sense for any real $\nu \ge 0$.

Lemma 3.6 *Suppose* (3.9). *Let $\beta_m \in [0, (1+\sqrt{\sigma_m})^{-1}]$ and let $d_m = 1 - \beta_m(1-\sqrt{\sigma_m})$. If*

$$0 \le \widehat{M}_{l-1,2m+i} \le \xi_{l-1,2m+i}\, I\,, \quad 0 \le \xi_{l-1,2m+i} \le 1\,, \quad i = 0, 1\,, \tag{3.15}$$

then

$$0 \le \widehat{M}_{l,m} \le \xi_{l,m}\, I\,, \quad \xi_{l,m} = \min_{d_m(1-\overline{\xi}_{l,m})+\overline{\xi}_{l,m} \le y \le 1} \ \max_{0 \le x \le 1} f(x,y)\,, \quad (3.16)$$

with $\overline{\xi}_{l,m} = \max\{\xi_{l-1,2m}, \xi_{l-1,2m+1}\}$ *and* $f(x,y) = (1-x)^\nu \Big(y + (1-y)Cx/(1-d_m)\Big)$ *where* C *is the constant from* (3.13)*.*

Proof: The proof is similar to the proof of Lemma 7.2.1 in [10].

Since $\widehat{M}_{l,m} = \widehat{S}^{\nu/2}_{l,m} Q_{l,m} \widehat{S}^{\nu/2}_{l,m} + \beta_m \widehat{S}^{\nu/2}_{l,m} \sum_{i=0}^{1} \widehat{p}_i^t \widehat{M}_{l-1,2m+i} \widehat{p}_i \widehat{S}^{\nu/2}_{l,m}$ and since $\widehat{M}_{l-1,2m+i} \ge 0$, $i = 0,1$, (3.10) proves $\widehat{M}_{l,m} \ge 0$. Using (3.15) and (3.10) we obtain

$$\begin{aligned} \widehat{M}_{l,m} &\le \widehat{S}^{\nu/2}_{l,m} \left\{ I - \beta_m\, \widehat{p}_0^t \widehat{p}_0\, (1-\xi_{l-1,2m}) - \beta_m\, \widehat{p}_1^t \widehat{p}_1\, (1-\xi_{l-1,2m+1}) \right\} \widehat{S}^{\nu/2}_{l,m} \\ &\le \widehat{S}^{\nu/2}_{l,m} \left\{ I - (1-\overline{\xi}_{l,m})\, \beta_m\, (\widehat{p}_0^t \widehat{p}_0 + \widehat{p}_1^t \widehat{p}_1) \right\} \widehat{S}^{\nu/2}_{l,m} \\ &= \widehat{S}^{\nu/2}_{l,m} \{ \overline{\xi}_{l,m}\, I + (1-\overline{\xi}_{l,m})\, Q_{m,l} \} \widehat{S}^{\nu/2}_{l,m}\,. \end{aligned}$$

By (3.14) and (3.10) we get $0 \le Q_{l,m} \le \alpha C X_{l,m} + (\alpha\, d_m + (1-\alpha))I$ for all $\alpha \in [0,1]$. Hence,

$$\begin{aligned} \widehat{M}_{l,m} &\le \widehat{S}^{\nu/2}_{l,m} \left\{ \alpha\, (1-\overline{\xi}_{l,m})\, C\, X_{l,m} + (1 - \alpha\, (1-\overline{\xi}_{l,m})\, (1-d_m))\, I \right\} \widehat{S}^{\nu/2}_{l,m} \\ &= \widehat{S}^{\nu/2}_{l,m} \Big(\frac{1-y}{1-d_m}\, C\, X_{l,m} + y\, I \Big) \widehat{S}^{\nu/2}_{l,m} \end{aligned}$$

for all $y \in [d_m(1-\overline{\xi}_{l,m}) + \overline{\xi}_{l,m},\ 1]$. The statement (3.16) follows by $0 \le X_{l,m} \le I$.

