Vanishing Moments and Biorthogonal Wavelet Systems

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Abstract

We consider biorthogonal wavelet systems with the vanishing of moments equally distributed between scaling functions and wavelet functions. Such wavelet systems provide good wavelet sampling approximation with exponential decay. Using a time domain design method, the closed form (minimum length) solution is obtained. An attractive feature behind it is that all filter coefficients are dyadic rational, which means that we can implement a very fast multiplication-free discrete wavelet transform, which consists of only addition and shift operations, on digital computers. Moreover these wavelet systems converge to the sinc wavelet system, and half of them are symmetric.

1 Introduction

In a wavelet system, vanishing moments are a necessary condition for the smoothness of the wavelet function [1]. In 1988, I. Daubechies, in her celebrated paper [2], introduced a class of compactly supported orthonormal wavelet systems in general, as well as a family with growing smoothness for increasing support, *Daubechies wavelet systems*. Daubechies wavelet systems put all freedom of parameters into vanishing moments of the wavelet function. One year later, R. Coifman suggested that it might be worthwhile to construct orthogonal wavelet systems with vanishing moments not only for wavelet functions, but also for scaling functions. Such orthogonal wavelet systems with vanishing moments equally distributed between scaling functions and wavelet functions (which are called *Coiflets* by Daubechies in [3]) are closer to being symmetric than Daubechies wavelet systems, which suggests that with vanishing moments on scaling functions, we might be able to design more symmetric wavelet systems. This is important since in signal processing, symmetry corresponds to linear phase. A method to construct Coiflets was proposed in [3] but the general existence problem is still open.

In [4], we presented a sampling approximation theorem. It says that for an optimal wavelet sampling approximation, vanishing moments should be equally distributed between the scaling function and the wavelet function, and the approximation error converges to zero with exponential decay.

Thus vanishing moments of the wavelet function is necessary for the smoothness of the wavelet system; vanishing moments of the scaling function will improve the symmetry; and equal vanishing moments distribution will provide an optimal wavelet sampling approximation with exponential decay. Based on these observations, we introduce *biorthogonal Coifman wavelet systems*, a family of compactly supported biorthogonal wavelet systems with the vanishing of moments equally distributed between scaling functions and wavelet functions. Biorthogonal Coifman wavelet systems provide multiplication-free discrete wavelet transform, and converge to the noncompactly supported sinc wavelet system. We want to thank Howard Resnikoff and Xiaodong Zhou for very helpful discussion concerning the convergence property.

2 A Sampling Approximation Theorem

The sampling approximation theorem is a quite general result. It is not only valid for a wavelet system (orthogonal, semiorthogonal, or biorthogonal), but also holds for an L^2 function which satisfies a two-scale difference equation.

Theorem 1. (Sampling Approximation Theorem) *If an $L^2(\mathbf{R})$ function $\phi(x)$ is a solution of the two-scale difference equation*

$$\phi(x) \;=\; \sum_{k\in\mathbf{Z}} a_k \phi(2x-k)\,,$$

and it is normalized

$$\int_{-\infty}^{\infty} \phi(x)\,dx \;=\; 1\,,$$

where $\{a_k\}$ is a real sequence with only finite nonzero elements, satisfying

$$\sum_{k\in\mathbf{Z}} (2k)^n a_{2k} \;=\; \sum_{k\in\mathbf{Z}} (2k+1)^n a_{2k+1} \;=\; 0\,,$$

for $n = 1, \cdots, N$, *and*

$$\sum_{k\in\mathbf{Z}} a_{2k} \;=\; \sum_{k\in\mathbf{Z}} a_{2k+1} \;=\; 1\,,$$

for $f(x) \in C_0^{N,1}(\mathbf{R}), j \in \mathbf{N}$, define

$$S^j(f)(x) \;:=\; 2^{-j/2} \sum_{k\in\mathbf{Z}} f\left(\frac{k}{2^j}\right) \phi_{j,k}(x)\,,$$

where $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$. *Then*

$$||f(x) - S^j(f)(x)||_{L^2} \ \leq \ C 2^{-j(N+1)},$$

where C *is a constant, depending only on* $f(x)$ *and* $\{a_k\}$.
If in addition, $\phi(x) \in C^m(\mathbf{R})$, *where* $m \in \mathbf{Z}, 0 \leq m \leq N$, *then*

$$||f(x) - S^j(f)(x)||_{H^m} \ \leq \ C 2^{-j(N+1-m)},$$

where C *is a constant, depending only on* $f(x)$ *and* $\{a_k\}$.

In the case of a wavelet system, the linear conditions on the sequence $\{a_k\}$ (which is the scaling vector) in the above theorem are exactly the vanishing moments conditions on the scaling function and the wavelet function [5, 6]. Thus, with vanishing moments equally distributed between the scaling function and the wavelet function, if we use sample values of a smooth function as scaling function coefficients at a fine scale, then the resulting wavelet series approximates the underlying smooth function with increasing accuracy in L^2 (or H^m) type norm as the number N of vanishing moments gets larger or the scale j gets larger (which corresponds to a finer scale). In addition, for an optimal wavelet sampling approximation, vanishing moments should be equally distributed between the scaling function and the wavelet function. The proof can be found in [4].

