Adaptive Wave Propagation Modeling

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ABSTRACT

This paper discusses several current attempts to use acoustic and electromagnetic wave propagation for modeling physical phenomena and the role that wavelet analysis is playing in these efforts. The first problem involves recent application of wavelets to computational fluid dynamics. The second problem involves geophysical modeling of the ocean floor, using acoustic waves, and wavelets have recently been shown to play an important role here already. The third problem involves modeling of SAR radar images in the context of automatic target recognition efforts. The fourth problem is global illumination in computer graphics, i.e., simulation of reflected and absorbed light for everyday environments. The role of wavelets is more embryonic in these latter two areas, but there are some common principles in all of these modeling efforts, and the methodology of wavelets seems well suited to certain aspects of these problems.

1 INTRODUCTION AND OVERVIEW

In 1988 Daubechies published her seminal paper on compactly supported wavelets [8]. Since that time there have been major developments in the applications of wavelets to many fields of science and engineering. In this paper we want to discuss certain problems concerning wave propagation modeling in which wavelets are playing (or in some cases seem likely to play) a significant role. The problems we want to discuss are:

- Static problems in fluid flow
- Acoustic modeling of the ocean floor
- Modeling Synthetic Aperture Radar (SAR) images
- Modeling illuminated surfaces for computer graphics

Each of these areas has its own physical models and equations of varying levels of generality governing the behavior of the phenomena in question.

In the fluid flow problem the geometry and boundary conditions are often prescribed and the resulting fluid flow is computed or simulated. More specifically, one tries to change, adapt or modify the geometry to optimize some flow considerations and the simulation attempts to predict outcomes without expensive wind-tunnel or turbulence experiments (although these will never be dispensed with!). Numerical analysts have worked for decades on various types of modeling of such phenomena (finite-difference schemes, finite-element methods, spectral methods, and many more).

An important problem for geophysicsts is to effectively model the reflection from sonar signals in a manner which allows one to understand the ocean floor topography. This problem is related to the seismic inverse scattering problems, which are a cornerstone of the search for oil and gas by the oil industry. There are various approaches to this type of problem and we will discuss a recent study which uses wavelets as a tool for solving this problem in a particular computational domain near the ocean floor and represents the ocean floor as a viscoelastic medium.

Synthetic aperture radar (SAR) is a type of radar image captured by a moving plane over a region of the earth and capturing a sequence of radar snapshots of the traditional kind and integrating this information into a single image which is a compilation of a sequence of the moving images. This is a tomographic-type composite image similar to the image of a CAT-scan machine being formed from a composite of a number of X-ray cross sections. This gives a far superior resolution than would be available with a single radar image for the size of the radar antenna available on planes. There have been major developments in simulating such images over the past decade, and we will discuss a future attempt to simulate such images by adapting the computational model developed for ocean floors as mentioned in the previous paragraph.

Finally, the problem of representing on a computer screen the reflection of light from lighted objects of variable geometry and variable light-reflecting ability in a common background has been studied by a number of computer scientists in the past few years. One particular means of modeling this problem, the radiosity model, turns out to be very amenable to using wavelets for a more efficient solution of the computational problem which arises.

In Section 2 we will present some background information about wavelets with references to current sources with much more extensive information.

2 ADAPTIVE MULTISCALE REPRESENTATION OF FUNCTIONS

In this section we will briefly recall the Daubechies compactly supported wavelets. Let g be a positive integer, and consider a $2 \times 2g$ matrix of the form

$$a = \begin{pmatrix} a_0^0 & \dots & a_{2g-1}^0 \\ a_0^1 & \dots & a_{2g-1}^1 \end{pmatrix}, \tag{2.1}$$

where

$$\sum_{k=0}^{2g-1} a_k^r a_{k+2l}^s = 2\delta_{l,0}\delta^{r,s}, \tag{2.2}$$

$$\sum_{k=0}^{2g-1} a_k^r = 2\delta^{r,0}, \tag{2.3}$$

where

$$\delta^{l,m} = \delta_{l,m} = \begin{cases} 0 & l \neq m \\ 1 & l = m, \quad l, m \in \mathbf{Z} \end{cases}$$

is the Kronecker delta function. This is a *wavelet matrix* of *rank* 2 and *genus* g (see [18]). By choosing

