Multiscale Applications of Wavelets to Solutions of Partial Differential Equations

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In 1988 Daubechies' fundamental paper on wavelets [1] appeared. In this paper we find for the first time a parametrized family of orthonormal systems of functions in $L^2(\mathbf{R})$ with certain important complementary properties. Each system of functions has the following properties:

- each system is generated from a scaling function $\varphi(x)$ and a wavelet function $\psi(x)$ by rescalings (by powers of an integer) and translations (e.g., $\varphi_{j,k}(x) := 2^{j/2}\varphi(2^j x - k)$ and $\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k)$. The wavelet system

$$\{\psi_{0,k}(x), \psi_{j,k}(x), \quad j,k \in \mathbf{Z}, \quad j \geq 0\} \tag{1}$$

  is an orthonormal basis for $L^2(\mathbf{R})$.

- each element in a given system has compact support and is continuous or can be chosen to be smooth up to a given finite order and by the rescaling above, the supports of the basis functions becomes very small for large scaling index j.

- There are fast algorithms for computing the coefficients of the expansion of a given digitized (sampled function). This is the *discrete wavelet transform* (from the sampled function to the wavelet expansion coefficients), and it is an O(N) algorithm.

If we consider such a wavelet system, and assume that there is a certain amount of smoothness (C^2, for instance), then we can try to use these functions as basis elements in a Galerkin/finite element type of numerical solutions of partial differential equations. Suppose we consider only finite scale expansions in terms of wavelets or scaling functions (which is what happens in practice). Namely, if we let for fixed $J \in \mathbf{N}$,

$$\tilde{f}(x) = \sum_k c_{Jk}\varphi_{Jk}(x), \tag{2}$$

where c_{Jk} represents a sampling of a given function $f(x)$ at the points $x = k/2^J$, then $\tilde{f}$ is a smooth wavelet interpolation of our original sampled $f(x)$ *at the scale* J (or, what is the same thing, on a mesh with mesh size $h = 1/2^J$). Mallat [5] showed that from the scaling equations defining φ and ψ one can reexpress $\tilde{f}$ in terms of scaling and wavelet functions at coarser scales, namely:

$$\tilde{f}(x) = \sum_k c_{Jk}\varphi_{Jk}(x) = \sum_k c_{0k}\varphi_{0k}(x) + \sum_k \sum_{j=0}^{J-1} d_{Jk}\psi_{jk}(x). \tag{3}$$

In (3) we see that the left hand side (LHS)represents the data at a single "fine" scale J, while the right hand side (RHS) gives a multiscale representation of the data at the coarser scales $\{0, 1, \ldots, J-1\}$. The Mallat transform consists of mapping the coefficients at the single scale on the LHS of (3) to the multiscale coefficients on the RHS of (3), and conversely (inverse Mallat transform). This transform consists of convolution with the filters which define the scaling and wavelet functions along with downsampling (and upsampling for the inverse transform).

The wavelet represention of a sampled function of the form (2) allows one to use the scaling functions at a given scale (in this case at the arbitrary scale J corresponding to a mesh size of $h = 1/2^J$) as finite-element or Galerkin-type basis elements in a discrete approximation to some continuous problem (e.g., solving a partial differential equation numerically). In a number of recent papers these ideas have been carried out for various types of elliptic boundary value problems [9,8,7,3,2]. In addition one can use the multiscale represntation of data as given in (3) to implement multigrid iterative schemes for solving such elliptic boundary problems where the solution by direct methods or by iterative methods at a single scale is prohibitive. In particular, one obtains an efficient multiscale algorithm for solving the model problem involving Laplace's equation for a domain with a very general boundary [4]. Moreover, a second model problem involving anisotropic coefficients in two dimensions with periodic boundary values admits a robust multigrid algorithm whose condition number is independent of the mesh size and of the anisotropy parameter [6]. In these multigrid applications of wavelets to numerical analysis the linear Mallat algorithms (transform from single scale to multiscale and conversely) play a major role. They allow one to map simply from one adjacent scale to another in a very effective manner, and that, along with the implicit orthogonality (and hence lack of redundancy), is one of the keys to their success in this applications (which is also true in their application to digital signal processing).

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