Vanishing Moments and Wavelet Approximation

Jun Tian and Raymond O. Wells, Jr.

January 9, 1995

Abstract

We consider wavelet systems with vanishing moments equally distributed for the scaling functions and the wavelet functions (Coifman wavelet systems). It is shown that the convergence rate of the natural wavelet approximation will be exponentially fast for such wavelet systems as a function of the degree of the vanishing moments. In the biorthogonal case, a formula is given for all degrees, and it is shown that the order of regularity grows linearly with the degree of the vanishing moments.

Contents

1 Introduction

Wavelet analysis has been developed rapidly in signal processing, numerical analysis, and mathematical modeling in the last ten years since the terminology "wavelet" was introduced in 1984, by A. Grossmann and J. Morlet in [10]. In 1992, R. O. Wells, Jr. and X. Zhou gave the following wavelet approximation theorem in [22], which is true in $\mathbf{R^n}$ as well.

Theorem 1.1 *Let $\phi(x)$ be the scaling function of an orthonormal wavelet system with a finite length scaling vector $\{a_k, k \in \mathbf{Z}\}$, $a_k = 0$ when $|k| > K$ for some positive integer K,*

$$\phi(x) \ = \ \sum_{k\in\mathbf{Z}} a_k \phi(2x-k)\,.$$

We define a constant c by

$$c \ := \ \frac{1}{2}\sum_{k\in\mathbf{Z}} k a_k\,.$$

Assume the function $f(x,y) \in C^2\left(\bar{\Omega}\right)$, where Ω is a bounded open set in $\mathbf{R^2}$. Let, for $j \in \mathbf{N}$,

$$f^j(x,y) \ := \ \frac{1}{2^j}\sum_{(k,l)\in\Lambda} f\left(\frac{k+c}{2^j}, \frac{l+c}{2^j}\right)\phi_{j,k}(x)\phi_{j,l}(y)\,, \quad (x,y)\in\Omega\,,$$

where the index set $\Lambda = \{(k,l) \in \mathbf{Z^2} : (\mathrm{supp}(\phi_{j,k}(x)) \times \mathrm{supp}(\phi_{j,l}(y))) \cap \Omega \neq \emptyset\}$, and $\phi_{j,i}(x) = 2^{j/2}\phi(2^j x - i)$. Then

$$||f(x,y) - f^j(x,y)||_{L^2(\Omega)} \ \leq \ C(1/2^j)^2\,,$$

and if $\phi(x) \in C^1(\mathbf{R})$,

$$||f(x,y) - f^j(x,y)||_{H^1(\Omega)} \leq C/2^j\,,$$

where C is a constant depending only on K, the diameter of Ω, and the maximum modulus of the first and second order derivatives of $f(x,y)$ on $\bar{\Omega}$.

There are two possible ways to get faster approximation. One is to choose some special wavelet systems in the approximation procedure; the other one is to take the weighted sampling values which involve higher degree moments. The second point of view is discussed in [20]. In this paper, we are mainly interested in the first one. More precisely, we will look at the wavelet systems with half vanishing moments for the scaling function and half vanishing moments for the wavelet function, which was proposed by R. Coifman in 1989 (see [1] and [4]). In Section 2, Lemma 2.2 gives an equivalent condition of the vanishing moments property, where Theorems 3.1 and 3.2 describe the fast approximation properties of the orthogonal and biorthogonal Coifman wavelet systems. Section 4 discusses the construction of the orthogonal Coifman wavelet systems and lists the numerical computation results of the first 10 scaling vectors; an example of symbolic computation is also included. Section 5 concerns the biorthogonal Coifman wavelet systems; exact formulas of the synthesis scaling vectors and the analysis scaling vectors are given. Section 6 discusses comparisons among the orthogonal, biorthogonal Coifman wavelet systems and the Daubechies wavelet systems, and their symmetry and Sobolev smoothness.

2 Orthogonal Coifman wavelet systems

First, we give a brief overview of orthonormal wavelet systems and multiresolution analysis. For more details, we refer to [3], [15], [16], and [19]. An $L^2(\mathbf{R})$ function $\psi(x)$ induces an orthonormal wavelet system of $L^2(\mathbf{R})$ if the dilations and translations of $\psi(x)$ constitute an orthonormal basis of $L^2(\mathbf{R})$, i.e., $\{\psi_{j,k}(x) : j,k \in \mathbf{Z}\}$ is an orthonormal basis, where $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$. We call $\psi(x)$ the wavelet function. Moreover, if $\psi(x)$ is compactly supported, P. G. Lemarié showed in [14] that this orthonormal wavelet system is associated with a multiresolution analysis. A multiresolution analysis consists of a sequence of embedded closed subspaces $V_j \subset L^2(\mathbf{R}), j \in \mathbf{Z}$, satisfying $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R})$ and $\bigcap_{j\in\mathbf{Z}} V_j = \{0\}$. Moreover, there should be an $L^2(\mathbf{R})$ function $\phi(x)$ such that $\{\phi_{j,k}(x) : k \in \mathbf{Z}\}$ is an orthonormal basis for V_j for all $j \in \mathbf{Z}$, where $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$. Since $\phi(x) \in V_0 \subset V_1$, we have

$$\phi(x) \;=\; \sum_{k\in\mathbf{Z}} a_k \phi(2x-k)\,, \tag{1}$$

then $\psi(x)$ and $\phi(x)$ are related by

$$\psi(x) \;=\; \sum_{k\in\mathbf{Z}} (-1)^k a_{-k+1}\phi(2x-k)\,. \tag{2}$$

We call $\alpha := \{a_k : k \in \mathbf{Z}\}$ and $\phi(x)$ the scaling vector and the scaling function of the wavelet system, respectively. As one knows, it is straightforward to define an orthonormal wavelet system by (2) from a multiresolution analysis.

We are mainly interested in compactly supported wavelet systems since they are easy to implement and have great applications in practice. For this, let's assume the scaling vector α has finite length, $a_k = 0$, for $k < N_1$, or $k > N_2$, where $N_1, N_2 \in \mathbf{Z}$, and $N_1 < N_2$. And we will assume that α has finite length all through the rest of the paper. It then follows (see [5] and [6]) that $\phi(x)$ has compact support. By (2), $\psi(x)$ also has compact support. In an orthonormal wavelet system, the scaling vector α satisfies

$$\sum_{k=N_1}^{N_2} a_k \;=\; 2\,, \tag{3}$$

$$\sum_{k=N_1}^{N_2} a_k a_{k+2l} \;=\; 2\delta_{0,l}\,, \quad l \in \mathbf{Z}\,. \tag{4}$$

As usual, we define two discrete Fourier transforms

$$F_0(\xi) \;=\; \frac{1}{2}\sum_{k=N_1}^{N_2} a_k e^{ik\xi}\,, \quad F_1(\xi) \;=\; \frac{1}{2}\sum_{k=1-N_2}^{1-N_1} (-1)^k a_{-k+1} e^{ik\xi}\,,$$

and we use the Fourier transform

$$\widehat{f}(\xi) \;=\; \int_{\mathbf{R}} e^{ix\xi} f(x)\,dx\,.$$

From (1) and (2), the Fourier transforms of $\phi(x)$ and $\psi(x)$ satisfy the relations

$$\widehat{\phi}(\xi) \;=\; F_0(\xi/2)\widehat{\phi}(\xi/2)\,, \tag{5}$$

$$\widehat{\psi}(\xi) \;=\; F_1(\xi/2)\widehat{\phi}(\xi/2)\,. \tag{6}$$

For a nonnegative integer p, the p-th moments of the scaling function and the wavelet function are defined by

$$\mathrm{Mom}_p(\phi) \;:=\; \int_{\mathbf{R}} x^p \phi(x)\,dx\,, \tag{7}$$

$$\mathrm{Mom}_p(\psi) \;:=\; \int_{\mathbf{R}} x^p \psi(x)\, dx\,, \tag{8}$$

respectively when the integrals make sense. As we know, for an orthonormal wavelet system, $\mathrm{Mom}_0(\phi) = 1, \mathrm{Mom}_0(\psi) = 0$. It follows that (see [19])

$$\sum_{k\in\mathbf{Z}} \phi(x-k) \;=\; 1\,. \tag{9}$$

It is natural to ask, with more vanishing moments on $\phi(x)$ and $\psi(x)$, what will be the counterpart of (9)? But first, let's switch to the definition of an *orthogonal Coifman wavelet system* (I. Daubechies called these functions *Coiflets* in [4]).

Definition 2.1 *An orthonormal wavelet system with compact support is called an* orthogonal Coifman wavelet system *of degree N if the vanishing moments of $\phi(x)$ and $\psi(x)$ are both of degree N, i.e.,*

$$\mathrm{Mom}_0(\phi) \;=\; \int_{\mathbf{R}} \phi(x)\, dx \;=\; 1\,, \tag{10}$$

$$\mathrm{Mom}_p(\phi) \;=\; \int_{\mathbf{R}} x^p \phi(x)\, dx \;=\; 0\,, \quad \textit{for } p = 1, \cdots, N\,, \tag{11}$$

$$\mathrm{Mom}_p(\psi) \;=\; \int_{\mathbf{R}} x^p \psi(x)\, dx \;=\; 0\,, \quad \textit{for } p = 0, \cdots, N\,. \tag{12}$$

It is easy to see that (10), (11), and (12) are equivalent to the following conditions on the scaling vector α,

$$\sum_{k\in\mathbf{Z}} (2k)^p a_{2k} \;=\; \sum_{k\in\mathbf{Z}} (2k+1)^p a_{2k+1} \;=\; 0\,, \quad \text{for } p = 1, \cdots, N\,, \tag{13}$$

$$\sum_{k\in\mathbf{Z}} a_{2k} \;=\; \sum_{k\in\mathbf{Z}} a_{2k+1} \;=\; 1\,. \tag{14}$$

One of the nice property of the orthogonal Coifman wavelet system is that this wavelet system extends (9) to higher degrees.

Lemma 2.2 *For an orthogonal Coifman wavelet system of degree N,*

$$\sum_{k\in\mathbf{Z}} (x-k)^p \phi(x-k) \;=\; \delta_{0,p}\,, \quad \textit{for } p = 0, \cdots, N\,. \tag{15}$$

Proof: Define

$$f(x) \;=\; \sum_{k\in\mathbf{Z}} (x-k)^p \phi(x-k)\,.$$

Since $\phi(x)$ has compact support, $f(x)$ is well-defined and periodic with 1 as a period. We have

$$\begin{aligned}
2^p f(x) &= 2^p \sum_{k\in\mathbf{Z}} (x-k)^p \phi(x-k) \\
&= 2^p \sum_{k\in\mathbf{Z}} \sum_{m\in\mathbf{Z}} \left(a_m (x-k)^p \phi(2x-2k-m)\right) \\
&= 2^p \sum_{k\in\mathbf{Z}} \sum_{i\in\mathbf{Z}} \left(a_{i-2k} (x-k)^p \phi(2x-i)\right) \\
&= \sum_{i\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} \left(a_{i-2k} (2x-i+i-2k)^p \phi(2x-i)\right) \\
&= \sum_{i\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} \sum_{l=0}^{p} \left(a_{i-2k} \begin{pmatrix} p \\ l \end{pmatrix} (2x-i)^l (i-2k)^{p-l} \phi(2x-i)\right) \\
&= \sum_{i\in\mathbf{Z}} \sum_{l=0}^{p} \left(\begin{pmatrix} p \\ l \end{pmatrix} \left(\sum_{k\in\mathbf{Z}} (i-2k)^{p-l} a_{i-2k}\right) (2x-i)^l \phi(2x-i)\right)
\end{aligned}$$

Applying (13) and (14),

$$\sum_{k\in\mathbf{Z}} (i-2k)^{p-l} a_{i-2k} = \delta_{p,l}\,.$$

Thus

$$2^p f(x) \;=\; \sum_{i\in\mathbf{Z}} (2x-i)^p \phi(2x-i) \;=\; f(2x)\,.$$

For any $i, j \in \mathbf{Z}, j \geq 0$,

$$\begin{aligned}
\int_{\frac{i}{2^j}}^{\frac{i+1}{2^j}} f(x)\,dx &= \int_{\frac{i}{2^j}}^{\frac{i+1}{2^j}} 2^{-p} f(2x)\,dx \\
&= 2^{-p-1} \int_{\frac{i}{2^{(j-1)}}}^{\frac{i+1}{2^{(j-1)}}} f(x)\,dx \\
&= \cdots \\
&= 2^{(-p-1)j} \int_{i}^{i+1} f(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
&= 2^{(-p-1)j} \int_i^{i+1} \left(\sum_{k \in \mathbf{Z}} (x-k)^p \phi(x-k) \right) dx \\
&= 2^{(-p-1)j} \int_{\mathbf{R}} x^p \phi(x)\, dx \\
&= 2^{(-p-1)j} \delta_{0,p}\,.
\end{aligned}
$$

As we know, the restriction of the Haar wavelet system on [0,1] is an orthonormal basis of $L^2([0,1])$, and $\int_{i/2^j}^{(i+1)/2^j} f(x)\, dx$ is exactly the coefficients for the wavelet orthogonal projection at the level j. So it follows immediately that

$$
f(x) \;=\; \sum_{k \in \mathbf{Z}} (x-k)^p \phi(x-k) \;=\; \delta_{0,p}\,.
$$

□

Remark 1 The equality (15) holds in the L^2 sense, but not for every $x \in \mathbf{R}$. In the pointwise sense, it is true for almost all $x \in \mathbf{R}$. So if $\phi(x)$ is continuous, (15) is valid for every $x \in \mathbf{R}$.