Corollary 3.7 *We adopt the assumptions and notations of Lemma* 3.6*. Further, let* $\beta_m = (1+\sqrt{\sigma_m})^{-1}$*. Then, there exists a positive integer* $\overline{\nu}$ *independent of* ϵ*,* δ_l *and* m *such that*

$$\begin{aligned} \varrho(\widehat{M}_{l,m}) \le \xi_{l,m} &= 1 - \frac{1-\sqrt{\sigma_m}}{1+\sqrt{\sigma_m}} \min\{1-\xi_{l-1,2m}, 1-\xi_{l-1,2m+1}\} \\ &= 1 - \frac{1-\sqrt{\sigma_m}}{1+\sqrt{\sigma_m}} \left(1 - \max\{\xi_{l-1,2m}, \xi_{l-1,2m+1}\}\right) \end{aligned}$$

for all $\nu \ge \overline{\nu}$*.*

Proof: The partial derivative $f_x(x,y) = (1-x)^{\nu-1}(-(\nu+1)(1-y)Cx/(1-d_m) - \nu\, y + (1-y)C/(1-d_m))$ shows that $f_x(x,y) \le 0$ for $y \in [d_m(1-\overline{\xi}_{l,m}) + \overline{\xi}_{l,m}, 1]$ if $\nu \ge \overline{\nu}$ with $\overline{\nu}$ sufficiently large. The lower bound $\overline{\nu}$ does not depend on m because the σ_m's are uniformly bounded smaller than 1, see (3.8). The function f is monotonically decreasing in x. Using the statement of Lemma 3.6 we estimate

$$\begin{aligned} \xi_{l,m} &= \min_{d_m(1-\overline{\xi}_{l,m})+\overline{\xi}_{l,m} \le y \le 1} f(0,y) = \min_{d_m(1-\overline{\xi}_{l,m})+\overline{\xi}_{l,m} \le y \le 1} y \\ &= 1 - \beta_m\,(1-\sqrt{\sigma_m})\,(1-\overline{\xi}_{l,m})\,. \end{aligned}$$

□

Now, we are able to formulate and to prove our main result which shows that the convergence rate of the damped V-cycle of the FDMGM depends at most on the number of levels.

Theorem 3.8 *Let $M_{L,0}(\nu,\nu)$ be the iteration matrix of the iteration process* (3.4) *with damping factors $\beta_m = (1+\sqrt{\sigma_m})^{-1}$. Further, let $\overline{\nu}$ be the positive integer determined in Corollary* 3.7*. If $2\nu \ge \overline{\nu}$ then*

$$\varrho(\, M_{L,0}(\nu,\nu)\,) = \|M_{L,0}(\nu,\nu)\|_{L,0} \le 1 - \left(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\right)^{L-L_e},$$

where $0 \le \sigma < 1$ is defined in (3.8)*.*

Proof: We have that $\varrho(\, M_{l,m}(\nu,\nu)\,) = \varrho(\, \widehat{M}_{l,m}(\nu,\nu)\,) = \|M_{l,m}(\nu,\nu)\|_{l,m}$. Since $(1-\sqrt{\sigma_m})/(1+\sqrt{\sigma_m}) \ge (1-\sqrt{\sigma})/(1+\sqrt{\sigma})$ for all m and since $\xi_{L_e,m} = 0$, $0 \le m \le 2^{L-L_e} - 1$, an inductive application of Corollary 3.7 proves Theorem 3.8.

Remark: The statement of Theorem 3.8 holds also true if we damp uniformly, i.e. $\beta_m \equiv (1+\sqrt{\sigma})^{-1}$.

In Figure 2 the convergence rates of the FDMGM (3.4) are plotted for $L \in \{3,4,5\}$ with $L_e = 0$ and for the uniform damping parameter $\beta = \beta_m = 0.9$. Again we used the Daubechies wavelet packets with $N = 3$ and the damped Jacobi iteration as BIM with $\nu_1 = 2$, $\nu_2 = 0$. The choice $\nu_1 = \nu_2 = 1$ leads to the same convergence rates.

