On the Wavelet Frequency Decomposition Method

Andreas Rieder Raymond O. Wells, Jr. Xiaodong Zhou

Computational Mathematics Laboratory
Mathematics Department, Rice University
Houston, TX 77251

Abstract

A wavelet variation of the Frequency decomposition multigrid method of Hackbusch is presented. The perfect reconstruction property of the wavelet system enable us to perform the convergence analysis of the frequency decomposition method. Some applications of this method are also presented.

1 Introduction

The main purpose if this paper is to describe a wavelet variation of the frequency decomposition method of Hackbusch [4]. Our method is applied to an anisotropic elliptic model problem with periodic boundary conditions. We are able to achieve robustness of the resulting two-level method with respect to any intermediate level in the corresponding recursive multilevel process. This is an improvement over the results given in [4] where only the robustness of the two-level method of the starting level is demonstrated. The main tool we use is perfect reconstruction property associated with the wavelet system.

In Section 2, we describe the main aspects of the Wavelet-Galerkin spaces, the wavelet packets and an frequency decomposition multilevel algorithm. This includes the basic theoretical robustness estimate, which includes, in particular, strong Cauch estimate for the L^2 product of derivatives of wavelet packets. In Section 3 we consider another application of this method to a problem in which the solution has large gradients in the interior of the domain, which seems to be quite effective.

2 Anisotropic Equations

In this section, we present a wavelet variation of the "*Frequency decomposition multigrid method*" of Hackbusch [4] to solve the following model problem:

$$-\epsilon \frac{\partial^2}{\partial x^2} u(x,y) - \frac{\partial^2}{\partial y^2} u(x,y) + u(x,y) = f(x,y) \quad \text{in} \quad \Omega = (0,s)^2 , \tag{1}$$

$$u \quad \text{periodic} , \tag{2}$$

where we assume $0 < \epsilon \leq 1$, and s is a positive integer.

The convergence rate of the standard wavelet multilevel method deteriorates as ϵ approaches zero, (see Figure 1). We are seeking a *robust* multilevel method in the sense that the convergence rate is not only independent of the discretization step size, but also of the anisotropic parameter ϵ.

There are essentially two ways to obtain robust multigrid methods. One way is to choose a sophisticated *basic iterative method* (BIM) as done e.g. in [12] and [14]. However, the performance of such improved BIM (incomplete LU decomposition [14], block versions of the Gauß-Seidel relaxation [12]) can be limited, see Chapter 10 in [4], especially for 3D applications.

A modification of the coarse grid correction opens another way towards robustness, [4] and [9]. It allows one to use simple pointwise smoothers which in general can be parallelized. Also, it can be extended to the 3D case in a relatively straightforward manner.

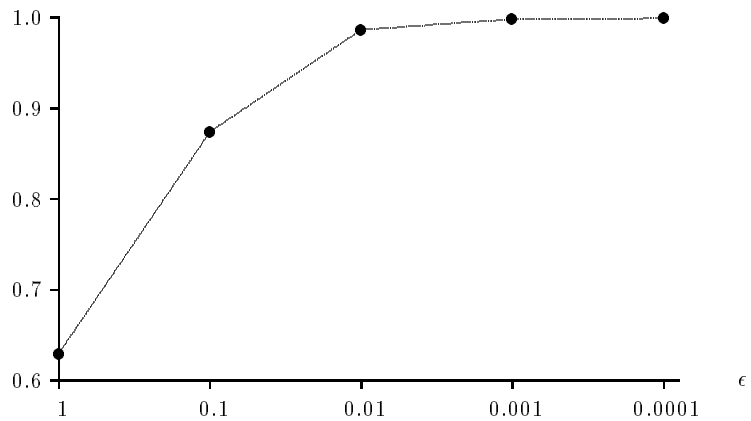

Figure 1: Convergence rate of the wavelet multigrid method for several ϵ

Let ψ^0 be the Daubechies scaling function of order N [2].Then we can define the wavelet system in the following way: for *binary number* κ and $x \in (-\infty, \infty)$,

$$\psi^{\kappa 0}(x) := \sum_{n=0}^{2N-1} a_n \, \psi^{\kappa}(2x-n),$$

$$\psi^{\kappa 1}(x) := \sum_{n=0}^{2N-1} b_n \, \psi^{\kappa}(2x-n).$$

Then we introduce the wavelet-Galerkin spaces at level l by

$$V_{p,l}^{\kappa} := \{v \in L^2(0,s) \quad : \quad v(x) = \sum_{k \in \mathbf{Z}} c_k \psi_{l,k}^{\kappa}(x), \; x \in [0,s], \\ \text{and } c_k = c_{k+2^l s}\}. \tag{3}$$