If we know some regularity conditions about the scaling function $\phi(x)$, then the above result can be improved.

Corollary 2.3 *For an orthogonal Coifman wavelet system of order N, if the scaling function $\phi(x) \in C^n(\mathbf{R})$, where n is a nonnegative integer, then*

$$
\sum_{k \in \mathbf{Z}} (x-k)^p \phi^{(m)}(x-k) \;=\; (-1)^p (p!) \delta_{m,p}\,, \quad \text{for } p = 0, \cdots, N,\ 0 \le m \le n. \tag{16}
$$

Proof: We prove this by induction. Assume (16) holds for $0 \le m \le l$, where $l \le n-1$. We want to prove that when $m = l+1$, (16) is still true. Define

$$
f(x,p,m) \;=\; \sum_{k \in \mathbf{Z}} (x-k)^p \phi^{(m)}(x-k)\,. \tag{17}
$$

Again, since $\phi(x)$ has compact support, $f(x,p,m)$ is well-defined. And we can interchange the sum and the differentiation because for any point x, only finite terms in the right hand side sum of (17) count. So

$$
f(x,p,l+1) \;=\; \sum_{k \in \mathbf{Z}} (x-k)^p \phi^{(l+1)}(x-k)
$$

$$
\begin{aligned}
&= \sum_{k\in\mathbf{Z}} \left(\left((x-k)^p \phi^{(l)}(x-k) \right)' - p \cdot (x-k)^{(p-1)} \phi^{(l)}(x-k) \right) \\
&= \sum_{k\in\mathbf{Z}} \left((x-k)^p \phi^{(l)}(x-k) \right)' - \sum_{k\in\mathbf{Z}} p \cdot (x-k)^{(p-1)} \phi^{(l)}(x-k) \\
&= \left(\sum_{k\in\mathbf{Z}} (x-k)^p \phi^{(l)}(x-k) \right)' - p \sum_{k\in\mathbf{Z}} (x-k)^{(p-1)} \phi^{(l)}(x-k) \\
&= (f(x,p,l))' - p \cdot f(x,p-1,l) \\
&= ((-1)^p (p!) \delta_{l,p})' - p \cdot (-1)^{(p-1)} ((p-1)!) \delta_{l,p-1} \\
&= 0 + (-1)^p (p!) \delta_{l,p-1} \\
&= (-1)^p (p!) \delta_{l+1,p} \,.
\end{aligned}
$$

The corollary is proved. $\square$

3 Sampling, Reconstruction and Approximation

Given samples equally spaced in time of a continuous signal, is it possible to reconstruct the signal? Or how close can the original signal be approximated from the knowledge of the samples? In [22], R. O. Wells, Jr. and X. Zhou proved a wavelet approximation theorem which can be applied to any wavelet system. Imposing the vanishing moments on the wavelet system, we can improve the convergent rate for Coifman wavelet systems.

Theorem 3.1 *For an orthogonal Coifman wavelet system of degree N with the scaling function $\phi(x)$ and the scaling vector α, assume α has finite length. If $f(x) \in C_0^N(\mathbf{R})$, define, for $j \in \mathbf{N}$,*

$$
f^j(x) := 2^{-j/2} \sum_{k\in\mathbf{Z}} f\left(\frac{k}{2^j}\right) \phi_{j,k}(x)\,, \tag{18}
$$

then

$$
\|f(x) - f^j(x)\|_{L^2} \leq C 2^{-jN}\,, \tag{19}
$$

where C depends only on $f(x)$ and the scaling vector α.

Proof: Using the Taylor expansion of f at the point x,

$$f\left(\frac{k}{2^j}\right) = \sum_{p=0}^{N-1}\left(\frac{1}{p!}f^{(p)}(x)\left(\frac{k}{2^j}-x\right)^p\right)+\frac{1}{N!}f^{(N)}(\theta_k)\left(\frac{k}{2^j}-x\right)^N,$$

for some θ_k on the line segment connecting x and $\frac{k}{2^j}$. From Lemma 2.2, for $1 \le p \le N$,

$$\sum_{k\in\mathbf{Z}}\left(\frac{k}{2^j}-x\right)^p\phi_{j,k}(x) = 2^{j/2-jp}\sum_{k\in\mathbf{Z}}\left(k-2^jx\right)^p\phi\left(2^jx-k\right) = 0,$$

and

$$\sum_{k\in\mathbf{Z}}\phi_{j,k}(x) = 2^{j/2}\sum_{k\in\mathbf{Z}}\phi\left(2^jx-k\right) = 2^{j/2}.$$

Assume $\mathrm{supp}(f)\subset[-A,A], \mathrm{supp}(\phi)\subset[-A,A]$ for some positive number A, then

$$\sum_{|k|\le(2^j+1)A}\left(\frac{k}{2^j}-x\right)^p f^{(p)}(x)\,\phi_{j,k}(x) = 0, \quad \text{for } 1\le p\le N,$$

$$\sum_{|k|\le(2^j+1)A} f(x)\,\phi_{j,k}(x) = 2^{j/2}f(x),$$

$$f^j(x) = 2^{-j/2}\sum_{k\in\mathbf{Z}}f\left(\frac{k}{2^j}\right)\phi_{j,k}(x) = 2^{-j/2}\sum_{|k|\le(2^j+1)A}f\left(\frac{k}{2^j}\right)\phi_{j,k}(x).$$

Back to the Taylor expansion, we have

$$\begin{aligned}
f^j(x) &= 2^{-j/2}\sum_{|k|\le(2^j+1)A}\left(\sum_{p=0}^{N-1}\left(\frac{1}{p!}f^{(p)}(x)\left(\frac{k}{2^j}-x\right)^p\right)\right.\\
&\qquad\left.+\frac{1}{N!}f^{(N)}(\theta_k)\left(\frac{k}{2^j}-x\right)^N\right)\phi_{j,k}(x)\\
&= 2^{-j/2}\left(\sum_{p=0}^{N-1}\sum_{|k|\le(2^j+1)A}\frac{1}{p!}f^{(p)}(x)\left(\frac{k}{2^j}-x\right)^p\phi_{j,k}(x)\right)\\
&\qquad+2^{-j/2}\sum_{|k|\le(2^j+1)A}\left(\frac{1}{N!}f^{(N)}(\theta_k)\left(\frac{k}{2^j}-x\right)^N\phi_{j,k}(x)\right)\\
&= f(x)+2^{-j/2}\sum_{|k|\le(2^j+1)A}\left(\frac{1}{N!}f^{(N)}(\theta_k)\left(\frac{k}{2^j}-x\right)^N\phi_{j,k}(x)\right).
\end{aligned}$$

Thus,

$$f^j(x) - f(x) \;=\; 2^{-j/2} \sum_{|k| \le (2^j+1)A} \left(\frac{1}{N!} f^{(N)}(\theta_k) \left(\frac{k}{2^j} - x \right)^N \phi_{j,k}(x) \right) ,$$

and

$$\begin{aligned}
&||f(x) - f^j(x)||_{L^2} \\
&\quad = \left\| 2^{-j/2} \sum_{|k| \le (2^j+1)A} \left(\frac{1}{N!} f^{(N)}(\theta_k) \left(\frac{k}{2^j} - x \right)^N \phi_{j,k}(x) \right) \right\|_{L^2} \\
&\quad = \frac{2^{-j(1/2+N)}}{N!} \left\| \sum_{|k| \le (2^j+1)A} \left(f^{(N)}(\theta_k) (k-y)^N \phi(y-k) \right) \right\|_{L^2} ,
\end{aligned}$$

where we make the substitution $y = 2^j x$. For convenience, define

$$g_k(y) \;:=\; f^{(N)}(\theta_k) (k-y)^N \phi(y-k) ,$$

then $g_k(y)$ has compact support $[-A+k, A+k]$, and $||g_k(y)||_{L^2}$ is uniformly bounded, $||g_k(y)||_{L^2} \le C$, where C depends only on $f(x)$ and $\phi(x)$. Using this we obtain

$$\begin{aligned}
&||f(x) - f^j(x)||_{L^2} \\
&= \frac{2^{-j(1/2+N)}}{N!} \left(\int_{\mathbf{R}} \left(\sum_{|k_1|,|k_2| \le (2^j+1)A} g_{k_1}(y) g_{k_2}(y) \right) dy \right)^{1/2} \\
&= \frac{2^{-j(1/2+N)}}{N!} \left(\int_{\mathbf{R}} \left(\sum_{|k_1|,|k_2| \le (2^j+1)A, |k_1-k_2| \le 2A} g_{k_1}(y) g_{k_2}(y) \right) dy \right)^{1/2} \\
&= \frac{2^{-j(1/2+N)}}{N!} \left(2\left(2^j+1\right) A \cdot 4A \cdot \left(\max_{k_1,k_2 \in \mathbf{Z}} \left\{ \left| \int_{\mathbf{R}} g_{k_1}(y) g_{k_2}(y)\, dy \right| \right\} \right) \right)^{1/2} \\
&= \frac{2^{-j(1/2+N)}}{N!} \left(8\left(2^j+1\right) A^2 \cdot C^2 \right)^{1/2} \\
&\le C 2^{-jN} ,
\end{aligned}$$

where C depends only on f and $\phi(x)$, i.e., C depends only on f and α. $\square$

Remark 1 We call $f^j(x)$ the *wavelet sampling approximation* of the function $f(x)$ *at the level* j. It is similar to but distinct from the *wavelet orthogonal projection*

$$P^j(f) := \sum_{k\in\mathbf{Z}} \left(\int_{\mathbf{R}} f(x)\,\phi_{j,k}(x)\,dx\right)\cdot\phi_{j,k}(x)\,,$$

which has been studied by various authors (see [1], [8], [9], [20], and others). The wavelet sampling approximation is what is used in most applications of wavelets, as it is the easiest approximation to compute. (Simply let the sample values of the given function be the corresponding expansion coefficients.) The value of the above result is that for Coifman wavelet systems, the degree of approximation is comparable to that obtained using Daubechies wavelet systems and the orthogonal projection (see the above references).
Remark 2 If Ω is a bounded open set in $\mathbf{R}$, a function $f\in C^N\left(\bar{\Omega}\right)$ can be extended to $\tilde{f}\in C_0^N(\mathbf{R})$. So the previous result on $C_0^N(\mathbf{R})$ function is certainly true for $C^N\left(\bar{\Omega}\right)$ function while the $L^2(\mathbf{R})$ norm is replaced by the $L^2(\Omega)$ norm.
Remark 3 As it can be seen in the above proof, we didn't use the orthogonality of the orthogonal Coifman wavelet systems anywhere. All we needed was the vanishing moments conditions (13) and (14). It implies that the above result still holds for non-orthogonal wavelet system. We formulate this result as the following theorem.

Theorem 3.2 *Suppose* $\{a_k\}$ *is a sequence with finite length. Assume that* $\{a_k\}$ *satisfies the vanishing moments condition up to degree* N*, i.e.,*

$$\sum_{k\in\mathbf{Z}} (2k)^p a_{2k} = \sum_{k\in\mathbf{Z}} (2k+1)^p a_{2k+1} = 0\,, \quad \textit{for } p=1,\cdots,N\,, \tag{20}$$

$$\sum_{k\in\mathbf{Z}} a_{2k} = \sum_{k\in\mathbf{Z}} a_{2k+1} = 1\,. \tag{21}$$

Define

$$F_0(\xi) = \frac{1}{2}\sum_{k\in\mathbf{Z}} a_k e^{ik\xi}\,,$$

and

$$\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} F_0\left(2^{-j}\xi\right)\,.$$

If $\phi(x) \in L^2(\mathbf{R})$, then for any function $f(x) \in C_0^N(\mathbf{R})$,

$$||f(x) - f^j(x)||_{L^2} \ \leq \ C2^{-jN}, \tag{22}$$

where C depends only on f and the sequence $\{a_k\}$, and $f^j(x)$ is defined by

$$f^j(x) \ := \ 2^{-j/2} \sum_{k \in \mathbf{Z}} f\left(\frac{k}{2^j}\right) \phi_{j,k}(x) \ .$$

In practice the smoothness (either the Hölder smoothness or the Sobolev smoothness) of the scaling function $\phi(x)$ can be calculated when we know the scaling vector $\alpha = \{a_k\}$. So using Corollary 2.3, we can measure the approximation error in $H^1(\mathbf{R}), H^2(\mathbf{R})$ norm or even higher, depending on how smooth the scaling function $\phi(x)$ is.