With a look to (3.12) we see that a damping parameter which is uniform in m, ϵ and δ_l is by far not optimal. For instance, if δ_l is fixed then $r_m(\epsilon,\delta_l)$

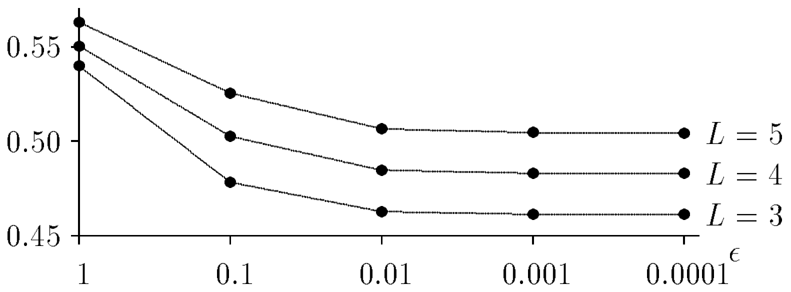

Figure 2: Damped V-cycle convergence rates of the wavelet variation of the FDMGM for different discretization step sizes and for a uniform damping parameter $\beta = 0.9$

tends to zero as $\epsilon \to 0$. Further, an explicit expression of r_m given in [15] shows that the assumption $r_m(\epsilon, \delta_l) = 0$ is reasonable if $\epsilon \ll \delta_l^2$. Moreover, in the extreme case $\epsilon = 0$ where no damping is needed the proofs in Chapter 7.2 of [10] can be transfered to the FDMGM without any modifications and yield the contraction number

$$\|M_{L,0}(\nu_1, \nu_2)\|_{L,0} \;\leq\; \frac{C}{\sqrt{C+\nu_1}\,\sqrt{C+\nu_2}}\,.$$

Appendix

A Uniform boundedness of the σ_m's

In this appendix the proof of (3.8) will be given.

We define

$$H(\omega) \;=\; \frac{1}{2}\sum_{k=0}^{2N-1} a_k\, e^{-i\,k\,\omega}\,, \quad i = \sqrt{-1}\,, \tag{A.1}$$

with the coefficients a_k from (2.1). The trigonometric polynomial H satisfies

$$H^2(\omega) \;+\; H^2(\omega+\pi) \;=\; 1\,, \tag{A.2}$$

see [6] or [7]. Further we will need the positive cosine series

$$\lambda^m(\omega) \;=\; \Gamma_0^m \;+\; 2\sum_{k=1}^{2N-2} \Gamma_k^m\, \cos(k\,\omega) \tag{A.3}$$

with the second order connection coefficients Γ_k^m (2.4). We have the following recursion formula

$$
\begin{aligned}
\lambda^{2m}(\omega) &= 4\Big(|H(\omega/2)|^2\lambda^m(\omega/2) \;+\; |H(\omega/2+\pi)|^2\lambda^m(\omega/2+\pi)\Big), && \text{(A.4)}\\
\lambda^{2m+1}(\omega) &= 4\Big(|H(\omega/2+\pi)|^2\lambda^m(\omega/2) \;+\; |H(\omega/2)|^2\lambda^m(\omega/2+\pi)\Big), && \text{(A.5)}
\end{aligned}
$$

see [15]. With these definitions we are able to express the numbers σ_m from Proposition 3.2 by

$$\sigma_m \;=\; \sup_{0\leq\,\omega\,\leq 2\pi} \frac{R^m(\omega)}{R^m(\omega) \;+\; \lambda^m(\omega/2)\lambda^m(\omega/2+\pi)} \tag{A.6}$$

where $R^m(\omega) \;=\; |H(\omega/2)|^2|H(\omega/2\,+\,\pi)|^2\,(\lambda^m(\omega/2)\,-\,\lambda^m(\omega/2\,+\,\pi))^2$, see also [15].

In the lemma below the bracket expression $[\cdot]$ denotes the 'greatest integer' and ld denotes the logarithm with respect to the basis 2.