Then we have

$$\begin{aligned} V_{p,l}^0 &= V_{p,l-1}^0 \oplus V_{p,l-1}^1 \\ &= V_{p,l-2}^0 \oplus V_{p,l-2}^1 \oplus V_{p,l-2}^{10} \oplus V_{p,l-2}^{11} \\ &= \cdots\cdots\cdots \end{aligned}$$

The Wavelet-Galerkin approximation $u_l \in V_l^0 \otimes V_l^0$ to $u \in H_p^1\Big((0,s)^2\Big)$ is the unique solution of

$$\mathcal{A}(u_l, v_l) = \int_{\Omega} f v_l \, dxdy, \qquad \text{for all } v_l \in V_l^0 \otimes V_l^0. \tag{4}$$

where $\mathcal{A}$ is the elliptic bilinear form

$$\mathcal{A}(u,v) = \int_{\Omega} (\epsilon \, u_x v_x + u_y v_y + uv) \, dxdy. \tag{5}$$

A convergence proof of $u_l \to u$ is given in [13].

By applying the Galerkin method to (4), one derives the following linear system

$$A_l^0 U_l = F_l, \tag{6}$$

where A_l^0 is the stiffness matrix of $\mathcal{A}$ in $V_l^0 \otimes V_l^0$. Note that the matrix A_l^0 (as well as the stiffness matrix A_l^{κ} of $\mathcal{A}$ in $V_l^{\kappa} \otimes V_l^{\kappa}$) is determined by $2N-1$ connection coefficients [1], [7]

$$\Gamma_k := \int \psi^{\kappa}(x)' \psi^{\kappa}(x-k)' \, dx, \quad k = 0, \ldots, 2N-1.$$

To derive a robust multilevel method of solving (6), we make the following observation. Let $U^{\kappa}_{l,\nu}$ be an approximate solution at level l in the space $V^{\kappa}_l \otimes V^{\kappa}_l$ by applying some basic iterative methods (BIM) ν times. In general, the BIM only reduces error components well in the direction of eigenvectors corresponding to large eigenvalues. In the presence of strong anisotropy, that is, ϵ small compared to 1, this means the error may contain not only a part which is low frequency in both the x and y-directions, but also a part which is high frequency in x-direction. Consequently, we approximate the error in both spaces $V^{\kappa 0}_{l-1} \otimes V^{\kappa 0}_{l-1}$ and $V^{\kappa 0}_{l-1} \otimes V^{\kappa 1}_{l-1}$ instead only in $V^{\kappa}_{l-1} \otimes V^{\kappa}_{l-1}$ as in the standard multigrid procedure. In the abstract setting of [15] and [16], the resulting coarse grid correction for the two-level method

$$U^{\kappa}_{l,\nu} \leftarrow U^{\kappa}_{l,\nu} + \Big[(h^t \otimes h^t)(A^{\kappa 0}_{l-1})^{-1}(h \otimes h) + (g^t \otimes h^t)(A^{\kappa 1}_{l-1})^{-1}(g \otimes h)\Big](F_l - A^{\kappa}_l U^{\kappa}_{l,\nu}) \tag{7}$$

may be viewed as an *additive subspace correction* with respect to the orthogonal subspaces $V^{\kappa 0}_{l-1} \otimes V^{\kappa 0}_{l-1}$ and $V^{\kappa 0}_{l-1} \otimes V^{\kappa 1}_{l-1}$. Here h and g are the periodic *Mallat Transformations* [8].