Theorem 3.3 *Assume the same conditions as in Theorem 3.2, if in addition $\phi(x) \in C^n(\mathbf{R})$, where $n \in \mathbf{Z}, 0 \leq n \leq N$, then*

$$||f(x) - f^j(x)||_{H^n} \ \leq \ C2^{-j(N-n)}, \tag{23}$$

where C depends only on f and the sequence $\{a_k\}$.

Proof: We will prove the case $n = 1$ here. The proof for the general case will be then apparent.

We now show that

$$||f'(x) - \left(f^j(x)\right)'||_{L^2} \ \leq \ C2^{-j(N-1)}, \tag{24}$$

for some constant C, independent of j. Using the same notation as in the proof of Theorem 3.2, we have

$$\begin{aligned}
&\left(f^j(x)\right)' \\
&= \left(2^{-j/2} \sum_{k \in \mathbf{Z}} f\left(\frac{k}{2^j}\right) \phi_{j,k}(x)\right)' \\
&= \left(2^{-j/2} \sum_{|k| \leq (2^j+1)A} f\left(\frac{k}{2^j}\right) \phi_{j,k}(x)\right)' \\
&= 2^{-j/2} \sum_{|k| \leq (2^j+1)A} f\left(\frac{k}{2^j}\right) \left(\phi_{j,k}(x)\right)'
\end{aligned}$$

$$
\begin{aligned}
&= 2^j \sum_{|k|\le (2^j+1)A} f\left(\frac{k}{2^j}\right)\phi'\left(2^j x - k\right) \\
&= 2^j \sum_{|k|\le (2^j+1)A} \left(\sum_{p=0}^{N-1}\left(\frac{1}{p!}f^{(p)}(x)\left(\frac{k}{2^j}-x\right)^p\right)\right. \\
&\quad \left. + \frac{1}{N!}f^{(N)}(\theta_k)\left(\frac{k}{2^j}-x\right)^N\right)\phi'\left(2^j x - k\right) \\
&= 2^j \left(\sum_{p=0}^{N-1}\sum_{|k|\le (2^j+1)A}\left(\frac{1}{p!}f^{(p)}(x)\left(\frac{k}{2^j}-x\right)^p\phi'\left(2^j x-k\right)\right)\right. \\
&\quad \left. + \sum_{|k|\le (2^j+1)A}\left(\frac{1}{N!}f^{(N)}(\theta_k)\left(\frac{k}{2^j}-x\right)^N\phi'\left(2^j x-k\right)\right)\right) \\
&= 2^j \sum_{|k|\le (2^j+1)A}\left(f(x)\,\phi'\left(2^j x-k\right)\right) \\
&\quad + 2^j \sum_{|k|\le (2^j+1)A}\left(f'(x)\left(\frac{k}{2^j}-x\right)\phi'\left(2^j x-k\right)\right) \\
&\quad + 2^j \left(\sum_{p=2}^{N-1}\sum_{|k|\le (2^j+1)A}\frac{1}{p!}f^{(p)}(x)\left(\frac{k}{2^j}-x\right)^p\phi'\left(2^j x-k\right)\right) \\
&\quad + 2^j \sum_{|k|\le (2^j+1)A}\left(\frac{1}{N!}f^{(N)}(\theta_k)\left(\frac{k}{2^j}-x\right)^N\phi'\left(2^j x-k\right)\right) \\
&= f'(x) + 2^j \sum_{|k|\le (2^j+1)A}\left(\frac{1}{N!}f^{(N)}(\theta_k)\left(\frac{k}{2^j}-x\right)^N\phi'\left(2^j x-k\right)\right)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left\|f'(x) - \left(f^j(x)\right)'\right\|_{L^2} \\
&\quad = 2^j \left\|\sum_{|k|\le (2^j+1)A}\left(\frac{1}{N!}f^{(N)}(\theta_k)\left(\frac{k}{2^j}-x\right)^N\phi'\left(2^j x-k\right)\right)\right\|_{L^2} \\
&\quad = \frac{2^{-j(-1/2+N)}}{N!}\left\|\sum_{|k|\le (2^j+1)A} f^{(N)}(\theta_k)(k-y)^N\phi'(y-k)\right\|_{L^2}
\end{aligned}
$$

Now applying the same estimate as in Theorem 3.2, we get

$$||f'(x) - \left(f^j(x)\right)'||_{L^2} \;\le\; C2^{-j(N-1)}\,,$$

where C depends only on f and $\{a_k\}$. Then (23) follows readily from above and Theorem 3.2. $\Box$

Taking the tensor product of these one dimensional wavelet systems, it is straightforward to generalize these two theorems to higher dimensions. And the proof is an easy modification of the one dimensional case.

Theorem 3.4 *Suppose $\{a_k^1, k \in \mathbf{Z}\}, \{a_k^2, k \in \mathbf{Z}\}, \cdots, \{a_k^m, k \in \mathbf{Z}\}$ are a family of sequences with finite length (i.e., there exists a positive integer K, $a_k^l = 0$ for $1 \le l \le m$, and $|k| > K$). Assume that $\{a_k^l\}$ satisfies the vanishing moments conditions up to degree N, i.e., for $1 \le l \le m$*

$$\sum_{k\in\mathbf{Z}} (2k)^p a_{2k}^l \;=\; \sum_{k\in\mathbf{Z}} (2k+1)^p a_{2k+1}^l \;=\; 0\,, \quad \text{for } p = 1, \cdots, N\,, \tag{25}$$

$$\sum_{k\in\mathbf{Z}} a_{2k}^l \;=\; \sum_{k\in\mathbf{Z}} a_{2k+1}^l \;=\; 1\,. \tag{26}$$

And for $1 \le l \le m$, define

$$F_0^l(\xi) \;=\; \frac{1}{2}\sum_{k\in\mathbf{Z}} a_k^l e^{ik\xi}\,,$$

$$\widehat{\phi^l}(\xi) \;=\; \prod_{j=1}^{\infty} F_0^l\left(2^{-j}\xi\right)\,.$$

If $\phi^l(x) \in L^2(\mathbf{R})$ for all $1 \le l \le m$, then for any function $f(x_1, x_2, \cdots, x_m) \in C_0^N(\mathbf{R}^m)$,

$$||f(x_1, x_2, \cdots, x_m) \;-\; f^j(x_1, x_2, \cdots, x_m)||_{L^2} \;\le\; C2^{-jN}\,, \tag{27}$$

where C depends only on f and the family of sequences $\{a_k^l\}$, and

$$\begin{aligned} f^j(x_1, x_2, \cdots, x_m) \;&:=\; 2^{-jm/2} \sum_{k_1,k_2,\cdots,k_m\in\mathbf{Z}} f\left(\frac{k_1}{2^j}, \frac{k_2}{2^j}, \cdots, \frac{k_m}{2^j}\right) \\ &\quad \cdot \phi_{j,k_1}^1(x_1)\cdot\phi_{j,k_2}^2(x_2)\cdots\phi_{j,k_m}^n(x_m)\,. \end{aligned}$$

If in addition, $\phi^l(x) \in C^n(\mathbf{R})$ for all $1 \le l \le m$, where n is a nonnegative integer not greater than N, then

$$||f(x_1, x_2, \cdots, x_m) - f^j(x_1, x_2, \cdots, x_m)||_{H^n} \le C2^{-j(N-n)}.$$

For example, we can choose $\{a_k^l\}$ to be the same sequence, then $\phi^1(x) = \phi^2(x) = \cdots = \phi^m(x)$.

4 Newton's method and the problem of constructing the orthogonal Coifman scaling function

In this section, we look at the construction of orthogonal Coifman wavelet systems. I. Daubechies discussed the orthogonal Coifman wavelet systems of odd degrees (which are even orders in her definition) with a preimposed form in [4]. Here we like to treat them in a different manner. In general, the solutions of a given degree are a finite set. We will apply Newton's method to derive one particular solution for each degree. Without knowing the existence of the solution, Newton's method may fail to prove the existence even though numerically the Newton's approximation sequence converges. On the other hand, the Newton-Kantorovich theorem (see [11], [12], and [17]) tells us that if the Newton's approximation sequence satisfies certain conditions (the Kantorovich constant is less than or equal to one half), then the limit of such sequence must be a solution.

As seen in Section 2, both the scaling function and the wavelet function are determined by the scaling vector $\alpha = \{a_k\}$. So we will focus on the construction of the discrete Fourier transform $F_0(\xi)$ of α. Going back to the definition of orthogonal Coifman wavelet systems in Section 2, the conditions (13) and (14) are equivalent to the following conditions which set the values of the derivatives of $F_0(\xi)$ at 0 and π,

$$F_0^{(p)}(0) = \delta_{0,p}, \quad \text{for } p = 0, 1, \cdots, N, \tag{28}$$

$$F_0^{(p)}(\pi) = 0, \quad \text{for } p = 0, 1, \cdots, N. \tag{29}$$

We want to find all the $F_0(\xi)$ satisfying (28), (29) and the quadratic condition,

$$|F_0(\xi)|^2 + |F_0(\xi + \pi)|^2 = 1. \tag{30}$$

According to the Lawton-Cohen theorem (see [13]), $\psi(x)$ defined by (6) with such an $F_0(\xi)$ will be the wavelet function of an orthonormal wavelet system if and only if the multiresolution operator T_α has 1 as a nondegenerate eigenvalue, where T_α is an $(2(N_2 - N_1) + 1) \times (2(N_2 - N_1) + 1)$ matrix defined by

$$(T_\alpha)_{l,k} \;=\; \frac{1}{2} \sum_n a_n a_{k-2l+n}\,, \quad -N_2 + N_1 + 1 \le l, k \le N_2 - N_1 - 1\,. \tag{31}$$

First, what will be the minimal length L of $F_0(\xi), L = N_2 - N_1 + 1$, for a given degree N? The quadratic equation (30), which is equivalent to (4), gives $L/2$ conditions. The equations (28) and (29) give $2N + 2$ conditions, but actually they are redundant. From (30),

$$F_0(\xi)F_0(-\xi) + F_0(\xi + \pi)F_0(-\xi + \pi) \;=\; 1\,,$$

and hence the even order derivatives of $F_0(0)$ can be calculated from the knowledge of the lower order derivatives of $F_0(0)$ and (29). So (28) and (29) give $(3N + 3)/2$ or $(3N + 2)/2$ conditions if N is odd or even, respectively. So all together there are $(L + 3N + 3)/2$ $(-1/2$ if N is even) conditions for L variables $\{a_{N_1}, \cdots, a_{N_2}\}$. Then L should be at least $(L + 3N + 3)/2$ $(-1/2$ if N is even). It follows that the minimal length for the degree N is $3N + 3$ if N is odd, and $(3N + 2)$ if N is even. Since we want the length of $F_0(\xi)$ to be as small as possible, L will always be the minimal length in the remaining parts.

Define a shifted version of the Fourier transform $F_0(\xi)$,

$$F_s(\xi) \;:=\; \frac{1}{2} \sum_{k=0}^{L-1} a_{k+N_1} e^{ik\xi} \;=\; F_0(\xi) e^{iN_1\xi}\,. \tag{32}$$

Then (28)-(30) are equivalent to

$$F_s^{(p)}(0) \;=\; (iN_1)^p\,, \quad \text{for } p = 0, 1, \cdots, N\,, \tag{33}$$

$$F_s^{(p)}(\pi) \;=\; 0\,, \quad \text{for } p = 0, 1, \cdots, N\,, \tag{34}$$

$$|F_s(\xi)|^2 + |F_s(\xi + \pi)|^2 \;=\; 1\,. \tag{35}$$

The multiresolution operator matrix is the same as T_α, since $F_s(\xi)$ is just a shifted version of $F_0(\xi)$. It follows from (34) that $F_s(\xi)$ is of the form

$$F_s(\xi) \;=\; \left(\frac{1 + e^{i\xi}}{2}\right)^{N+1} Q_s(\xi)\,, \tag{36}$$

where $Q_s(\xi)$ is a trigonometric polynomial.

Now we want to solve (33)-(35) for $F_s(\xi)$. For convenience, set

$$M_s(x) \;=\; \frac{1}{2}\sum_{k=0}^{L-1} a_{k+N_1} x^k\,. \tag{37}$$

There is a nice formula for the first $N+1$ derivatives of $M_s(x)$ at 1.