Lemma A.1 *The inclusion*

$$0 \;<\; \min_{0\leq\,\omega\,\leq 2\pi}\,\lambda^1(\omega) \;\leq\; 4^{-[\mathrm{ld}\, m]}\lambda^m(\omega) \;\leq\; \max_{0\leq\,\omega\,\leq 2\pi}\,\lambda^1(\omega) \;<\; \infty \tag{A.7}$$

holds true for all $m\geq 1$.

Proof: Both inner inequalities follow inductively from (A.4), (A.5) and (A.2). The leftmost inequality was shown in Lemma 2.3 (iv) in [15].

Lemma A.2 *The supremum of the σ_m's* (A.6) *is bounded smaller than* 1*,*

$$\sup\,\{\sigma_m \mid m\in\mathbf{N}_0\} \;<\; 1\,.$$

Proof: We assume that $\sup\{\sigma_m \mid m\geq 1\} = 1$. Then, for any $\alpha > 0$ there exists an integer $m_\alpha \geq 1$ and an $\omega_\alpha \in [0,2\pi)$ such that

$$\sigma_{m_\alpha} \;=\; \frac{R^{m_\alpha}(\omega_\alpha)}{R^{m_\alpha}(\omega_\alpha) \;+\; \lambda^{m_\alpha}(\omega_\alpha/2)\,\lambda^{m_\alpha}(\omega_\alpha/2+\pi)} \;\geq\; 1-\alpha\,,$$

which is equivalent to

$$\alpha \;\geq\; \frac{16^{-[\mathrm{ld}\, m_\alpha]}\lambda^{m_\alpha}(\omega_\alpha/2)\,\lambda^{m_\alpha}(\omega_\alpha/2+\pi)}{16^{-[\mathrm{ld}\, m_\alpha]}R^{m_\alpha}(\omega_\alpha) \;+\; 16^{-[\mathrm{ld}\, m_\alpha]}\lambda^{m_\alpha}(\omega_\alpha/2)\,\lambda^{m_\alpha}(\omega_\alpha/2+\pi)} \;>\; 0\,. \tag{A.8}$$

By (A.7) and by $|H(\omega/2)|^2|H(\omega/2\,+\,\pi)|^2 \;<\; 1/4$ which follows from (A.2) we derive $0 \;<\; 16^{-[\mathrm{ld}\, m_\alpha]}R^{m_\alpha}(\omega_\alpha) \;\leq\; (\max_{0\leq\,\omega\,\leq 2\pi}\lambda^1(\omega)\,)^2$. Therefore, (A.8) implies that $16^{-[\mathrm{ld}\, m_\alpha]}\lambda^{m_\alpha}(\omega_\alpha/2)\,\lambda^{m_\alpha}(\omega_\alpha/2\,+\,\pi) \;\to\; 0$ as α tends to zero. However, this contradicts $4^{-[\mathrm{ld}\, m]}\lambda^m(\omega) \geq \min_{0\leq\,\omega\,\leq 2\pi}\lambda^1(\omega) > 0$, see (A.7). Hence, $\sup\{\sigma_m \mid m\geq 1\} < 1$. Since $\sigma_0 < 1$ Lemma A.2 is proved.

B An approximation property

Our goal is to verify the approximation property stated in Lemma B.1.

Lemma B.1 *Let the matrices $A_{l,m}$ be defined as in* (3.1) *and the mapping U as in* (3.11). *Then,*

$$\|A_{l,m}^{-1} - U^t(UA_{l,m}U^t)^{-1}U\| \leq C_A\, \delta_l^{-2}\,,$$

where C_A is a constant being independent of ϵ, δ_l and m.

First we note that $\lambda_\mu^m = \delta_l^{-2}\lambda^m(2\pi\mu/n_l)$, $\mu = 0,\ldots,n_l-1$, are the eigenvalues of c_l^m (3.2) where the function λ^m is given in (A.3). All circulant matrices of the dimension n_l share the same system of l^2-orthonormal eigenvectors v_μ, $\mu = 0,\ldots,n_l-1$,

$$(v_\mu)_k \;=\; \frac{1}{\sqrt{n_l}}\, e^{-i\,2\pi\,\mu k/n_l},$$

see [8]. Consequently, the matrices $A_{l,m}$ have the eigenvalues

$$\Lambda_{m;\mu,\nu} \;=\; \delta_l^{-2}(\,\epsilon\,\lambda^m(2\pi\,\mu/n_l) \;+\; \lambda^0(2\pi\,\nu/n_l)\,) + 1\,,$$

$0 \leq \mu,\nu \leq n_l - 1$, with corresponding eigenvectors $v_{\mu,\nu} = v_\mu \otimes v_\nu$.