Theorem 2.1 *Let S_l be the iteration matrix of some BIM, then the iteration matrix, $M^{\kappa}_l(\nu)$, of the two-level method with ν presmoothing steps, can be written as*

$$\begin{aligned} M^{\kappa}_l(\nu) \;=\; & \Big[I - (h^t \otimes h^t)(A^{\kappa 0}_{l-1})^{-1}(h \otimes h)A_l \\ & - (g^t \otimes h^t)(A^{\kappa 1}_{l-1})^{-1}(g \otimes h)A^{\kappa}_l\Big] S^{\nu}_l, \end{aligned} \tag{8}$$

whose energy norm satisfies

$$||M^{\kappa}_l(\nu)||_{A^{\kappa}_l} \le C\eta(\nu) + r_{\kappa}, \tag{9}$$

where C is a positive constants independent of the discretization step size δ_l, the anisotropic parameter ϵ and κ. Moreover, $\sup_{\kappa} r_{\kappa} < 1$ and $\eta(\nu) \to 0$, as $\nu \to \infty$.

Remark: The constant r_{κ} in the above theorem is, in fact, the best constant for which the following strong Cauchy inequality holds,

$$\Big|\int v'(x)\, w'(x)\, dx\Big|^2 \le r_{\kappa} \Big(\int |v'(x)|^2\, dx\Big)\Big(\int |w'(x)|^2\, dx\Big) \tag{10}$$

for any $v \in V^{\kappa 0}_l$ and $w \in V^{\kappa 1}_l$.

For the multilevel convergence analysis of this wavelet frequency decomposition method, the V-cycle fails to be convergent with the usual technique [6]. Indeed, the numerical experiements seem to indicate this fact. With some quantitative estimates of the r_{κ}'s, one might be able to prove the W-cycle convergence, see [10] for detailed discussions. In [11], we are able to prove level-dependent convergence of a *damped* V-cycle, that is, the convergence rate is independent of the discretization step size and ϵ, it depends only the number of levels. More specifically, by a *damped* two-level iteration with damping parameter β, we mean that the corresponding iteration matrix is of the following form:

$$\begin{aligned} M^{\kappa}_l(\nu) \;=\; & \Big[I - \beta\,(h^t \otimes h^t)(A^{\kappa 0}_{l-1})^{-1}(h \otimes h)A_l \\ & - \beta\,(g^t \otimes h^t)(A^{\kappa 1}_{l-1})^{-1}(g \otimes h)A^{\kappa}_l\Big] S^{\nu}_l. \end{aligned} \tag{11}$$

Then we have

Theorem 2.2 *Let $M_{L,0}(\nu,\nu)$ be the iteration matrix of the damped V-cycle with damping factor $\beta = (1+\sqrt{\sigma})^{-1}$, where $\sigma = \sup_{\kappa} r_{\kappa} < 1$. L is the number of levels. Then the spectral radius $\rho(M_{L,0}(\nu,\nu))$ satisfies*

$$\rho(M_{L,0}(\nu,\nu)) \le 1 - \Big(\frac{1-\sqrt{\sigma}}{1+\sqrt{\sigma}}\Big)^L,$$

for $\nu \ge \bar{\nu}$, where $\bar{\nu}$ is a positive integer independent of ϵ and discretization step size.

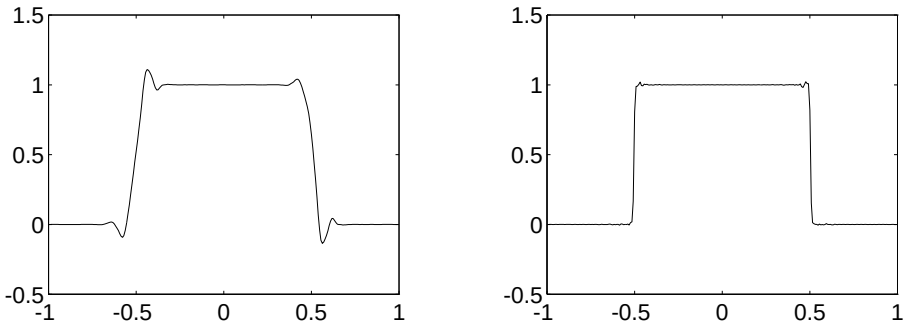

Figure 2: Left: 3 V-cycles with low pass filter only. Right: 3 V-cycles with both low and high pass filters. $\epsilon = 10^{-5}$.