Lemma 4.1 *Let $M_s(x)$ be as above, then*

$$M_s^{(p)}(1) \;=\; (-1)^p p! \begin{pmatrix} N_1-1+p \\ p \end{pmatrix}, \quad \text{for } 0 \le p \le N\,,$$

where

$$\begin{pmatrix} N_1-1+p \\ p \end{pmatrix} \;=\; \frac{N_1(N_1+1)\cdots(N_1-1+p)}{1\cdot 2\cdots p}\,.$$

Proof: Define

$$M_0(x) \;=\; \frac{1}{2}\sum_{k=N_1}^{N_2} a_k x^k\,.$$

As we know, $F_0^{(p)}(0) = \delta_{0,p}$, for $0 \le p \le N$. It follows that $M_0^{(p)}(1) = \delta_{0,p}$, for $0 \le p \le N$. Now $M_s(x) = M_0(x)x^{-N_1}$, so

$$M_s^{(p)}(1) \;=\; (x^{-N_1})^{(p)}(1) \;=\; (-1)^p p! \begin{pmatrix} N_1-1+p \\ p \end{pmatrix}.$$

The lemma is proved. □

Corollary 4.2 *Assume $F_s(\xi)$ is the shifted version of $F_0(\xi)$ as in (32), we have*

$$F_s(\xi) \;=\; \sum_{p=0}^{N} \begin{pmatrix} N_1-1+p \\ p \end{pmatrix} (1-e^{i\xi})^p + (1-e^{i\xi})^{N+1} R_s(\xi)\,, \tag{38}$$

where $R_s(\xi)$ is a trigonometric polynomial.

Proof: Applying Lemma 4.1, one obtain

$$M_s(x) \;=\; \sum_{p=0}^{N} \begin{pmatrix} N_1 - 1 + p \\ p \end{pmatrix} (1-x)^p + (1-x)^{N+1} R(x)\,,$$

where $R(x)$ is a polynomial. From the definition of $M_s(x)$, we get (38). $\square$

Theorem 4.3 *Let $F_0(\xi)$ satisfy (28), (29), and (30), then*

$$\begin{aligned} F_0(\xi) \;&=\; \left(\left(\sum_{p=0}^{N} \begin{pmatrix} N_1 - 1 + p \\ p \end{pmatrix} (1-e^{i\xi})^p \right) \left(\sum_{l=0}^{N} \begin{pmatrix} N + l \\ l \end{pmatrix} \left(\frac{1-e^{i\xi}}{2} \right)^l \right) \right. \\ &\quad \left. + (1-e^{i\xi})^{N+1} R_0(\xi) \right) e^{iN_1\xi} \left(\frac{1+e^{i\xi}}{2} \right)^{N+1}, \end{aligned}$$

where $R_0(\xi)$ is a trigonometric polynomial of degree N.

Proof: From (36) and (38),

$$\begin{aligned} F_s(\xi) \;&=\; \left(\frac{1+e^{i\xi}}{2} \right)^{N+1} Q_s(\xi) \\ &=\; \sum_{p=0}^{N} \begin{pmatrix} N_1 - 1 + p \\ p \end{pmatrix} (1-e^{i\xi})^p + (1-e^{i\xi})^{N+1} R_s(\xi)\,. \end{aligned}$$

By a standard polynomial argument, we get

$$\begin{aligned} F_s(\xi) \;&=\; \left(\left(\sum_{p=0}^{N} \begin{pmatrix} N_1 - 1 + p \\ p \end{pmatrix} (1-e^{i\xi})^p \right) \left(\sum_{l=0}^{N} \begin{pmatrix} N + l \\ l \end{pmatrix} \left(\frac{1-e^{i\xi}}{2} \right)^l \right) \right. \\ &\quad \left. + (1-e^{i\xi})^{N+1} R_0(\xi) \right) \left(\frac{1+e^{i\xi}}{2} \right)^{N+1}, \end{aligned}$$

where $R_0(\xi)$ is a trigonometric polynomial. By (32), Theorem 4.3 is proved. $\square$

So does an orthogonal Coifman wavelet system of degree N exist? And how can we construct it? Our first approach is Newton's method. In particular, we will prove the existence of orthogonal Coifman wavelet systems of degree up to 9 by utilizing a fundamental result due to L. Kantorovich. We let $B(x_0, r)$ denote the open ball of radius r about the point $x_0 \in \mathbf{R}^n$.

Theorem 4.4 (Newton-Kantorovich Theorem) *Assume that* $f : D \subset \mathbf{R}^n \to \mathbf{R}^n$ *is* C^1 *on a convex set* $D_0 \subset D$ *such that*

$$||f'(x) - f'(y)|| \leq \gamma \, ||x - y|| , \quad \forall x, y \in D_0 .$$

Suppose that there exists an $x_0 \in D_0$ *such that* $||f'(x_0)^{-1}|| \leq \beta$ *and* $\alpha = \beta\gamma\eta \leq 1/2$*, where* $\eta \geq ||f'(x_0)^{-1} f(x_0)||$*. Set*

$$t^* = (\beta\gamma)^{-1}[1 - (1 - 2\alpha)^{1/2}] , \quad t^{**} = (\beta\gamma)^{-1}[1 + (1 - 2\alpha)^{1/2}] ,$$

and assume that $\overline{B}(x_0, t^*) \subset D_0$*. Then the Newton iterates*

$$x_{k+1} = x_k - f'(x_k)^{-1} f(x_k) , \quad k = 0, 1, \cdots ,$$

are well-defined, remain in $\overline{B}(x_0, t^*)$ *and converge to a solution* x^* *of* $f(x) = 0$ *which is unique in* $B(x_0, t^{**}) \cap D_0$*.*

The basic idea of the proof is to construct a majorizing sequence for x_k. For a complete proof and some applications of the Newton-Kantorovich theorem, see [11],[12], and [17].

It is clear from the definition that the degree 0 orthogonal Coifman wavelet system is exactly the Haar wavelet system, $\{a_0 = 1, a_1 = 1\}$. Let's try to compute other degree orthogonal Coifman wavelet systems using Newton's method.

a. N = 1

Take the Haar wavelet system as the initial point, i.e.,

$$x_0 = \begin{pmatrix} a_{-3} \\ a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} .$$

We compute the Newton iterates by Matlab,

$$x_1 = \begin{pmatrix} 0e+0 \\ -2.5e-1 \\ 5e-1 \\ 1.5e+0 \\ 5e-1 \\ -2.5e-1 \end{pmatrix}, \; x_2 = \begin{pmatrix} 0e+0 \\ -1.25e-1 \\ 5e-1 \\ 1.25e+0 \\ 5e-1 \\ -1.25e-1 \end{pmatrix}, \; x_3 = \begin{pmatrix} -2.083333333333334e-2 \\ -1.041666666666667e-1 \\ 5.416666666666666e-1 \\ 1.208333333333333e+0 \\ 4.791666666666667e-1 \\ -1.041666666666667e-1 \end{pmatrix},$$

$$x_4 = \begin{pmatrix} -2.213541666666667e-2 \\ -1.028645833333333e-1 \\ 5.442708333333334e-1 \\ 1.205729166666667e+0 \\ 4.778645833333334e-1 \\ -1.028645833333334e-1 \end{pmatrix}, \; x_5 = \begin{pmatrix} -2.214054297900262e-2 \\ -1.028594570209974e-1 \\ 5.442810859580053e-1 \\ 1.205718914041995e+0 \\ 4.778594570209974e-1 \\ -1.028594570209974e-1 \end{pmatrix},$$

$$x_6 = \begin{pmatrix} -2.214054305846308e-2 \\ -1.028594569415369e-1 \\ 5.442810861169262e-1 \\ 1.205718913883074e+0 \\ 4.778594569415370e-1 \\ -1.028594569415369e-1 \end{pmatrix}, \; x_7 = \begin{pmatrix} -2.214054305846309e-2 \\ -1.028594569415369e-1 \\ 5.442810861169262e-1 \\ 1.205718913883074e+0 \\ 4.778594569415369e-1 \\ -1.028594569415369e-1 \end{pmatrix}, \cdots$$

From the above computation, we can't tell whether the orthogonal Coifman wavelet system of degree 1 exists or not. So we want to apply the Newton-Kantorovich theorem prove the existence. Define the mapping $f : \mathbf{R}^6 \longrightarrow \mathbf{R}^6$ by

$$f(a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2) \;=\; \begin{pmatrix} a_{-3}a_{-1} + a_{-2}a_0 + a_{-1}a_1 + a_0a_2 \\ a_{-3}a_1 + a_{-2}a_2 \\ a_{-3} + a_{-1} + a_1 - 1 \\ -a_{-3} + a_{-2} - a_{-1} + a_0 - a_1 + a_2 \\ -3a_{-3} - 2a_{-2} - a_{-1} + a_1 + 2a_2 \\ 3a_{-3} - 2a_{-2} + a_{-1} - a_1 + 2a_2 \end{pmatrix}. \tag{39}$$

It follows from the previous discussion that the solutions of $f(x) = 0$ are exactly the orthogonal Coifman wavelet systems of degree 1 if 1 is a nondegenerate eigenvalue of the multiresolution matrix T_α. The mapping $f(x)$ is C^1 (actually C^∞) from $\mathbf{R}^6$ to $\mathbf{R}^6$. Moreover, $f'(x)$ is Lipschitz with Lipschitz constant 1. Since $||f'(x_7)^{-1}|| < 10$, $||f'(x_7)^{-1}f(x_7)|| < 1/100$, if we set

$$y_0 \;=\; x_7\,, \quad y_{k+1} \;=\; y_k - f'(y_k)^{-1}f(y_k)\,, \quad k = 0, 1, \cdots,$$

then Theorem 4.4 implies that $\{y_k\}$ (i.e., $\{x_k\}$) converges to a solution of $f(x) = 0$. So the orthogonal Coifman wavelet system of degree 1 does exist and the Newton method will give us a fast approximation algorithm. Actually it can be seen as above that there is almost no numerical difference between x_6 and x_7. And it can be easily checked that for $\alpha = x_7$, 1 is not a degenerate eigenvalue for the corresponding multiresolution matrix T_α. For numerical applications, we simply take x_7 as an approximate orthogonal

Coifman wavelet system of degree 1, i.e.,

$$\begin{pmatrix} a_{-3} \\ a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -2.214054305846309e-2 \\ -1.028594569415369e-1 \\ 5.442810861169262e-1 \\ 1.205718913883074e+0 \\ 4.778594569415369e-1 \\ -1.028594569415369e-1 \end{pmatrix}.$$

b. N = 2

Take the previous orthogonal Coifman wavelet system of degree 2 as the initial point, i.e.,

$$x_0 = \begin{pmatrix} a_{-4} \\ a_{-3} \\ a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0e+0 \\ -2.214054305846309e-2 \\ -1.028594569415369e-1 \\ 5.442810861169262e-1 \\ 1.205718913883074e+0 \\ 4.778594569415369e-1 \\ -1.028594569415369e-1 \\ 0e+0 \end{pmatrix}.$$

After 10 Newton's iteration, x_{10} is equal to

$$x_{10} = \begin{pmatrix} a_{-4} \\ a_{-3} \\ a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 4.687500000000001e-2 \\ -2.116013576461578e-2 \\ -1.406250000000000e-1 \\ 4.384804072938474e-1 \\ 1.140625000000000e+0 \\ 6.865195927061527e-1 \\ -4.687500000000001e-2 \\ -1.038398642353842e-1 \end{pmatrix}.$$

Here

$$f(a_{-4}, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3) = \begin{pmatrix} a_{-4}a_{-2} + a_{-3}a_{-1} + a_{-2}a_0 + a_{-1}a_1 + a_0a_2 + a_1a_3 \\ a_{-4}a_0 + a_{-3}a_1 + a_{-2}a_2 + a_{-1}a_3 \\ a_{-4}a_2 + a_{-3}a_3 \\ a_{-4} + a_{-2} + a_0 + a_2 - 1 \\ -a_{-4} + a_{-3} - a_{-2} + a_{-1} - a_0 + a_1 - a_2 + a_3 \\ -4a_{-4} - 3a_{-3} - 2a_{-2} - a_{-1} + a_1 + 2a_2 + 3a_3 \\ 4a_{-4} - 3a_{-3} + 2a_{-2} - a_{-1} + a_1 - 2a_2 + 3a_3 \\ -16a_{-4} + 9a_{-3} - 4a_{-2} + a_{-1} + a_1 - 4a_2 + 9a_3 \end{pmatrix}.$$

Since $f'(x)$ is Lipschitz with Lipschitz constant 1, $||f'(x_{10})^{-1}|| < 200$, and $||f'(x_{10})^{-1}f(x_{10})|| < 1/1000$, again by Theorem 4.4, $\{x_k\}$ converges to a solution of $f(x) = 0$. Also 1 is not a degenerate eigenvalue of the multiresolution matrix T_α when $\alpha = x_{10}$. For numerical application, we simply take x_{10} to be an approximate orthogonal Coifman wavelet system of degree 2.

The significance of the Newton-Kantorovich theorem is that even if we don't know the existence of the solution of $f(x) = 0$, we can still apply the Newton's iteration method. If the numerical results are good enough (i.e., the conditions of the Newton-Kantorovich theorem are satisfied), then it follows that $f(x) = 0$ has a solution and the numerical results will give a fast approximation.

So to get an orthogonal Coifman wavelet system of degree N, here is our Newton's method algorithm:

1. Take an orthogonal Coifman wavelet system of degree $N-1$ as the initial point. More precisely, assume the scaling vector of such system is α^{N-1}.