Lemma B.2 *We have that*

$$\|U^t(UA_{l,m}U^t)^{-1}UA_{l,m}\| \;\leq\; C\,,$$

where the constant C does not depend on ϵ, δ_l or m.

Proof: The proof will be very technical and we will use results proved in [15].

The matrix $UA_{l,m}U^t$ has the following block structure

$$UA_{l,m}U^t \;=\; \begin{pmatrix} A_{l-1,2m} & B \\ B^t & A_{l-1,2m+1} \end{pmatrix}$$

with $B = \epsilon\, c_{l-1}^{m,c} \otimes I$ and $c_{l-1}^{m,c} = hc_l^m g^t$. Again, $c_{l-1}^{m,c}$ is a circulant matrix,

$$c_{l-1}^{m,c} \;=\; \delta_{l-1}^{-2}\,\mathrm{Cir}_{n_{l-1}}\,(\Gamma_0^{m,c}\;\Gamma_1^{m,c}\;\cdots\;\Gamma_{2N-2}^{m,c}\;0\;\cdots\;0\;\Gamma_{2-2N}^{m,c}\;\cdots\;\Gamma_{-1}^{m,c})\,,$$

determined by the (mixed) connection coefficients

$$\Gamma_k^{m,c} \;=\; \int_{\mathbf{R}} (\psi^{2m})'(x-k)\,(\psi^{2m+1})'(x)\,dx\,.$$

The eigenvalues of $c_{l-1}^{m,c}$ are denoted by $\lambda_\mu^{m,c} = \delta_{l-1}^{-2}\lambda^{m,c}(2\pi\,\mu/n_{l-1})$, $0 \le \mu \le n_{l-1}-1$, where $\lambda^{m,c}(\omega) = \sum_{k=2-2N}^{2N-2} \Gamma_k^{m,c}\, e^{-i\,k\omega}$ satisfying

$$\lambda^{m,c}(\omega) \;=\; H(\omega/2)\,\overline{H(\omega/2+\pi)}\,(\,\lambda^m(\omega/2) - \lambda^m(\omega/2+\pi)\,), \tag{B.1}$$

with H as in (A.1). The inverse of $UA_{l,m}U^t$ can be expressed by its block entries

$$(UA_{l,m}U^t)^{-1} \;=\; \begin{pmatrix} A_0^{-1}(I - BA_1^{-1}\Sigma^{-1}B^tA_0^{-1}) & -A_0^{-1}BA_1^{-1}\Sigma^{-1} \\ -A_1^{-1}\Sigma^{-1}B^tA_0^{-1} & A_1^{-1}\Sigma^{-1} \end{pmatrix} \tag{B.2}$$

where $\Sigma \;=\; I - A_1^{-1}B^tA_0^{-1}B$. For convenience we set $A_0 := A_{l-1,2m}$ and $A_1 := A_{l-1,2m+1}$. The matrix Σ is invertible because

$$\varrho(I-\Sigma) \;\le\; \sigma \;<\; 1 \tag{B.3}$$

due to (3.7) and (3.8). Let f be in $\mathbf{R}^{n_l^2}$. Then,

$$\|U^t(UA_{l,m}U^t)^{-1}UA_{l,m}f\|^2 \;=\; \sum_{\mu,\nu}\sum_{\alpha,\beta} \Lambda_{m;\mu,\nu}\,\Lambda_{m;\alpha,\beta}$$

$$\times\,\langle\,(UA_{l,m}U^t)^{-1}Uv_{\mu,\nu}\,,\,(UA_{l,m}U^t)^{-1}U\overline{v}_{\alpha,\beta}\,\rangle\,\langle\,f\,,\,v_{\mu,\nu}\,\rangle\,\langle\,f\,,\,\overline{v}_{\alpha,\beta}\,\rangle\,.$$