$$a_k^1 = (-1)^{k+1} a_{2g-1-k}^0,$$

we see that the orthogonality conditions in (2.2) and (2.3) are satisfied if they are satisfied for the first row. Given any wavelet matrix of the form (2.1), we can find compactly supported scaling functions and wavelet functions which are in $L^2(\mathbf{R})$ and which satisfy

$$\varphi(x) = \sum_{k=0}^{2g-1} a_k^0 \varphi(2x-k),$$

$$\psi(x) = \sum_{k=0}^{2g-1} a_k^1 \varphi(2x-k).$$

More generally we can choose an $m \times mg$ wavelet matrix satisfying conditions similar to (2.1) and (2.2) above (see [18]). A major problem in adaptive wavelet analysis is to choose the right *rank* (m is the rank) for the problem. In digital signal processing, for instance, one uses rank $m = 2$ for compression of images, $m = 32$ for compression of audio signals, $m = 512$ for digital transmission of telecommunication signals, and so forth. For a fixed rank choosing a suitable wavelet system (or more generally biorthogonal wavelet system) is a difficult optimization problem. Most of the problems we discuss in this paper involve the initial steps of finding the first wavelet implementation of an algorithm, and the optimization phase is yet to be done.

The function $\varphi(x)$ is called the *scaling function* associated with a, and $\psi(x)$ is called the *wavelet function* associated with a. There is a $(g-1)$-dimensional parameter space of choice of wavelet matrices ([25], [37]). The functions φ and ψ both have support in the interval $[0, 2g-1]$, and we define the *wavelet system* associated with a by

$$\varphi_k^j(x) := 2^{\frac{j}{2}} \varphi(2^j x - k),$$

$$\psi_k^j(x) := 2^{\frac{j}{2}} \psi(2^j x - k), \tag{2.4}$$

where $k \in \mathbf{Z}$, $j \in \mathbf{Z}^+$ (the nonnegative integers). The fundamental theorem of Daubechies and Lawton is that

$$\varphi_k^0(x), \psi_k^j(x), \quad k \in \mathbf{Z}, j \in \mathbf{Z}^+, \tag{2.5}$$

is a tight frame for $L^2(\mathbf{R})$ for any wavelet matrix a (see [8], [20]), i.e., any $f \in L^2(\mathbf{R})$ has an L^2 expansion of the form

$$f = \sum f_k \varphi_k^0(x) + \sum f_k^j \psi_k^j(x), \tag{2.6}$$

where

$$f_k = \int f(x)\varphi_k^0(x)dx, \qquad f_k^j = \int f(x)\psi_k^j(x)dx. \tag{2.7}$$

For a generic wavelet matrix, (2.5) is an orthonormal system, and (2.6) is the usual orthonormal expansion (cf. Lawton [21]).

Daubechies chooses a very specific wavelet matrix a_{Dg}, for each genus $g \geq 1$, which maximizes the vanishing moments of ψ, i.e., she chooses a so that

$$\int_{\mathbf{R}} x^n \psi(x)dx = 0, \quad m = 0, \ldots, g-1.$$

For each g, there are only a finite number of such choices, and we use the choice made by Daubechies. Her choice leads to a specific countable family of scaling functions and wavelet functions

$$\varphi_{Dg}, \psi_{Dg}, \quad g = 1, 2, \ldots,$$

with a number of properties. In Daubechies' paper she denoted these functions by ${}_N\varphi(x)$, ${}_N\psi(x)$, where $N = g$. We call φ_{Dg} the *Daubechies scaling function* of genus g, and ψ_{Dg} the *Daubechies wavelet function* of genus g, for $g \in \mathbf{N}$ (the positive integers). Daubechies' recent book [9] contains a thorough discussion of all of these issues. In particular, the Daubechies wavelets have specific differentiability properties, namely, $\varphi \in C^{\alpha(g)}$, where α is affine linear, and for $g \geq 3$, $\varphi \in C^1$.