   - if N is odd (as in the case **a.** above),

   $$x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \alpha^{N-1} \\ 0 \end{pmatrix}.$$

   - if N is even (as in the case **b.** above),

   $$x_0 = \begin{pmatrix} 0 \\ \alpha^{N-1} \\ 0 \end{pmatrix}.$$

2. Compute Newton iterates starting from x_0.

3. If for some x_{k_0}, the Newton-Kantorovich condition

   $$\gamma \cdot ||f'(x_{k_0})^{-1}|| \cdot ||f'(x_{k_0})^{-1}f(x_{k_0})|| \leq 1/2$$

   is satisfied, where γ is the Lipschitz constant of $f'(x)$, then define $y_0 = x_{k_0}$.

4. An error estimate is needed if we are trying to prove the above inequality from numerical results.

5. From the Newton-Kantorovich theorem, the Newton iterates starting from y_0 converge to a solution of $f(x) = 0$.

6. Check that 1 is not a degenerate eigenvalue of the corresponding multiresolution operator T_α. Then the iterate y_k we choose will be a good approximation to the scaling vector α^N of the orthogonal Coifman wavelet system of degree N.

Based on this algorithm, here we give the coefficients of the scaling vectors $\{\alpha^0, \alpha^1, \cdots, \alpha^9\}$ for the orthogonal Coifman wavelet systems of degree through 0 to 9.

$$
\begin{aligned}
\alpha^0 &= \{a_0 = 1 \quad a_1 = 1\} \\
\alpha^1 &= \left\{ \begin{array}{cc}
a_{-3} = -2.214054305846309e-2 & a_{-2} = -1.028594569415369e-1 \\
a_{-1} = 5.442810861169262e-1 & a_0 = 1.205718913883074e+0 \\
a_1 = 4.778594569415369e-1 & a_2 = -1.028594569415369e-1
\end{array} \right\} \\
\alpha^2 &= \left\{ \begin{array}{cc}
a_{-4} = 4.687500000000001e-2 & a_{-3} = -2.116013576461578e-2 \\
a_{-2} = -1.406250000000000e-1 & a{-1} = 4.384804072938474e-1 \\
a_0 = 1.140625000000000e+0 & a_1 = 6.865195927061527e-1 \\
a_2 = -4.687500000000001e-2 & a_3 = -1.038398642353842e-1
\end{array} \right\} \\
\alpha^3 &= \left\{ \begin{array}{cc}
a_{-7} = -1.019010798215288e-3 & a_{-6} = -2.578406712281320e-3 \\
a_{-5} = 7.935767225924024e-3 & a_{-4} = 3.348882032655902e-2 \\
a_{-3} = -8.405296092154323e-2 & a_{-2} = -1.081712141834229e-1 \\
a_{-1} = 5.897343873912384e-1 & a_0 = 1.149364787713728e+0 \\
a_1 = 5.460420930695333e-1 & a_2 = -9.527918062201626e-2 \\
a_3 = -5.864027596693714e-2 & a_4 = 2.317519347743374e-2
\end{array} \right\} \\
\alpha^4 &= \left\{ \begin{array}{cc}
a_{-8} = 1.100684225065227e-3 & a_{-7} = -7.406275252609123e-4 \\
a_{-6} = -1.708984375000000e-2 & a_{-5} = 9.921465811937182e-3 \\
a_{-4} = 6.893895537402157e-2 & a_{-3} = -7.756041618077222e-2 \\
a_{-2} = -1.268710684973909e-1 & a_{-1} = 5.383395571089353e-1 \\
a_0 = 1.121367647372065e+0 & a_1 = 6.372385805173691e-1 \\
a_2 = -5.903279734843452e-2 & a_3 = -1.244177315465760e-1 \\
a_4 = 1.158642262467386e-2 & a_5 = 1.721917181436737e-2
\end{array} \right\}
\end{aligned}
$$

$$\alpha^5 = \left\{ \begin{array}{cc} a_{-11} = -4.893146851061624e-5 & a_{-10} = -1.003855491065533e-4 \\ a_{-9} = 6.593303475863657e-4 & a_{-8} = 1.580410201915188e-3 \\ a_{-7} = -3.640917831132484e-3 & a_{-6} = -1.273920202209764e-2 \\ a_{-5} = 2.245848192407572e-2 & a_{-4} = 4.886818864233392e-2 \\ a_{-3} = -1.163925015231710e-1 & a_{-2} = -1.015402815097777e-1 \\ a_{-1} = 6.059671435456483e-1 & a_{0} = 1.122570513740660e+0 \\ a_{1} = 5.730066705472950e-1 & a_{2} = -8.644152712042383e-2 \\ a_{3} = -9.301552895745398e-2 & a_{4} = 3.316712095834069e-2 \\ a_{5} = 1.100625341566277e-2 & a_{6} = -5.364837341844136e-3 \end{array} \right\}$$

$$\alpha^6 = \left\{ \begin{array}{cc} a_{-12} = 4.606896825041038e-5 & a_{-11} = -3.823393326605808e-5 \\ a_{-10} = -7.400929411026428e-4 & a_{-9} = 5.445459664138592e-4 \\ a_{-8} = 6.648554460215593e-3 & a_{-7} = -4.301103695860125e-3 \\ a_{-6} = -2.968707220366963e-2 & a_{-5} = 2.490802670578170e-2 \\ a_{-4} = 7.414330425968660e-2 & a_{-3} = -1.121458311492803e-1 \\ a_{-2} = -1.121080284733010e-1 & a_{-1} = 5.777841424573339e-1 \\ a_{0} = 1.105616520630899e+0 & a_{1} = 6.229169131781114e-1 \\ a_{2} = -6.088387476537906e-2 & a_{3} = -1.361653712212063e-1 \\ a_{4} = 1.972245504790986e-2 & a_{5} = 2.981989212955940e-2 \\ a_{6} = -2.757834983508583e-3 & a_{7} = -3.322980437587264e-3 \end{array} \right\}$$

$$\alpha^7 = \left\{ \begin{array}{cc} a_{-15} = -2.524358359989246e-6 & a_{-14} = -4.609838325353006e-6 \\ a_{-13} = 4.416569382461668e-5 & a_{-12} = 8.816045323194410e-5 \\ a_{-11} = -3.676592334272509e-4 & a_{-10} = -8.330003901883414e-4 \\ a_{-9} = 1.791187855390569e-3 & a_{-8} = 5.305329827060999e-3 \\ a_{-7} = -8.002021689901074e-3 & a_{-6} = -2.151263231017449e-2 \\ a_{-5} = 3.547166284540619e-2 & a_{-4} = 5.562727391693902e-2 \\ a_{-3} = -1.360762293560412e-1 & a_{-2} = -9.422547504779143e-2 \\ a_{-1} = 6.143146193357714e-1 & a_{0} = 1.106252910079098e+0 \\ a_{1} = 5.873348100322010e-1 & a_{2} = -7.930530592489823e-2 \\ a_{3} = -1.149284838038542e-1 & a_{4} = 3.773447713912610e-2 \\ a_{5} = 2.272492296652961e-2 & a_{6} = -1.038905032694064e-2 \\ a_{7} = -2.304450287539944e-3 & a_{8} = 1.261922422861893e-3 \end{array} \right\}$$

$$\alpha^8 = \left\{ \begin{array}{cc} a_{-16} = 2.253190780495695e-6 & a_{-15} = -2.152143904068564e-6 \\ a_{-14} = -4.295842247192477e-5 & a_{-13} = 3.740552476712784e-5 \\ a_{-12} = 4.112285429205518e-4 & a_{-11} = -3.211387783790904e-4 \\ a_{-10} = -2.797279987693231e-3 & a_{-9} = 1.941769422898988e-3 \\ a_{-8} = 1.264304572933647e-2 & a_{-7} = -9.504571148718980e-3 \\ a_{-6} = -3.739795911061904e-2 & a_{-5} = 3.842472941974922e-2 \\ a_{-4} = 7.415880120667900e-2 & a_{-3} = -1.343151858130729e-1 \\ a_{-2} = -1.007162770426715e-1 & a_{-1} = 5.980087017878338e-1 \\ a_0 = 1.094274969346707e+0 & a_1 = 6.175739132097274e-1 \\ a_2 = -5.997859151294635e-2 & a_3 = -1.444589178644569e-1 \\ a_4 = 2.482936945667842e-2 & a_5 = 3.941500144032918e-2 \\ a_6 = -6.044177030563314e-3 & a_7 = -7.488005003312453e-3 \\ a_8 = 6.575756338630860e-4 & a_9 = 6.884499465383870e-4 \end{array} \right\}$$

$$\alpha^9 = \left\{ \begin{array}{cc} a_{-19} = -1.358212135464472e-7 & a_{-18} = -2.296399299805941e-7 \\ a_{-17} = 2.915005842710325e-6 & a_{-16} = 5.233619320026196e-6 \\ a_{-15} = -3.008063596395375e-5 & a_{-14} = -5.829368777244230e-5 \\ a_{-13} = 1.984938227974709e-4 & a_{-12} = 4.268915950172564e-4 \\ a_{-11} = -9.016444801392472e-4 & a_{-10} = -2.349895868827066e-3 \\ a_{-9} = 3.438766968771003e-3 & a_{-8} = 9.562233598261282e-3 \\ a_{-7} = -1.295349950301171e-2 & a_{-6} = -2.794258537276394e-2 \\ a_{-5} = 4.620914455413364e-2 & a_{-4} = 5.838938555056139e-2 \\ a_{-3} = -1.492888402656793e-1 & a_{-2} = -8.773462965647225e-2 \\ a_{-1} = 6.194005181568405e-1 & a_0 = 1.095016542708073e+0 \\ a_1 = 5.961918029174383e-1 & a_2 = -7.360510694893753e-2 \\ a_3 = -1.299967565094465e-1 & a_4 = 3.983803439596872e-2 \\ a_5 = 3.310436661298578e-2 & a_6 = -1.432822469882016e-2 \\ a_7 = -5.882156328071385e-3 & a_8 = 3.080573451990462e-3 \\ a_9 = 5.071055047160665e-4 & a_{10} = -2.999290456691924e-4 \end{array} \right\}$$

Remark 1 Compared the above numerical results with Table 2 in [4], the coefficients of the odd degrees 1, 3, 5, 7 and 9 coincide I Daubechies' computation. Note that in our case the sum of the coefficients is 2.
Some illustrations of the orthogonal Coifman wavelet systems are included in Fig. 1 and 2.

Another approach to obtain orthogonal Coifman wavelet systems is given by applying Theorem 4.3. In what follows, we give an example of how to use Theorem 4.3 to construct orthogonal Coifman wavelet systems.

N = 1

According to Theorem 4.3, (32) and (36), we have

$$Q_s(\xi) = \left(\sum_{p=0}^{1} \binom{N_1-1+p}{p} (1-e^{i\xi})^p\right) \left(\sum_{l=0}^{1} \binom{1+l}{l} \left(\frac{1-e^{i\xi}}{2}\right)^l\right)$$

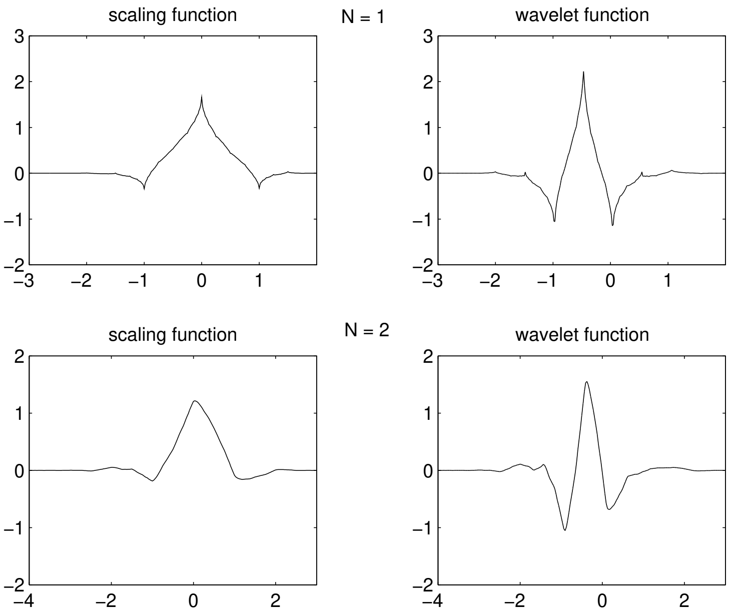

Figure 1: the orthogonal Coifman wavelet system of degree 1 and 2

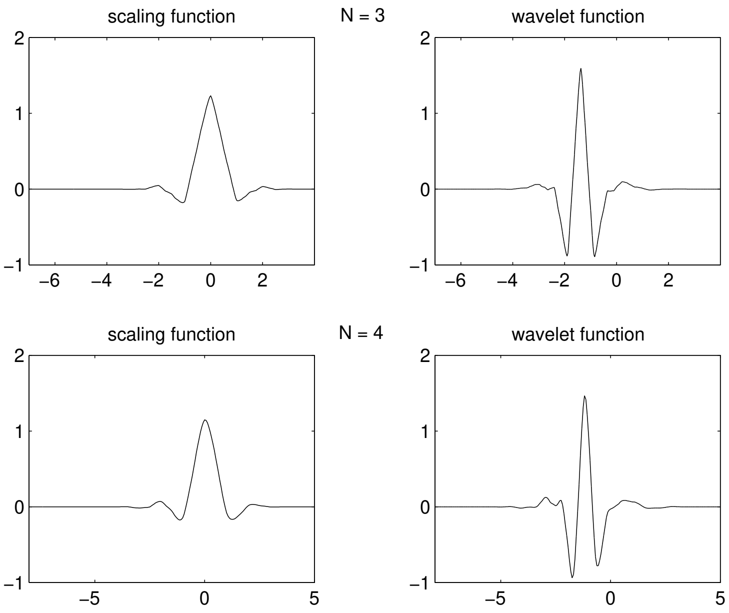

Figure 2: the orthogonal Coifman wavelet system of degree 3 and 4

$$
\begin{aligned}
&+ \ \left(1-e^{i\xi}\right)^2\left(r_0+r_1e^{i\xi}\right) \\
= \ & \left(1+N_1(1-e^{i\xi})\right)\left(2-e^{i\xi}\right)+\left(\frac{1-e^{i\xi}}{2}\right)^2\left(r_0+r_1e^{i\xi}\right) .
\end{aligned}
$$