A simple calculation shows that

$$Uv_{\mu,\nu} \;=\; \begin{pmatrix} H(\omega_\mu)\,H(\omega_\nu)\,\widetilde{v}_{2\mu,2\nu} \\ G(\omega_\mu)\,H(\omega_\nu)\,\widetilde{v}_{2\mu,2\nu} \end{pmatrix}. \tag{B.4}$$

Here, $\omega_\mu = 2\pi\,\mu/n_l$ and $G(\omega) = 2^{-1}\sum_{k=0}^{2N-1} b_k e^{-i\,k\omega}$ with the b_k's from (2.3). The vectors $\widetilde{v}_{2\mu,2\nu}$ are tensor products, $\widetilde{v}_{2\mu,2\nu} = \widetilde{v}_{2\mu}\otimes\widetilde{v}_{2\nu}$, where $(\widetilde{v}_{2\mu})_k = \sqrt{2}\,e^{-i\,2\pi\,2\mu k/n_l}/\sqrt{n_l}$, $k = 0,\ldots,n_{l-1}-1$. Note that $\widetilde{v}_{2\mu,2\nu}$ is an eigenvector of the circulant matrices A_0, A_1, B and B^t, respectively. Using (B.4) as well as (B.2) leads to

$$(UA_{l,m}U^t)^{-1}Uv_{\mu,\nu} \;=\; \begin{pmatrix} r_{\mu,\nu}\,\widetilde{v}_{2\mu,2\nu} \\ t_{\mu,\nu}\,\widetilde{v}_{2\mu,2\nu} \end{pmatrix}$$

with

$$
\begin{aligned}
r_{\mu,\nu} &= H(\omega_\mu)\, H(\omega_\nu)\, \Lambda^{-1}_{2m;2\mu,2\nu} \Big(1 + \Lambda^{-1}_{\Sigma;2\mu,2\nu} \\
&\qquad \times \big(\Lambda^{-1}_{2m+1;2\mu,2\nu}\, |\Lambda^{c}_{2m;2\mu,2\nu}|^2\, \Lambda^{-1}_{2m;2\mu,2\nu} - \Lambda^{c}_{2m;2\mu,2\nu}\, \Lambda^{-1}_{2m+1;2\mu,2\nu} \big) \Big), \\
t_{\mu,\nu} &= G(\omega_\mu)\, H(\omega_\nu)\, \Lambda^{-1}_{2m+1;2\mu,2\nu}\, \Lambda^{-1}_{\Sigma;2\mu,2\nu} \big(-\overline{\Lambda}^{c}_{2m;2\mu,2\nu}\, \Lambda^{-1}_{2m;2\mu,2\nu} + 1 \big),
\end{aligned}
$$

where $\Lambda^{c}_{2m;2\mu,2\nu} = \epsilon\, \delta^{-2}_{l-1} \lambda^{m,c}(2\omega_\mu)$ and $\Lambda_{\Sigma;2\mu,2\nu} = 1 - \Lambda^{-1}_{2m;2\mu,2\nu}\, |\Lambda^{c}_{2m;2\mu,2\nu}|^2$ $\Lambda^{-1}_{2m+1;2\mu,2\nu}$. Since

$$
\begin{aligned}
&\langle\, (UA_{l,m}U^t)^{-1} U v_{\mu,\nu}\,,\, (UA_{l,m}U^t)^{-1} U \overline{v}_{\alpha,\beta} \,\rangle = \\
&(r_{\mu,\nu}\, \overline{r}_{\alpha,\beta} + t_{\mu,\nu}\, \overline{t}_{\alpha,\beta})\, \langle\, \tilde{v}_{2\mu,2\nu}\,,\, \overline{\tilde{v}}_{2\alpha,2\beta} \,\rangle = (r_{\mu,\nu}\, \overline{r}_{\alpha,\beta} + t_{\mu,\nu}\, \overline{t}_{\alpha,\beta})\, \Delta^{\mu,\alpha}_{n_l}\, \Delta^{\nu,\beta}_{n_l}
\end{aligned}
$$