The Daubechies wavelet system gives a specific multiresolution analysis of $L^2(\mathbf{R})$ (cf. Mallat [22]). Namely, if we let (in the topology of $L^2(\mathbf{R})$)

$$V_j := \text{closure (span } \{\varphi_k^j : k \in \mathbf{Z}\}), \tag{2.8}$$

and let W_j be the orthogonal complement of V_j in V_{j+1}, then for fixed $j \in \mathbf{Z}$, $\{\varphi_k^j\}_{k\in\mathbf{Z}}$ is an orthonormal basis for V_j, and $\{\psi_k^j\}_{k\in\mathbf{Z}}$ is an orthonormal basis for W_j. Moreover, for any $j \in \mathbf{Z}$,

$$L^2(\mathbf{R}) = V_j \oplus \bigoplus_{n\geq j} W_n = \bigoplus_{n\in\mathbf{Z}} W_n.$$

In addition, if we let P_j be the orthonormal projection $L^2(\mathbf{R}) \xrightarrow{P_j} V_j$, then we see that

$$\begin{aligned} \bigcup_{j\in\mathbf{Z}} V_j &= L^2(\mathbf{R}), \\ \bigcap_{j\in\mathbf{Z}} V_j &= 0, \end{aligned}$$

in the sense that

$$\begin{aligned} \lim_{j\to\infty} P_j f &= f, \\ \lim_{j\to-\infty} P_j f &= 0. \end{aligned}$$

The Mallat identity asserts that for a fixed *level* J (corresponding to a scale width of 2^{-J}),

$$V_J = V_0 \oplus W_0 \oplus \cdots \oplus W_{J-1}, \tag{2.9}$$

where W_j is the orthogonal complement of V_j in V_{j-1}, and one has that W_j are spanned by ψ_k^j. The formulas relating these two representations are given by ([22], see also [40]): if we assume an expansion of the form

$$f^j(x) = \sum_{k \in \mathbf{Z}} f_k^{j,0} \varphi_k^j(x) \in V_j, \tag{2.10}$$

(where $f_*^{*,0}$ denotes coefficients of φ, and f_*^{*1} denotes coefficients of ψ),

$$f^j(x) = \sum_{k \in \mathbf{Z}} f_k^{0,0} \varphi_k(x) + \sum_{i=0}^{j-1} \sum_{k \in \mathbf{Z}} f_k^{i,1} \psi_{ik}(x) \in V_0 \oplus W_0 \oplus \cdots + W_{j-1}, \tag{2.11}$$

where

$$\begin{aligned} f_k^{i,0} &= \frac{1}{\sqrt{2}} \sum_l f_l^{i+1,0} a_{l-2k}, \quad 0 \le i \le j, \\ f_k^{i,1} &= \frac{1}{\sqrt{2}} \sum_l f_l^{i+1,0} b_{l-2k}, \quad 0 \le i \le j. \end{aligned} \tag{2.12}$$

Conversely, if we know the coefficients of the expansion (2.11), then the coefficients in (2.10) are determined recursively by

$$f_k^{i,0} = \frac{1}{\sqrt{2}} \sum_l \left(f_l^{i-1,0} a_{k-2l} + f_L^{i-1,1} b_{k-2l} \right), \quad 0 \le i \le j. \tag{2.13}$$

Tensor products of the L^2 basis $\varphi_n(x), \psi_k^j(x)$ are used in the 2-dimensional setting, and tensor products of these formulas are used as well.

We went through this in some detail to illustrate the nature of the wavelet systems being used in other parts of this paper. They are very effective bases for representing data as well as potential solutions to specific problems in a wide variety of areas as we will see in the next few sections. Equation (2.9) shows that data at a single scale (represented by the index J) can be decomposed into a direct sum of data at coarser scales ($J-1, \ldots, 0$), where each decrease in index corresponds to a coarsening of the data by a factor of 2. The formulas of Mallat (2.12) and (2.13) allow one to move easily from one scale to another, and this is important in a number of applications.

3 COMPUTATIONAL FLUID DYNAMICS

The fundamental equations of fluid dynamics are the classical Navier–Stokes equations. These equations have a number of special cases which serve as model problems for new computational techniques and these include the Euler equation, the Stokes equations, and the linear Laplace's equation. For backgound on these topics we refer to the monograph by Chorin and Marsden [6]. Wavelets can be used at a single scale as a variant of the more classical finite-element method, and the work done by Weiss and some of his coworkers indicates its usefulness for representing turbulent flow for Euler flow [35], [36], [26]. This was preceded by earlier work on Burgers' equation (see, e.g., [19]). The overlapping nature of the wavelet representation as well as the longer-length wavelet-type filters which enter into these papers in a computational manner seem to help control the stability of the shock waves and vortex flows which arise in the Burgers and Euler flow settings.