The norm square of $Q_s(\xi)$ is equal to

$$
\begin{aligned}
|Q_s(\xi)|^2 \ &= \ Q_s(\xi)Q_s(-\xi) \\
&= \ 1+(1-\cos(\xi))\left(2{N_1}^2+2N_1-r_0-r_1+4\right) \\
&\quad + \ (1-\cos(\xi))^2\left(\frac{1}{4}{r_0}^2+\frac{1}{4}{r_1}^2+(3N_1+2)r_0+(N_1+3)r_1\right. \\
&\quad \left. +8{N_1}^2+8N_1+ \ \cos(\xi)(2r_0r_1+4(N_1+1)r_1)\right) .
\end{aligned}
$$

The condition (35) is equivalent to (see [3])

$$
|Q_s(\xi)|^2 \ = \ \sum_{p=0}^{N-1}\left(\begin{array}{c} N-1+p \\ p \end{array}\right)(\frac{1-\cos(\xi)}{2})^p+(1-\cos(\xi))^N R(\cos(\xi)) \, , \quad (40)
$$

where $R(x)$ is an odd polynomial, chosen such that the right-hand side of (40) is nonnegative. So for $N=1$,

$$
|Q_s(\xi)|^2 \ = \ 1+(1-\cos(\xi))+(1-\cos(\xi))^2 R(\cos(\xi)) \, . \quad (41)
$$

It follows immediately that

$$
2{N_1}^2+2N_1-r_0-r_1 \ = \ -3 \, , \quad (42)
$$

$$
\frac{{r_0}^2}{4}+\frac{{r_1}^2}{4}+(3N_1+2)r_0+(N_1+3)r_1+8{N_1}^2+8N_1 \ = \ 0 \, . \quad (43)
$$

Substituting $x=r_0+6N_1+4, \ y=r_1+2N_1+6$, we get

$$
x+y \ = \ 2{N_1}^2+10N_1+13 \, , \quad (44)
$$

$$
x^2+y^2 \ = \ 8{N_1}^2+40N_1+52 \, . \quad (45)
$$

The necessary and sufficient condition that (44) and (45) have real solutions is that

$$
\begin{aligned}
(2{N_1}^2+10N_1+13)^2 \ &\le \ 2(8{N_1}^2+40N_1+52) \\
&= \ 8(2{N_1}^2+10N_1+13) \, ,
\end{aligned}
$$

i.e.,

$$0 \;\le\; 2{N_1}^2 + 10N_1 + 13 \;\le\; 8\,.$$

The solution set of this inequality is {-1,-2,-3,-4}. We only consider N_1 = -1, or -2, since the other two are symmetric to these two.

1. $N_1 = -1$. There are two solutions of $\{r_0, r_1\}$,

$$r_0 \;=\; \frac{9 \mp \sqrt{15}}{2}, \quad r_1 \;=\; \frac{-3 \pm \sqrt{15}}{2},$$

which give two different scaling vectors α^1,

$$\begin{pmatrix} a_{-1} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} \frac{9\mp\sqrt{15}}{16} \\ \frac{13\pm\sqrt{15}}{16} \\ \frac{3\pm\sqrt{15}}{8} \\ \frac{3\mp\sqrt{15}}{8} \\ \frac{1\mp\sqrt{15}}{16} \\ \frac{-3\pm\sqrt{15}}{16} \end{pmatrix}.$$

It is easy to check that the second solution, i.e.,

$$r_0 \;=\; \frac{9 - \sqrt{15}}{2}, \quad r_1 \;=\; \frac{-3 + \sqrt{15}}{2},$$

corresponds to a continuous differentiable scaling function ϕ. And the derivative of ϕ has Hölder exponent $0.19181433454\cdots$ (see [4]).

2. $N_1 = -2$. There are also two solutions of $\{r_0, r_1\}$,

$$r_0 \;=\; \frac{17 \pm \sqrt{7}}{2}, \quad r_1 \;=\; \frac{-3 \mp \sqrt{7}}{2}.$$

These give the scaling vectors

$$\begin{pmatrix} a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \frac{1\mp\sqrt{7}}{16} \\ \frac{5\pm\sqrt{7}}{16} \\ \frac{7\pm\sqrt{7}}{8} \\ \frac{7\mp\sqrt{7}}{8} \\ \frac{1\mp\sqrt{7}}{16} \\ \frac{-3\pm\sqrt{7}}{16}. \end{pmatrix}.$$

And the numerical value of the first solution is

$$\begin{pmatrix} a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{7}}{16} \\ \frac{5+\sqrt{7}}{16} \\ \frac{7+\sqrt{7}}{8} \\ \frac{7-\sqrt{7}}{8} \\ \frac{1-\sqrt{7}}{16} \\ \frac{-3+\sqrt{7}}{16} \end{pmatrix} \approx \begin{pmatrix} -1.028594569415369e-1 \\ 4.778594569415369e-1 \\ 1.205718913883074e+0 \\ 5.442810861169262e-1 \\ -1.028594569415369e-1 \\ -2.214054305846308e-2 \end{pmatrix}$$

which is the symmetric version of α^1 we computed before using Newton's method algorithm.

5 Biorthogonal Coifman wavelet systems

In [2], A. Cohen, I. Dauchchies, and J. C. Feauveau introduced biorthogonal wavelet systems. It is well-known that compactly supported orthonormal wavelet functions can not be symmetric, except the Haar wavelet system. On the other hand, in the biorthogonal case, since we use different synthesis and analysis scaling vectors, exact reconstruction still holds, and both the synthesis and analysis wavelet functions can be symmetric. Such symmetric property enhances the performance of wavelet systems in signal processing and many other fields. For more details concerning biorthogonal wavelet systems, we refer to their original paper [2].

We summarize the relevant information here. Basically we have two scaling vectors, $\alpha = \{a_k\}$ and $\tilde{\alpha} = \{\tilde{a}_k\}$, satisfying

$$\sum_{k\in\mathbf{Z}} a_k \tilde{a}_{k+2l} \;=\; 2\delta_{0,l}\,, \quad l \in \mathbf{Z}\,. \tag{46}$$

Define

$$F_0(\xi) \;=\; \frac{1}{2}\sum_{k\in\mathbf{Z}} a_k e^{ik\xi}\,, \quad \tilde{F}_0(\xi) \;=\; \frac{1}{2}\sum_{k\in\mathbf{Z}} \tilde{a}_k e^{ik\xi}\,, \tag{47}$$

$$\widehat{\phi}(\xi) \;=\; \prod_{j=1}^{\infty} F_0(2^{-j}\xi)\,, \quad \widehat{\tilde{\phi}}(\xi) \;=\; \prod_{j=1}^{\infty} \tilde{F}_0(2^{-j}\xi)\,. \tag{48}$$

So particularly,

$$\widehat{\phi}(0) \;=\; \widehat{\tilde{\phi}}(0) \;=\; 1\,. \tag{49}$$

As before, assume both α and $\tilde{\alpha}$ have finite lengths, and

$$\sum_{k\in\mathbf{Z}} a_k \;=\; \sum_{k\in\mathbf{Z}} \tilde{a}_k \;=\; 2\,, \tag{50}$$

then again from [5] and [6], $\phi(x)$ and $\widehat{\phi}(x)$ are well-defined and have compact supports. Suppose that $\phi(x), \widehat{\phi}(x) \in L^2(\mathbf{R})$, define the wavelet functions by

$$\psi(x) \;=\; \sum_{k\in\mathbf{Z}} (-1)^k \tilde{a}_{-k-1}\phi(2x-k)\,, \quad \tilde{\psi}(x) \;=\; \sum_{k\in\mathbf{Z}} (-1)^k a_{-k-1}\tilde{\phi}(2x-k)\,.$$

It follows (see [5]) that for any $f(x), g(x) \in L^2(\mathbf{R})$,

$$\lim_{J\to\infty} \sum_{j=-J}^{J} \sum_{k\in\mathbf{Z}} < f, \psi_{j,k} >< \tilde{\psi}_{j,k}, g > = < f, g >\,,$$

or, in the weak L^2 sense,

$$f \;=\; \lim_{J\to\infty} \sum_{j=-J}^{J} \sum_{k\in\mathbf{Z}} < f, \psi_{j,k} > \tilde{\psi}_{j,k}\,, \tag{51}$$

where $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, and $\tilde{\psi}_{j,k}(x) = 2^{j/2}\tilde{\psi}(2^j x - k)$. If $\widehat{\phi}(\xi)$ and $\widehat{\tilde{\phi}}(\xi)$ decay fast at infinity,

$$|\widehat{\phi}(\xi)| \;\le\; C(1+|\xi|)^{-1/2-\epsilon}\,, \quad |\widehat{\tilde{\phi}}(\xi)| \;\le\; C(1+|\xi|)^{-1/2-\epsilon}\,,$$

for some positive numbers C and ϵ, then the above convergence (51) is strong convergence in the L^2 norm.

The orthogonal Coifman wavelet system has a natural extension to the biorthogonal wavelet system setting. For convenience, we call such systems biorthogonal Coifman wavelet systems.

Definition 5.1 *A biorthogonal wavelet system with compact support is called a* biorthogonal Coifman wavelet system *of degree N if the following two conditions are satisfied,*

1. *the vanishing moments of the scaling function $\tilde{\phi}(x)$ and the wavelet function $\tilde{\psi}(x)$ are both of degree N, i.e.,*

$$\mathrm{Mom}_p(\tilde{\phi}) \;=\; \int_{\mathbf{R}} x^p \tilde{\phi}(x)\,dx \;=\; \delta_{0,p}, \quad \textit{for } p = 0, \cdots, N, \tag{52}$$

$$\mathrm{Mom}_p(\tilde{\psi}) \;=\; \int_{\mathbf{R}} x^p \tilde{\psi}(x)\,dx \;=\; 0, \quad \textit{for } p = 0, \cdots, N, \tag{53}$$

2. the vanishing moment of the wavelet function $\psi(x)$ is of degree N,

$$\mathrm{Mom}_p(\psi) \;=\; \int_{\mathbf{R}} x^p \psi(x)\, dx \;=\; 0, \quad \textit{for } p = 0, \cdots, N, \tag{54}$$

As before, translating the vanishing moments into the conditions on the scaling vectors α and $\tilde{\alpha}$, (52), (53), and (54) are equivalent to

$$\sum_{k\in\mathbf{Z}} (2k)^p \tilde{a}_{2k} \;=\; \sum_{k\in\mathbf{Z}} (2k+1)^p \tilde{a}_{2k+1} \;=\; 0\,, \quad \text{for } p = 1, \cdots, N\,, \tag{55}$$

$$\sum_{k\in\mathbf{Z}} \tilde{a}_{2k} \;=\; \sum_{k\in\mathbf{Z}} \tilde{a}_{2k+1} \;=\; 1\,, \tag{56}$$

$$\sum_{k\in\mathbf{Z}} (-1)^k k^p a_k \;=\; 0 \quad \text{for } p = 0, \cdots, N\,. \tag{57}$$

An immediate result about the biorthogonal Coifman wavelet system is that the analysis scaling function $\phi(x)$ also has vanishing moments up to degree N.

