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A Preconditioned CG–Method for Wavelet–Galerkin Discretizations of Elliptic Problems

We present an efficient solver for a class of linear systems arising by a wavelet–Galerkin discretization of the following elliptic model problem

$$-\alpha\,\Delta u + u = f \quad \text{in} \quad \omega \subset \mathbb{R}^2,$$
$$u = g \quad \text{on} \quad \partial\omega,$$

via a penalty/fictitious domain formulation. The constant α is positive and ω is an open domain in $\mathbb{R}^2$.

1. Penalty/fictitious domain formulation

The *penalty/fictitious domain formulation* of the model problem becomes: let $\Omega = [0,R]^2$, $\omega \subset \Omega$, be the *fictitious domain* and $\epsilon > 0$ the *penalty parameter*. Then, we seek a $u^\epsilon \in H^1_p(\Omega)$, the Sobolev space of order 1 with periodic boundary conditions on Ω, such that

$$\int_\Omega (\alpha \nabla u^\epsilon \cdot \nabla v + u^\epsilon v) dxdy + \frac{1}{\epsilon}\int_{\partial\omega} u^\epsilon v\, ds = \int_\Omega \widetilde{f} v\, dxdy + \frac{1}{\epsilon}\int_{\partial\omega} gv\, ds, \tag{1}$$

for all $v \in H^1_p(\Omega)$, where $\widetilde{f}$ is an arbitrary L^2-extension of f in Ω. The solution u^ϵ converges to u, the weak solution of the model problem, in $H^1(\omega)$ for $\epsilon \to 0$, see [2].

2. Wavelet–Galerkin discretization

Using the *Daubechies scaling function* φ of order $N \geq 3$, [1], we define the periodic wavelet–Galerkin space at level L by ($R \in \mathbb{N}$)

$$V^p_L = V^p_L(0,R) := \{v \in L^2(0,R) \ : \ v(x) = \sum_k c_k \varphi^L_k(x),\ x \in [0,R],\ \text{with } c_k = c_{k+2^L R}\}$$

with $\varphi^L_k(x) = 2^{L/2}\varphi(2^L x - k)$, and we approximate $H^1_p(\Omega)$ by the tensor product $X_L = V^p_L \otimes V^p_L$. The penalty problem (1) restricted to X_L can be represented by the linear system,

$$A_{L,\epsilon}\, u^\epsilon_L = f_L + \epsilon^{-1} M_L g_L, \tag{2}$$

see [3], with appropriate right hand sides f_L, $g_L \in \mathbb{R}^{n^2}$, $n = n_L = \dim V^p_L = 2^L R$. The stiffness matrix $A_{L,\epsilon}$ is of the form $A_{L,\epsilon} = A_L + \epsilon^{-1} M_L \in \mathbb{R}^{n^2 \times n^2}$, where A_L is symmetric and positive–definite. The diagonal matrix M_L representing the boundary integrals in (1) has diagonal entries either 1 or 0.

3. A preconditioned conjugate gradient method

Because the entries of A_L increase of order $O(h_L^{-2})$, where $h_L = 2^{-L}$ is the discretization step-size, the discrete penalty formulation (2) is meaningful only if ϵ is much smaller than h^2_L. This means that ϵ^{-1} has the most effection on the condition number of $A_{L,\epsilon}$. Therefore, we first reduce the influence of the penalty parameter ϵ. The family of solutions $\{u^\epsilon_L\}_{\epsilon>0}$ has a limit which is given by

$$\lim_{\epsilon \to 0} u^\epsilon_L = u_L + M_L g_L,$$

where u_L is the unique solution of

in the range $\mathrm{R}(I - M_L)$ of $(I - M_L)$, see [3].

Instead of the ill–conditioned system (2) we choose to solve (3). Since $(I - M_L)A_L(I - M_L)$ is symmetric and positive–definite on $\mathrm{R}(I - M_L)$ we can use the conjugate gradient method (*cg*-method) for the iterative solution of (3). A natural choice for a preconditioner of the *cg*-method acting on (3) is the symmetric multigrid method for the matrix A_L developed in [3].

4. Numerical experiments

We consider the model problem with $f \equiv 1$ and $g \equiv 0$ on the disk ω with center $(0, 0)$ and radius 0.25. The fictitious domain Ω is the square $[-0.5, 0.5]^2$.

We will use the following abbreviations: CG^ϵ denotes the *cg*-method applied to (2) with $\epsilon = 10^{-8}$, CG denotes the *cg*-method acting on (3) and PCG stands for the *cg*-method applied to the system (3) with the above mentioned preconditioner.

Table 1 shows the number of iteration steps needed by the three methods to yield an Euclidean norm (not weighted with the discretization step-size) of the residue smaller than 0.01. The discretization step-size is $h_8 = 1/256$.

Table 1. Necessary numbers of iterations to yield an Euclidean norm of the residue smaller than 0.01.

	$\alpha = 1$	$\alpha = 0.01$	$\alpha = 0.0001$
CG^ϵ	1538	1219	136
CG	781	608	68
PCG	79	99	17

At the end we compare CG and PCG for $\alpha = 1$ and varying discretization step-sizes $h_L = 2^{-L}$, $L = 6, 7, 8, 9$, see Table 2. To get a meaningful result we have to adapt the stopping criterion to the discretization step-size. We stop CG and PCG if the Euclidean norm of their residues is smaller than $r_L = 0.01 \cdot 2^{L-8}$.

Table 2. Necessary numbers of iterations to yield an Euclidean norm of the residue smaller than $r_L = 0.01 \cdot 2^{L-8}$, $L = 6, 7, 8, 9$.

	$h_6 = 1/64$	$h_7 = 1/128$	$h_8 = 1/256$	$h_9 = 1/512$
CG	168	371	781	1599
PCG	36	54	79	118

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5. References

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