In another direction, there have been both single-scale and multiscale algorithms for solving elliptic boundary-value problems using wavelets a the fundamental representational tool. In [38] one finds a review of single-scale developments in this direction, and the review article [11] emphasizes the role of wavelets in computational fluid dynamics.

More recently, there have been some developments of algorithms which utilize the multiscale nature of wavelets. In this direction we note the paper by Rieder [28], which develops multigrid algorithms for boundary value problems in the one-dimensional case. The multiscale stucture of wavelets uses wavelet filters, convolution, and either upsampling or downsampling to map data from one scale to another these are the Mallat formulas (2.12) and (2.13)). In multigrid theory these processes of mapping from one scale to another are called restriction and prolongation operators. The wavelet represtation gives a convenient representation of these operators including the properties of being "canonical," which means that certain natural compatibility conditions are satisfied.

For higher-dimensional problems several new results have been developed quite recently. These include:

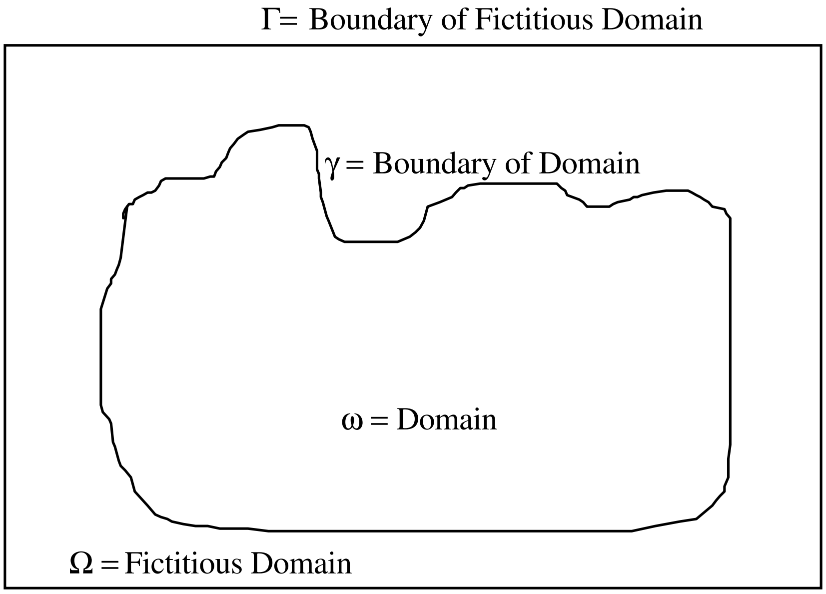

Figure 3.1: The general domain ω is embedded in a larger domain with simpler boundary Ω for the fictitious domain approach to solving boundary value problems.

- Multigrid algorithms for anisotropic model elliptic problems where the resulting multigrid iteration matrix has a spectral radius bounded by a constant smaller than one which is independent of the mesh size and of the anisotropy parameter [29].
- A multigrid preconditioner for a single-scale solver for a model elliptic boundary-value problem with a very general boundary [12].

Let us comment on these two developments here. In the anisotropic model problem, we considered the differential equation

$$-\varepsilon\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+u=f \tag{3.14}$$

where the unknown function u is required to belong to a suitable Sobolev space of functions (H^1) defined in a rectangular domain Ω in $\mathbf{R}^2$ and which is periodic on the boundary of $\partial\Omega$. The usual multigrid methods degenerate as ε goes to zero for a problem of this sort, and special techniques are needed. Hackbusch introduced in [16] a method of decomposing residual errors in the multigrid algorithm into high and low freqency parts depending on the spatial direction. Wavelet packets are a generalization of the original wavelet decomposition of Mallat and Daubechies as described in Section 2 (see [27] for a discussion of wavelet packets). One is able to refine the ideas of Hackbusch in the paper cited above and obtain a more precise result [29] where the orthogonality of the wavelet packets improves the representation of Hackbusch in which such orthogonality was not present. The references in [29] give more background for the general area of mulitgrid analysis. Let it suffice for us to say here that if one has an iteration scheme at a single level (fixed mesh size) for solving some induced system of linear equations (or nonlinear equations as well), then for the usual methods (Jacobi iterations, etc.) the spectral radius of the iteration matrix approaches 1 as the mesh size gets small, and consequently the iteration process slows down enormously for small mesh size. Multigrid methods, when they work, provide an alternative iteration procedure, utilizing information at multiple scales in a systematic fashion, and which has a spectral radius bounded by a constant smaller than 1, independent of the mesh size (this is a *robust* procedure).