Lemma 5.2 *For a biorthogonal Coifman wavelet system of degree N, the vanishing moments' degree of the scaling function $\phi(x)$ is also N,*

$$\mathrm{Mom}_p(\phi) \;=\; \int_{\mathbf{R}} x^p \phi(x)\, dx \;=\; \delta_{0,p}\,, \quad \textit{for } p = 0, \cdots, N\,, \tag{58}$$

i.e.,

$$\sum_{k\in\mathbf{Z}} k^p a_k \;=\; 2\delta_{0,p}\,, \quad \textit{for } p = 0, \cdots, N\,. \tag{59}$$

Proof: The bilinear relation (46) is equivalent to

$$F_0(\xi)\tilde{F}_0(-\xi) + F_0(\xi+\pi)\tilde{F}_0(-\xi+\pi) \;=\; 1\,. \tag{60}$$

From the definition of biorthogonal Coifman wavelet systems,

$$\tilde{F}_0^{(p)}(0) \;=\; \delta_{0,p}\,, \quad \tilde{F}_0^{(p)}(\pi) \;=\; 0\,, \quad \text{for } p = 0, 1, \cdots, N\,.$$

Take the p-th derivative on both side of (60) and evaluated at 0, we get

$$F_0^{(p)}(0) \;=\; \delta_{0,p}\,,$$

which is exactly (59). $\square$

Now we want to determine the biorthogonal Coifman wavelet system. We begin with the synthesis scaling vector $\tilde{\alpha}$. Like in the orthogonal case, we require $\tilde{\alpha}$ to have minimal length among all $\tilde{\alpha}$ satisfying (55) and (56). For symmetric reason, we distribute the nonzero terms of $\{\tilde{a}_k\}$ as symmetrically as possible. More precisely, $\tilde{\alpha}$ starts with $\tilde{a}_{-N}$, ends with $\tilde{a}_{N+1}$. And the linear conditions on $\{\tilde{a}_{-N}, \tilde{a}_{-N+1}, \cdots, \tilde{a}_{N+1}\}$ naturally divides into two parts, even and odd. For the even part, it is easy to see that the solution is exactly that $\tilde{a_0} = 1$ and all other even terms are zero. For the odd part, the solution always exists since the coefficient matrix is a Vandermonde's matrix. So in particular, it is a non-singular matrix. And using the determinant formula of a Vandermonde's matrix, it follows that the exact formulas for these odd coefficients are

- if N is even, $N = 2n$,

$$\tilde{a}_{2k+1} = \frac{\prod_{j\neq k, j=-n}^{n}(2j+1)}{2^N \prod_{j\neq k, j=-n}^{n}(j-k)} = \frac{(-1)^k}{2k+1}\begin{pmatrix} 2n-1 \\ n-1 \end{pmatrix}\begin{pmatrix} 2n \\ n+k \end{pmatrix}\frac{2n+1}{2^{4n-1}}$$

- if N is odd, $N = 2n-1$,

$$\tilde{a}_{2k+1} = \frac{\prod_{j\neq k, j=-n}^{n-1}(2j+1)}{2^N \prod_{j\neq k, j=-n}^{n-1}(j-k)} = \frac{(-1)^k}{2k+1}\begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix}\begin{pmatrix} 2n-1 \\ n+k \end{pmatrix}\frac{2n-1}{2^{4n-3}}$$

  here we define $\begin{pmatrix} m \\ l \end{pmatrix} := 0$ if $l > m$ or $l < 0$, which is standard in combinatorial theory.

Again, using a polynomial argument, it is easy to prove that

$$\tilde{F}_0(\xi) \;=\; \frac{1}{2}\sum_{k\in\mathbf{Z}} \tilde{a}_k e^{ik\xi} \;=\; \left(\frac{1+e^{i\xi}}{2}\right)^{N+1} \tilde{Q}(e^{i\xi}),$$

where $\tilde{Q}(x)$ is a polynomial of degree N and $\tilde{Q}(1-2x)$ is equal to the first $N+1$ terms of the polynomial

$$\left((1-2x)^N\left(\sum_{p=0}^{N}\begin{pmatrix} N+p \\ p \end{pmatrix} x^p\right)\right) .$$

As observed by H. L. Resnikoff, these synthesis scaling vectors are a smooth approximation to the scaling vector for $\mathrm{sinc}(\pi x) = \frac{\sin \pi x}{\pi x}$. We will discuss this further in a forthcoming paper [18].

This kind of synthesis scaling vector has a very special structure, namely, all even terms are zero, except $\tilde{a}_0 = 1$. And if N is odd, such a synthesis scaling vector is symmetric. The next theorem tells us how to construct the analysis scaling vector α from the synthesis scaling vector $\tilde{\alpha}$.

Theorem 5.3 *Assume $\tilde{a}_k$ to be defined as above, depending on whether N is even or odd. Set*

$$a_{2k+1} \;=\; \tilde{a}_{2k+1}\,, \tag{61}$$

and

$$a_{2k} = 2\delta_{0,k} - \sum_{l\in\mathbf{Z}} \tilde{a}_{2l+1}\tilde{a}_{2l+1-2k}\,. \tag{62}$$

Then the resulting pair α and $\tilde{\alpha}$ constitutes a biorthogonal Coifman wavelet system of degree N.

Proof: From (61) and (62), the bilinear relation (46) is satisfied. So it suffices to check that α satisfies (57). By (61), (55), and (56), it is enough to prove

$$\sum_{k\in\mathbf{Z}} (2k)^p a_{2k} \;=\; \delta_{0,p}\,, \quad \text{for } \; p = 0, 1, \cdots N\,. \tag{63}$$

We have

$$\begin{aligned}
&\sum_{k\in\mathbf{Z}} (2k)^p a_{2k} \\
&\quad= \sum_{k\in\mathbf{Z}} \left((2k)^p \left(2\delta_{0,k} - \sum_{n\in\mathbf{Z}} \tilde{a}_{2n+1}\tilde{a}_{2n+1-2k} \right) \right) \\
&\quad= 2\delta_{0,p} - \sum_{k,n\in\mathbf{Z}} (2k)^p \tilde{a}_{2n+1}\tilde{a}_{2n+1-2k} \\
&\quad= 2\delta_{0,p} - \sum_{m,n\in\mathbf{Z}} ((2n+1)-(2m+1))^p \tilde{a}_{2n+1}\tilde{a}_{2m+1} \\
&\quad= 2\delta_{0,p} - \sum_{m,n\in\mathbf{Z}} \sum_{l=0}^{p} \left((-1)^l \begin{pmatrix} p \\ l \end{pmatrix} (2n+1)^{p-l}(2m+1)^l \tilde{a}_{2n+1}\tilde{a}_{2m+1} \right) \\
&\quad= 2\delta_{0,p} - \sum_{l=0}^{p} (-1)^l \left(\begin{pmatrix} p \\ l \end{pmatrix} \sum_{m\in\mathbf{Z}} (2m+1)^l \tilde{a}_{2m+1} \sum_{n\in\mathbf{Z}} (2n+1)^{p-l} \tilde{a}_{2n+1} \right)
\end{aligned}$$

$$
\begin{aligned}
&= 2\delta_{0,p} - \sum_{l=0}^{p}(-1)^l \left(\begin{pmatrix} p \\ l \end{pmatrix} \sum_{m\in\mathbf{Z}} (2m+1)^l \tilde{a}_{2m+1}\delta_{l,p} \right) \\
&= 2\delta_{0,p} - (-1)^p \sum_{m\in\mathbf{Z}} (2m+1)^p \tilde{a}_{2m+1} \\
&= 2\delta_{0,p} - (-1)^p \delta_{0,p} \\
&= \delta_{0,p}
\end{aligned}
$$

□

Remark 1 One can prove (see [18]) that all the a_k's and $\tilde{a}_k$'s are dyadic fractions, i.e., all of them is of the form $\frac{2p+1}{2^q}$, for some $p, q \in \mathbf{Z}$. Such property will make the biorthogonal Coifman wavelet systems easier and faster to implement on a computer.

An immediate corollary is that the even coefficients of the analysis filter are symmetric.

Corollary 5.4 *In the biorthogonal Coifman wavelet system,* $a_{2k} = a_{-2k}$.

Since the odd degree synthesis scaling vector is symmetric, we derive the following theorem.

Theorem 5.5 *If N is odd, then the biorthogonal Coifman wavelet system of degree N is symmetric, i.e.,*

$$a_k = a_{-k}, \quad \tilde{a}_k = \tilde{a}_{-k}.$$

Here we list in Table 1 the coefficients of the analysis, synthesis scaling vectors, with the degree of the biorthogonal Coifman wavelet system $N = 0$, 1, 2, 3 and 4.

And we include some illustrations of the biorthogonal Coifman wavelet systems in Fig. 3, 4 and 5.

In [2], A. Cohen, I. Daubechies, and J. C. Feauveau proved that $\psi_{j,k}(x)$, $\tilde{\psi}_{j,k}(x)$ constitute two dual Riesz bases if and only if

$$\int_{\mathbf{R}} \phi(x)\tilde{\phi}(x-k)\,dx = \delta_{0,k}. \tag{64}$$

Now we will use Lawton's condition to verify (64) for the biorthogonal Coifman wavelet system.

Table 1: The Coefficients of Biorthogonal Coifman Wavelet Systems

N	a_k	$\tilde{a}_k$
N = 0	a_0 = 1	$\tilde{a}_0$ = 1
	a_1 = 1	$\tilde{a}_1$ = 1
N = 1	a_{-2} = -1/4	
	a_{-1} = 1/2	$\tilde{a}_{-1}$ = 1/2
	a_0 = 3/2	$\tilde{a}_0$ = 1
	a_1 = 1/2	$\tilde{a}_1$ = 1/2
	a_2 = -1/4	$\tilde{a}_2$ = 0
	a_3 = 0	
N = 2	a_{-4} = 3/64	
	a_{-3} = 0	
	a_{-2} = -3/16	$\tilde{a}_{-2}$ = 0
	a_{-1} = 3/8	$\tilde{a}_{-1}$ = 3/8
	a_0 = 41/32	$\tilde{a}_0$ = 1
	a_1 = 3/4	$\tilde{a}_1$ = 3/4
	a_2 = -3/16	$\tilde{a}_2$ = 0
	a_3 = -1/8	$\tilde{a}_3$ = -1/8
	a_4 = 3/64	
	a_5 = 0	
N = 3	a_{-6} = -1/256	
	a_{-5} = 0	
	a_{-4} = 9/128	
	a_{-3} = -1/16	$\tilde{a}_{-3}$ = -1/16
	a_{-2} = -63/256	$\tilde{a}_{-2}$ = 0
	a_{-1} = 9/16	$\tilde{a}_{-1}$ = 9/16
	a_0 = 87/64	$\tilde{a}_0$ = 1
N = 3	a_1 = 9/16	$\tilde{a}_1$ = 9/16
	a_2 = -63/256	$\tilde{a}_2$ = 0
	a_3 = -1/16	$\tilde{a}_3$ = -1/16
	a_4 = 9/128	$\tilde{a}_4$ = 0
	a_5 = 0	
	a_6 = -1/256	
	a_7 = 0	
N = 4	a_{-8} = 15/16384	
	a_{-7} = 0	
	a_{-6} = -35/2048	
	a_{-5} = 0	
	a_{-4} = 345/4096	$\tilde{a}_{-4}$ = 0
	a_{-3} = -5/128	$\tilde{a}_{-3}$ = -5/128
	a_{-2} = -405/2048	$\tilde{a}_{-2}$ = 0
	a_{-1} = 15/32	$\tilde{a}_{-1}$ = 15/32
	a_0 = 10317/8192	$\tilde{a}_0$ = 1
	a_1 = 45/64	$\tilde{a}_1$ = 45/64
	a_2 = -405/2048	$\tilde{a}_2$ = 0
	a_3 = -5/32	$\tilde{a}_3$ = -5/32
	a_4 = 345/4096	$\tilde{a}_4$ = 0
	a_5 = 3/128	$\tilde{a}_5$ = 3/128
	a_6 = -35/2048	
	a_7 = 0	
	a_8 = 15/16384	
	a_9 = 0	