(here $\Delta^{\mu,\alpha}_{n_l} = 1$ if $|\mu - \alpha| = k\, n_l/2$, $k \in \mathbf{Z}$, and $\Delta^{\mu,\alpha}_{n_l} = 0$ otherwise) we have that

$$
\|U^t (UA_{l,m}U^t)^{-1} U A_{l,m} f\| \le 3\, \|f\| \max_{\mu,\nu} \Lambda_{m;\mu,\nu} \big(|r_{\mu,\nu}| + |t_{\mu,\nu}| \big).
$$

In the last step of the proof we show that the maximum is independent of ϵ, δ_l and m. Therefore we supply a bunch of estimates. First,

$$
\Lambda_{\Sigma;2\mu,2\nu} \ge 1 - \sigma \quad \text{and} \quad \Lambda^{-1}_{2m;2\mu,2\nu}\, |\Lambda^{c}_{2m;2\mu,2\nu}|^2\, \Lambda^{-1}_{2m+1;2\mu,2\nu} \le \sigma \tag{B.5}
$$

by (B.3) and (3.7), (3.8), respectively. Further,

$$
\begin{aligned}
\frac{|H(\omega_\mu)\, H(\omega_\nu)|\, \Lambda_{m;\mu,\nu}}{\Lambda_{2m;2\mu,2\nu}} &\le 4 \sup_{0 \le \omega,\eta \le 2\pi} \frac{|H(\omega)\, H(\eta)|\, (\epsilon\, \lambda^m(\omega) + \lambda^0(\eta) + \delta_l^2)}{\epsilon\, \lambda^{2m}(2\omega) + \lambda^0(2\eta) + \delta_{l-1}^2} \\
&\le 4 \max \Big\{ \underbrace{\sup_{0 \le \omega \le 2\pi} \frac{|H(\omega)| \lambda^m(\omega)}{\lambda^{2m}(2\omega)}}_{=E_1},\, \underbrace{\sup_{0 \le \eta \le 2\pi} \frac{|H(\eta)| \lambda^0(\eta)}{\lambda^0(2\eta)}}_{=E_2},\, \frac{\delta_l^2}{\delta_{l-1}^2} \Big\}.
\end{aligned}
$$

If $m \ge 1$ then (A.7) implies that E_1 is uniformly bounded in m. For $m = 0$, E_1 equals E_2. Due to part (ii) of Lemma 2.3 in [15] $\lambda^0(2\omega^*)$ is equal to zero if and only if $\omega^* \in \{0, \pi, 2\pi\}$. For $\omega^* \in \{0, 2\pi\}$, the limit $\lim_{\omega \to \omega^*} \lambda^0(\omega)/\lambda^0(2\omega)$ exists. In the case $\omega^* = \pi$, the function $H(\omega)$ has a zero al least of order 3 ($N \ge 3$), see [6] or [7]. However, the zero of $\lambda^0(2\omega)$ in $\omega^* = \pi$ is only of order 2, Lemma 2.1 (i) in [15]. Consequently, E_1 and E_2 are finite and hence,

$$\frac{|H(\omega_\mu)\, H(\omega_\nu)|\, \Lambda_{m;\mu,\nu}}{\Lambda_{2m;2\mu,2\nu}} \le C_1\,, \qquad C_1 \ne C_1(\epsilon,\delta_l,m)\,. \tag{B.6}$$