The second problem mentioned above concerns an attempt to fashion a multigrid algorithm for an elliptic boundary-value problem with a very general boundary. The papers [40], [39], [13] give a systematic development of

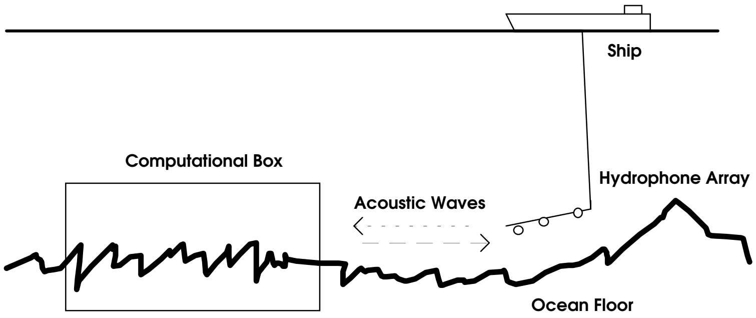

Figure 4.2: Low-angle acoustic scattering near the ocean floor and the computational box for simulation calculations.

single-scale wavelet solutions for model elliptic boundary-value problems for very general boundaries (including both Dirichlet and Neumann boundary conditions). One key to this analysis is the representation of both the boundary and the boundary data in terms of wavelets, and, in addition, to embed the domain with a general boundary into a larger domain with a simpler geometry (rectangular) on which one can employ fast periodic solvers. This is pictured in Figure 3.1. The methodology of penalty methods [40],[13], as well as Lagrange multipliers have also been used in this context (see [26] for an example of such a computation in the context of Euler flow).

4 ACOUSTIC MODELING OF THE OCEAN FLOOR

The modeling of the ocean floor has a long and varied history. A recent development in this direction is the viscoelastic type of model developed in [32,31]. In this model the ocean basement is modeled by a fractal boundary [14], and a finite-difference scheme is used for the solution of the viscoelastic equations in a region with absorbing boundary conditions simulating long-range asymptotic conditions in the ocean. Fractals were used to model the inherent multiscale structure of the ocean floor; and wavelets, by their very nature, have the same capacity for representing such a multiscale structure. They have also assumed a two-dimensional model exploiting symmetries for ocean bottoms near the mid-Atlantic ridge where some of their experimental data has come from. The experimental apparatus being modeled includes a ship with hydrophones lowered near the bottom of the ocean and sending (and receiving) sonar signals at a shallow angle towards the ocean floor, and receiving them with the same devices. This is pictured in Figure 4.2.

The simulation of the scattering experiments has been dicussed for various cases using high-order finite-difference schemes in [4,5,32,31]. A natural alternative is to use wavelet-based approximations (using a wavelet-Galerkin scheme, exploiting the orthogonality of the wavelet basis). It would yield the following advantages in principle:

- The orthogonality of the wavelets will allow the implementation of an explicit time-stepping in an efficient manner (see [30]).
- The scalable structure of the wavelet basis allows for easy implementation of multigrid to speed up the

computations; this uses the multiscale structure of wavelets (see [12,29]).

- High-order Daubechies wavelets induce implicitly high-order finite-difference operators in an automatic fashion in the wavelet-Galerkin setting (see [2,23]).
- Wavelets can be used to give a multiscale representation of the geometry of the problem, including the fractal interface.
- Wavelets could help in analyzing the scattered signal by giving a multiresolution of the scattered wave, which would help in the pattern recognition problem inherent in scattering problems.
- There are well-understood parametrizations of choices of wavelets which can allow for optimization of the choice of wavelets for a specific aspect of the problem. In particular, in [30], one finds an optimization of the wavelet-Galerkin coding used in inverse scattering problems which has a far superior efficiency compared to earlier codes optimizing over all finite-difference type schemes (which leads in principle to unconditioned stiffness matrices).

In the papers cited above a hybrid of fractal geometry, finite-difference calculations and Fourier transforms of the scattered signal were used for the simulation, and wavelet analysis could be involved in all parts of the problem and forms a common language for the data, as well as giving parallelizable code which could in conjunction with multigrid methods greatly speed up the computation of the solution. A very interesting first step for a special case of a wavelet solution of the general viscoelastic problem described above is presented in a recent preprint [30]. This paper indicates that the variable choice of wavelet basis can allow one to optimize to find a best choice of wavelets for a specific problem, and in the cases considered, the resulting wavelet solution was far superior to standard finite-difference techniques.