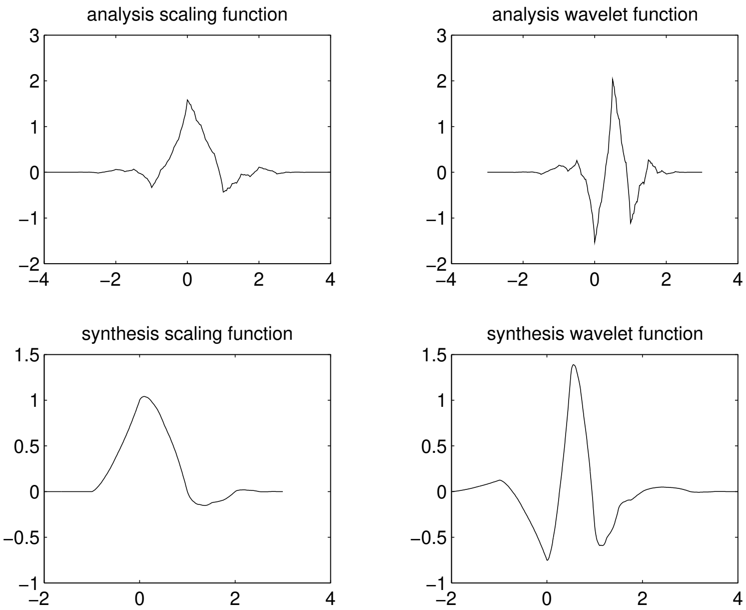

Figure 3: the biorthogonal Coifman wavelet system of degree 2

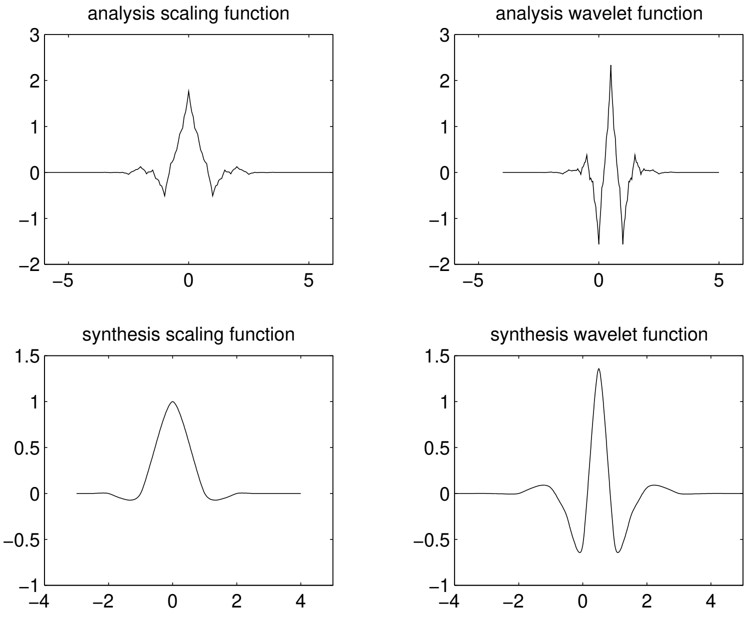

Figure 4: the biorthogonal Coifman wavelet system of degree 3

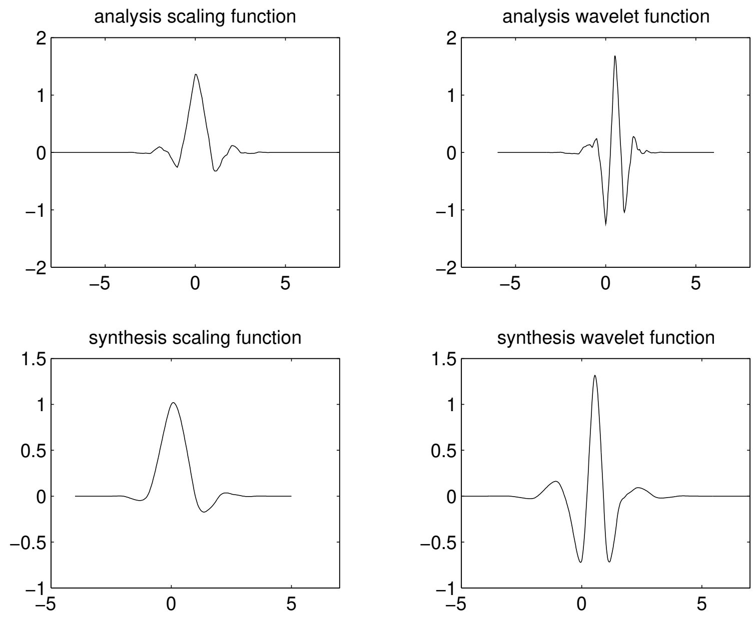

Figure 5: the biorthogonal Coifman wavelet system of degree 4

Lemma 5.6 *For a biorthogonal Coifman wavelet system, define a multiresolution matrix* $T = (t_{k,l}), -3N \leq k, l \leq 3N$,

$$t_{k,l} \;=\; \frac{1}{2} \sum_{m \in \mathbf{Z}} a_m \tilde{a}_{m+l-2k}\,.$$

If T *has 1 as a nondegenerate eigenvalue, then*

$$\int_{\mathbf{R}} \phi(x)\tilde{\phi}(x-k)\,dx \;=\; \delta_{0,k}\,.$$

Proof: Define

$$c_k \;=\; \int_{\mathbf{R}} \phi(x)\tilde{\phi}(x-k)dx\,.$$

Note that both $\phi(x)$ and $\tilde{\phi}(x)$ have compact support, then it follows that $c_k = 0$ for $|k| > 3N$. We have

$$c_k \;=\; \int_{\mathbf{R}} \left(\sum_{m \in \mathbf{Z}} a_m \phi(2x-m) \right) \left(\sum_{n \in \mathbf{Z}} \tilde{a}_n \tilde{\phi}(2x-2k-n) \right)\,dx$$

$$
\begin{aligned}
&= \sum_{m,n\in\mathbf{Z}} a_m \tilde{a}_n \int_{\mathbf{R}} \phi(2x-m)\tilde{\phi}(2x-2k-n)\,dx \\
&= \frac{1}{2}\sum_{m,n\in\mathbf{Z}} a_m \tilde{a}_n c_{2k-m+n} \\
&= \frac{1}{2}\sum_{m,l\in\mathbf{Z}} a_m \tilde{a}_{m+l-2k} c_l \\
&= \sum_{l\in\mathbf{Z}} t_{k,l} c_l \,,
\end{aligned}
$$

i.e.,

$$
Tc \;=\; c, \quad \text{where } c = (c_{-3N}, c_{-3N+1}, \cdots, c_{3N})'.
$$

Thus c is an eigenvector of the multiresolution matrix T with eigenvalue 1. Since 1 is a nondegenerate eigenvalue, and $(0,\cdots,0,1,0,\cdots,0)'$ is also an eigenvector with eigenvalue 1, we have $c = \gamma(0,\cdots,0,1,0,\cdots,0)'$, or $c_k = \gamma\delta_{0,k}$ for some constant γ. So all we need to do is to prove $\gamma = 1$. Since

$$
\begin{aligned}
c_k &= \int_{\mathbf{R}} \phi(x)\tilde{\phi}(x-k)dx \\
&= \int_{\mathbf{R}} \widehat{\phi}(\xi)\overline{\widehat{\tilde{\phi}}(\xi)} e^{ik\xi}dx \\
&= \int_0^{2\pi} \left(\sum_{l\in\mathbf{Z}} \widehat{\phi}(\xi+2l\pi)\overline{\widehat{\tilde{\phi}}(\xi+2l\pi)} \right) e^{ik\xi}dx \,,
\end{aligned}
$$

it follows that

$$
\sum_{l\in\mathbf{Z}} \widehat{\phi}(\xi+2l\pi)\overline{\widehat{\tilde{\phi}}(\xi+2l\pi)} \;=\; (2\pi)^{-1}\gamma \quad a.e.
$$

From (49), we know that

$$
\widehat{\phi}(0) \;=\; \widehat{\tilde{\phi}}(0) \;=\; 1\,.
$$

And since $F_0(\pi) = \tilde{F}_0(\pi) = 0$, which is from the vanishing moments conditions of $\tilde{\psi}(x)$ and $\psi(x)$, (48) implies $\widehat{\phi}(2l\pi) = \widehat{\tilde{\phi}}(2l\pi) = 0$ for $l \neq 0$. It follows that $\gamma = 1$. $\square$

Based on this criterion, it is easy to check that for the biorthogonal Coifman wavelet systems of degree 0, 1, 2, 3, and 4, $\psi(x)$ and $\tilde{\psi}(x)$ constitute dual Riesz bases. We will give a theoretical treatment of this problem in the general case in [18].

We can also calculate the asymptotic growth of the Sobolev smoothness, with respect to the degree of vanishing moments. It is a direct application of T. Eirola's method (see [7]), i.e., the Sobolev smoothness is determined by the spectral radius of the multiresolution operator T_α. And it turns out that for the synthesis scaling function of degree N, its Sobolev smoothness is

$$s_N \;=\; \left(1 - \frac{log(3)}{2log(2)}\right) N + O\left(log(N)\right) \;\approx\; 0.2075N.$$

6 Comparison with Daubechies wavelet systems

In this section, we give a short comparison of the orthogonal and biorthogonal Coifman wavelet systems with the Daubechies wavelet systems.

With the same degree N (the degree of the vanishing moments of the wavelet function), the number of the nonzero coefficients of the scaling vector of the Daubechies wavelet system is $2N + 2$, and the one of the orthogonal Coifman wavelet system scaling vector is $3N + 3$ if N is odd and $3N + 2$ if N is even; the synthesis scaling vector in the biorthogonal Coifman wavelet system has $N + 2$ nonzero coefficients, and the analysis scaling vector has $3N + 2$ nonzero coefficients. So the total number of the nonzero coefficients of the analysis and synthesis scaling vectors of the biorthogonal Coifman wavelet system is $4N + 4$, which is comparable to the Daubechies wavelet systems. For the orthogonal Coifman wavelet system, it is $6N + 4$ when N is even and $6N + 6$ when N is odd. about one half longer. Table 2 gives the number of the nonzero coefficients of the scaling vectors of these three wavelet systems.

As for symmetry, the Coifman wavelet systems (both of the orthogonal and biorthogonal ones) are more symmetric than the Daubechies wavelet systems. And Theorem 5.5 states that the biorthogonal Coifman wavelet system is really symmetric when N is odd. Even if N is even, the biorthogonal Coifman wavelet system is very close to be symmetric. The graphics in

Table 2: The Number of the Nonzero Coefficients of the Scaling Vectors of Three Wavelet Systems

N	Daubechies wavelet system	orthogonal Coifman wavelet system	biorthogonal Coifman wavelet system	
			analysis scaling vector	synthesis scaling vector
2	6	8	8	4
3	8	12	11	5
4	10	14	14	6
5	12	18	17	7
6	14	20	20	8
7	16	24	23	9
8	18	26	26	10
9	20	30	29	11

Section 5 show such "closed" symmetry.

To calculate the Sobolev smoothness of the orthogonal and biorthogonal Coifman wavelet systems, we use the program provided by Peter Heller in Aware Inc. And the Sobolev smoothness of these orthogonal and biorthogonal Coifman wavelet systems are given in Table 3.

In Fig. 6, the top curve is the synthesis scaling function of the biorthogonal Coifman wavelet system, the orthogonal Coifman wavelet system is the second, and the third is the Daubechies wavelet system, the last one is the analysis scaling function of the biorthogonal Coifman wavelet system.

Acknowledgment

We would like to thank Peter Heller and Howard L. Resnikoff in Aware Inc., Lucas Monzon of Yale University, Richard A. Tapia and Xiaodong Zhou of the Computational Mathematics Laboratory of Rice University, Mladen V. Wickerhauser of Washington University, and other colleagues in the Computational Mathematics Laboratory at Rice, for many helpful discussions and valuable assistance. After completing this work, we learned that some results in Section 2 and 3 have also been derived by W. Sweldens and R. Piessens in [21].

Table 3: Sobolev Smoothness of Three Wavelet Systems

N	Daubechies wavelet system	orthogonal Coifman wavelet system	biorthogonal Coifman wavelet system	
			analysis scaling function	synthesis scaling function
2	1.415	1.773	1.200	1.839
3	1.775	1.836	1.179	2.441
4	2.096	2.449	1.773	2.714
5	2.388	2.485	1.772	3.175
6	2.658	3.025	2.292	3.409
7	2.914	3.046	2.305	3.793
8	3.161	3.557	2.793	4.004
9	3.402	3.566	2.815	4.344

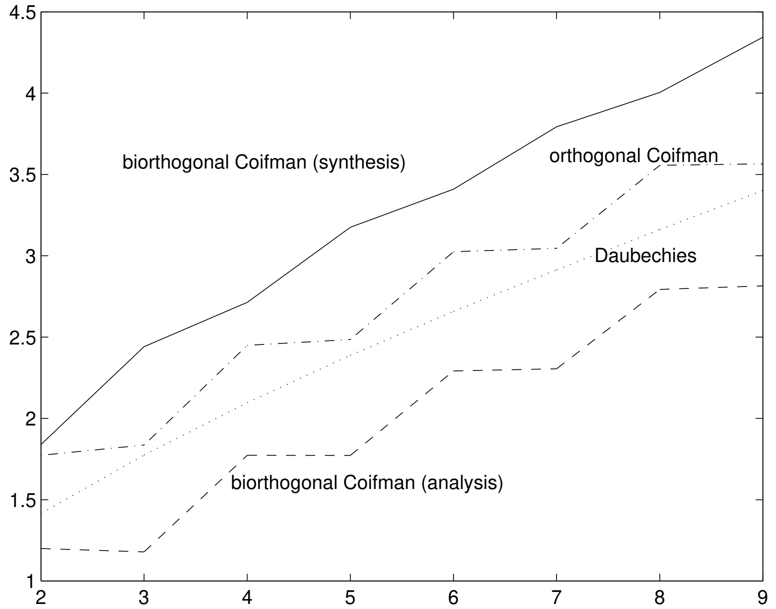

Figure 6: the comparison of the Sobolev smoothness

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Jun Tian
Department of Mathematics
Rice University
P. O. Box 1892
Houston, Texas 77251
juntian@rice.edu

Raymond O. Wells, Jr.
Department of Mathematics
Rice University
P. O. Box 1892
Houston, Texas 77251
wells@rice.edu