3 Large Gradient Problems

We use the following simple one dimensional model problem to illustrate another application of the wavelet frequency decomposition method for treating some problems whose solution posess large gradients in the interior of the domain. Consider for a small positive number ϵ,

$$-\epsilon\, u'' + u = \chi_{(-0.5,0.5)}, \quad \text{in } (-1,1), \tag{12}$$

$$u \text{ is periodic,}$$

where $\chi_{(-0.5,0.5)}$ is the characteristic function of the interval $(-0.5, 0.5)$.

By applying the standard wavelet based multilevel method [3], we see from the left picture in Figure 2 that near the boundary $x = \pm 0.5$, the approximate solution has large oscillations. It is simply because that the solution has large gradients near the boundary, in the standard multilevel process, only the smooth part of the error is corrected at the coarser level. But if we replace the BIM by an extra coarser grid correction as in [5], we are able to correct the high frequency part of the error at coarser level, as well. One can see the improvements from the right side picture in Figure 2.

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5 REFERENCES

[1] G. Beylkin. On the representation of operators in bases of compactly supported wavelets. *SIAM J. Numerical Analysis.* Vol. 6, No. 6, pp. 1716-1740, 1992.

[2] I. Daubechies. Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, 41:906–966, 1988.

[3] R. Glowinski, A. Rieder, R. O. Wells, Jr., and X. Zhou. A wavelet multilevel preconditioner for solving dirichlet boundary value problems in general domains. Technical Report 93-06, Rice University, 1993. Computational Mathematics Laboratory.

[4] W. Hackbusch. The frequency decomposition multi-grid method, part i: Application to anisotropic equations. *Numer. Math.* 56, pp. 229-245, 1989.

[5] W. Hackbusch. The frequency decomposition multi-grid method, part ii: Convergence analysis based on the additive schwarz method. *Numer. Math.* 63, pp. 433-453, 1992.

[6] W. Hackbusch. *Multi-Grid Methods and Applications.* Springer Series in Computational Mathematics. Springer-Verlag, New York, 1985.

[7] A. Latto, H. L. Resnikoff, and E. Tenenbaum. The evaluation of connection coefficients of compactly supported wavelets. In *Proceedings of the USA-French Workshop on Wavelets and Turbulence.* Princeton University, 1991.

[8] S. Mallat. Multiresolution approximation and wavelet orthonormal bases of $l^2(\mathbf{r})$. *Trans. Amer. Math. Soc.*, 315:69–87, 1989.

[9] N. H. Naik and J. V. Rosendale. The improved robustness of multigrid elliptic solvers based on multiple semicoarsened grids. *SIAM J. Numer. Anal.* Vol. 30, No. 1, pp. 215-229, 1993.

[10] A. Rieder, R. O. Wells, Jr., and X. Zhou. A wavelet approach to robust multilevel solvers for anisotropic elliptic problems. Technical Report 93-07, Rice University, 1993. Computational Mathematics Laboratory.

[11] A. Rieder and X. Zhou. On the damped v-cycle convergence of the wavelet frequency decomposition method. Computational Mathematics Laboratory TR93-10, 1993.

[12] K. Stüben and U. Trottenberg. Multigrid methods: fundamental algorithms, model problem analysis and applications. In W. Hackbusch and U. Trottenberg, editors, *Multigrid Methods*, New York, pp. 1-176, 1982. Lecture Notes 960, Springer-Verlag.

[13] R. O. Wells, Jr. and X. Zhou. Wavelet interpolation and approximate solutions of elliptic partial differential equations. Technical Report 92-03, Rice University, 1992. Computational Mathematics Laboratory.

[14] G. Wittum. On the robustness of ilu smoothing. *SIAM J. Sci. Stat. Comput.* Vol. 10, No. 4, pp. 699-717, 1989.

[15] J. Xu. Iterative methods by space decomposition and subspace correction. *SIAM Review.* 34, 1992.

[16] H. Yserentant. Old and new convergence proofs for multigrid methods. *Acta Numerica.* Cambridge University Press, New York, pp. 285-326, 1993.