5 MODELING SYNTHETIC APERATURE RADAR (SAR) IMAGES

5.1 Synthetic Aperture Radar

SAR imaging captures the reflectivity of the surface being imaged. A synthetic aperture antenna is created by creating the illusion of a large number of physically small antennas spread spatially. This spatial spreading is obtained by using the same antenna that is airborne. As the aircraft moves, the antenna has moved spatially. The transmission and reception are done by the same antenna.

The raw data consists of radar echoes (a scalar function of time) for each position of the antenna. This data is obtained by correlating the transmitted signal with the received signal. The transmited signal is usually an FM chirp that is modulated on a microwave frequency carrier. The received signal is demodulated before correlation.

A SAR image is generated in the following manner. For each point P on the ground and for each radar echo $E(t)$, the effective position (time instant t_0) such that $E(t_0)$ is a measure of the reflection from P is obtained. One integrates this information over all echoes to get a measure of the reflectivity of P. This process is repeated for all P on the ground to generate a SAR image.

5.2 Modeling SAR Signals by Solving Boundary-Value Problems

In the filter processing of a SAR signal a fundamental assumption that is normally made in practice is that the received signal looks exactly like the transmitted signal, but with an amplitude gain (due to reflectivity variation), and a time-delay depending on the distance between the position on the earth and the antenna. This model works satisfactorily and is the basis for most SAR analysis today.

However the limits of accuracy of the reflectivity and range information is limited by how much the received signal looks like the transmitted signal. The transmitted signal is converted by the antenna system to electromagnetic waves which are reflected by the target depending on its boundary conditions. An interesting problem is to study (as outlined below) the propagation of this electromagnetic field (EM field) to the extent that depending on the target and the input electromagnetic field one can predict the nature of the EM field and hence the reflected SAR signal at the receiving antenna. Assuming an appropriate model for the transduction of the EM field to the received scalar signal transduction, one gets a model for the received signal with respect to which one should correlate the true received signal. This would potentially give better resolution SAR images.

5.3 Electromagnetic Wave Propagation Boundary-Value Problems

There has been a lot of progress in solving complex boundary-value problems in the context of fluid mechanics (turbulent flow), EM wave propagation and acoustic wave propogation. In Sections 3 and 4 above we gave a brief report on progress in the area of using wavelets as a tool for computational fluid mechanics and acoustic waves. Moreover, it has become clear that the tools of computational fluid mechanics in particular have serious applications to EM wave propagation problems (see for instance [33], where an example of this is carried out quite explicitly). In the paper [10], we find specific examples of such fluid flow type calculations applied to wave propagation problems.

The goal of such a study is to obtain a detailed description of the reflected signal according to the nature of the geometry and physical characteristics of the illuminated area (radar footprint). In principle, this will require solving *Maxwells's equations* with appropriate boundary conditions and convenient absorbing boundary conditions to reduce the size of the computational domain.

Recently there have been solutions developed [10] for two-dimensional scattering problems of large wave number (i.e., $D/\lambda \gg 1$, where D is a characteristic length, the diameter of the reflector, for example, and λ is the wavelength of the carrier EM wave). The new methodology combines variational approximation and least squares based iterative methods. It is very well suited to domain decomposition and fictitious domain base solutions methods. This implies in turn that the finite-element methods which have been employed in [10] (see also [24]) can be improved by a wavelet-Galerkin adapation in the spirit of the research described in Section 3. A review of direct time integration methods for Maxwell's equations is given in [33]. The two-dimensional calcuation carried out in [10] is quite demanding (280,000 triangles, 140,000 grid points). Three-dimensional calculations will definitely require the use of massively parallel machines, since practical problems may include 10^6 to 10^9 grid points.

The use of wavelets for solving such problems will bring the following enhancements for such a large calculation:

- The natural multigrid speed-up using the multiscale structure of wavelets.
- The natural parallelizability of the wavelet coding and its simple adapation to parallel machines, as already demonstrated on simpler problems.