Next,

$$\begin{aligned} \frac{|G(\omega_\mu)\, H(\omega_\nu)|\, \Lambda_{m;\mu,\nu}}{\Lambda_{2m+1;2\mu,2\nu}} &\le 4 \sup_{0\le\omega,\eta\le 2\pi} \frac{|G(\omega)\, H(\eta)|\,(\epsilon\,\lambda^m(\omega)+\lambda^0(\eta)+\delta_l^2)}{\epsilon\,\lambda^{2m+1}(2\omega)+\lambda^0(2\eta)+\delta_{l-1}^2} \\ &\le 4\max\Big\{ \underbrace{\sup_{0\le\omega\le 2\pi} \frac{\lambda^m(\omega)}{\lambda^{2m+1}(2\omega)}}_{=E_3},\ \underbrace{\sup_{0\le\eta\le 2\pi} \frac{|H(\eta)|\lambda^0(\eta)}{\lambda^0(2\eta)}}_{=E_2},\ \frac{\delta_l^2}{\delta_{l-1}^2}\Big\}\,. \end{aligned}$$

Using (A.7) we are able to estimate $E_3 \le \max_\omega \lambda^1(\omega)/\min_\omega \lambda^1(\omega)$ for $m\ge 1$. Hence, E_3 is uniformly bounded in m and

$$\frac{|G(\omega_\mu)\, H(\omega_\nu)|\, \Lambda_{m;\mu,\nu}}{\Lambda_{2m+1;2\mu,2\nu}} \le C_2\,, \qquad C_2 \ne C_2(\epsilon,\delta_l,m)\,. \tag{B.7}$$

Finally, let j be 0 or 1, then

$$\begin{aligned} \frac{|\Lambda^c_{2m;2\mu,2\nu}|}{\Lambda_{2m+j;2\mu,2\nu}} &\le 4 \sup_{0\le\omega,\eta\le 2\pi} \frac{\epsilon\,|\lambda^{m,c}(2\omega)|}{\epsilon\,\lambda^{2m+j}(2\omega)+\lambda^0(2\eta)+\delta_{l-1}^2} \\ &\le 4\, \underbrace{\sup_{0\le\omega\le 2\pi} \frac{|\lambda^{m,c}(2\omega)|}{\lambda^{2m+j}(2\omega)}}_{=E_4}\,. \end{aligned}$$

We have that $|\lambda^{m,c}(2\omega)| \le \max_\omega \lambda^m(\omega)$ by (B.1) and by $|H(\omega)H(\omega+\pi)| \le 1/2$. The cases $j=1$ and $j=0$ with $m\ge 1$ yield $E_4 \le \max_\omega \lambda^1(\omega)/\min_\omega \lambda^1(\omega)$. It remains to consider $j=0$ and $m=0$. However, this was already done in Lemma 2.4 of [15]. Altogether we have shown that

$$\frac{|\Lambda^c_{2m;2\mu,2\nu}|}{\Lambda_{2m+j;2\mu,2\nu}} \le C_3\,, \qquad C_3 \ne C_3(\epsilon,\delta_l,m)\,, \tag{B.8}$$

for $j=0,1$. We finish the proof of Lemma B.2 by

$$\begin{aligned} \Lambda_{m;\mu,\nu}\,|r_{\mu,\nu}| &\le C_1\,(1 + (1-\sigma)^{-1}(\sigma + C_3))\,, \\ \Lambda_{m;\mu,\nu}\,|t_{\mu,\nu}| &\le C_2\,(1-\sigma)^{-1}(\,C_3+1\,)\,, \end{aligned}$$

where we have used (B.5), (B.6), (B.7) and (B.8).

Proof of Lemma B.1: We will need the estimate $\|(I - U^tU)A_{l,m}^{-1}\| \leq C'\delta_l^2$ which can be verified by a simple modification of the proof of Lemma 5.3 in [15]. The constant C' does not depend on ϵ, δ_l or m. Now,

$$\begin{aligned}\|A_{l,m}^{-1} - U^t(UA_{l,m}U^t)^{-1}U\| &= \|(I - U^t(UA_{l,m}U^t)^{-1}UA_{l,m})(I - U^tU)A_{l,m}^{-1}\| \\ &\leq \|(I - U^t(UA_{l,m}U^t)^{-1}UA_{l,m})\| \, \|(I - U^tU)A_{l,m}^{-1}\| \leq C_A\, \delta_l^2\end{aligned}$$

with $C_A = (1 + C)C'$ where the constant C is as in Lemma B.2.

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