Efforts are underway by various teams of researchers to carry out different aspects of the research program outlined in this section. Specific developments which are to be carried out in the course of the research and development are the use of an adaptive mesh technique (using the multiscale representation) to concentrate the calculations on the regions of the solution domain where the solution has sharper gradients, and to minimize calculations where unnecessary. This is similar to the process of image compression, a successful wavelet technique, where the amount of information necessary to adequately *represent* an image is dynamically varied from image to image, keeping only the essential information for a subjective interpretation of the image (cutting down on the amount of computational nodes if this is incorporated into a computational scheme as planned).

6 ILLUMINATION AND RADIOSITY IN COMPUTER GRAPHICS

The final topic in this paper is that of illumination and radiosity. There has been a number of contributions to this field over the past few years, and most recently it has been pointed out that wavelets (and in particular the work of the wavelet group at Yale [3],[1]) can play a significant role in reducing the computational time in the simulation of such images. Some models of reflection from surfaces use electromagnetic reflection models in the spirit of our discussion of SAR images in the preceding section [7],[17],[34], and these have their deficiencies for rough surfaces as pointed out by experts in this area.

A specific type of modeling global illumination is that of radiosity, which is based on the principle of conservation of energy. In particular, all energy from light which is emitted in an enclosure is tracked in some manner as it reflects off the various surfaces in the environment, until it dissipates into heat. The paper [41] gives a brief review of this model and develops higher-order finite-element solutions of the fundamental integral equation which has the form:

$$B(s,t) = E(s,t) + \int\int K(s,t,u,v)B(u,v)dudv, \tag{6.15}$$

where B is a vector-valued function of two variables, where each component of the vector represents a particular surface element, and where the independent variables represent a parametrization of the surface component. All of the key radiosity variables (radiosity, emittance, reflectivty, etc.) are represented as functions of two parameters of the surface elements. The kernel K is a matrix-valued function of the same parameters which includes specific information about the differential geometry, reflectivity and visibility of the surface elements. The solution of the equation combined with ray-tracing techniques will allow a computer graphics rendering of the resulting illuminated environment. The basic idea of a finite-element solution of such an integral equation is to represent all elements of the problem, the kernel and the unknown function B in terms of a finite basis of a particularly simple type (the finite elements), and in such a manner that the inner products (L^2 type inner products) of the finite elements are computable and the integral equation reduces to a linear equation for the unknown coefficients of the expansion in terms of the given basis. The usual finite-element bases consist of elementary spline functions of low order (piecewise linear, piecewise quadratic, etc.)

In a recent paper [15] it has been shown how the wavelet solution of an integral equation of the type (6.15) can be carried out in a manner similar to the finite-element approach described, but using wavelets of specific coefficient length (corresponding to a certain degree of smoothness). The hierarchical structure reduces the problem from computing $O(n^2)$ interaction coefficients for n linear equations to approximately $O(n)$ interaction coefficients, which greatly speeds up the process. We leave further discussion of these issues to the literature cited here.

7 CONCLUSION

We have discussed four rather distinct problems involving modeling the interaction of geometry and the motion of fluids or waves: fluid mechanics, ocean floor modeling, SAR images, and illumination of surfaces. For all but one of the examples it has been shown in the papers cited here that the use of wavelets has led to a significant improvement in the modeling process; and in the one remaining case, there is a research effort under way to see if this is also the case (the case of SAR images).

Efforts at Aware, Inc. in Cambridge, Yale, and Washington University (among others) are underway on the use of wavelets for representing and enhancing images in medicine (X-ray images, MRI, and tomography). We could have included this in our discussion, but we concentrated in this paper on those types of modeling which required the solutions of differential or integral equations and in which the use of wavelets enhanced the solution process as well as the resulting images. The use of wavelets for compression and image enhancement has become quite well established, and its role in solving equations is, in comparision, more embryonic.

For solving partial differential equations the well established methods of finite-difference codes, and the major developments in finite-element techniques over the past two decades as well as the classical spectral theory have proven to be powerful tools. It seems clear to this author that the role of wavelets for specific types of problems in differential equations and geometric modeling is at the beginning and that there will be significant developments in the next decade as more scientists and applied mathematicians begin to exploit the powerful multiscale representation with its orthogonal localized basis provided by wavelet theory. In particular, the wavelet formalism includes both aspects of high-order finite-difference techniques (see, e.g., [23]) as well as important aspects of finite-element techniques but adding important orthogonality properties (which itself is a reflection of the classical spectral approach). In some sense the wavelet approach to differential equations is a compromise blend of important aspects of all of three of the classical methods and includes its built in hierarchical structure which is not a natural part of the classical methods.

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