3 Biorthogonal Coifman Wavelet Systems

A big advantage of biorthogonal wavelet systems [7, 8] over orthonormal wavelet systems is that biorthogonal systems can be symmetric, while orthonormal systems can't, except for the *Haar wavelet system*.

3.1 Definition

Definition 2. *A biorthogonal wavelet system with compact support is called a* biorthogonal Coifman wavelet system *of degree* N *if the following two conditions are satisfied,*

- *the vanishing moments of the synthesis scaling function* $\tilde{\phi}(x)$ *and wavelet function* $\tilde{\psi}(x)$ *are of degree* N, *i.e.,*

$$\int_{-\infty}^{\infty} x^n \tilde{\phi}(x)\,dx \ = \ 0\,, \quad \text{for}\, n = 1, \cdots, N\,,$$

$$\int_{-\infty}^{\infty} x^n \tilde{\psi}(x)\,dx \ = \ 0\,, \quad \text{for}\, n = 0, \cdots, N\,.$$

- *the vanishing moment of the analysis wavelet function* $\psi(x)$ *is of degree* N,

$$\int_{-\infty}^{\infty} x^n \psi(x)\,dx \ = \ 0\,, \quad \text{for}\, n = 0, \cdots, N\,.$$

Note that in the definition of the biorthogonal Coifman wavelet system, although there is no vanishing moment requirement on the analysis scaling function $\phi(x)$, it turns out that $\phi(x)$ also has vanishing moments up to degree N, because of the perfect reconstruction condition

$$\sum_{k\in\mathbf{Z}} a_k \tilde{a}_{k+2l} \;=\; 2\delta_{0,l}\,, \quad \forall l \in \mathbf{Z}\,.$$

Lemma 3. *For a biorthogonal Coifman wavelet system of degree N, the vanishing moments' degree of the analysis scaling function $\phi(x)$ is also N,*

$$\int_{-\infty}^{\infty} x^n \phi(x)\, dx \;=\; 0\,, \quad \text{for}\, n = 1, \cdots, N\,.$$

The proof of Lemma 3 can be found in [9]. So based on Theorem 1, biorthogonal Coifman wavelet systems will provide very good wavelet sampling approximation.

3.2 Construction

Using a time domain design method, the closed form solutions of the minimum length biorthogonal Coifman wavelet systems of all degrees are obtained.

Theorem 4. *The minimum length biorthogonal Coifman wavelet systems of degree N have the analysis scaling vector $\{a_k\}$ and the synthesis scaling vector $\{\tilde{a}_k\}$ of the form*

$$\tilde{a}_0 \;=\; 1\,, \quad \tilde{a}_{2k} \;=\; 0\,\text{when}\, k \neq 0\,,$$

- *if N is even, $N = 2n$,*

$$a_{2k+1} \;=\; \tilde{a}_{2k+1} \;=\; \frac{(-1)^k(2n+1)}{2^{4n-1}(2k+1)} \binom{2n-1}{n-1} \binom{2n}{n+k}$$

- *if N is odd, $N = 2n-1$,*

$$a_{2k+1} \;=\; \tilde{a}_{2k+1} \;=\; \frac{(-1)^k(2n-1)}{2^{4n-3}(2k+1)} \binom{2n-2}{n-1} \binom{2n-1}{n+k}$$

and

$$a_{2k} \;=\; 2\delta_{0,k} - \sum_{l\in\mathbf{Z}} \tilde{a}_{2l+1}\tilde{a}_{2l+1-2k}\,.$$

The scaling vectors of the minimum length biorthogonal Coifman wavelet systems with degrees $N = 0, 1, 2, 3$ and 4 are listed in Table 1.

Table 1. The Coefficients of Biorthogonal Coifman Wavelet Systems

N	a_k	$\tilde{a}_k$
$N=0$	a_0 = 1	$\tilde{a}_0$ = 1
	a_1 = 1	$\tilde{a}_1$ = 1
	a_{-2} = -1/4	
	a_{-1} = 1/2	$\tilde{a}_{-1}$ = 1/2
$N=1$	a_0 = 3/2	$\tilde{a}_0$ = 1
	a_1 = 1/2	$\tilde{a}_1$ = 1/2
	a_2 = -1/4	
	a_{-4} = 3/64	
	a_{-3} = 0	
	a_{-2} = -3/16	
	a_{-1} = 3/8	$\tilde{a}_{-1}$ = 3/8
$N=2$	a_0 = 41/32	$\tilde{a}_0$ = 1
	a_1 = 3/4	$\tilde{a}_1$ = 3/4
	a_2 = -3/16	$\tilde{a}_2$ = 0
	a_3 = -1/8	$\tilde{a}_3$ = -1/8
	a_4 = 3/64	
	a_{-6} = -1/256	
	a_{-5} = 0	
	a_{-4} = 9/128	
$N=3$	a_{-3} = -1/16	$\tilde{a}_{-3}$ = -1/16
	a_{-2} = -63/256	$\tilde{a}_{-2}$ = 0
	a_{-1} = 9/16	$\tilde{a}_{-1}$ = 9/16
	a_0 = 87/64	$\tilde{a}_0$ = 1
	a_1 = 9/16	$\tilde{a}_1$ = 9/16
	a_2 = -63/256	$\tilde{a}_2$ = 0
$N=3$	a_3 = -1/16	$\tilde{a}_3$ = -1/16
	a_4 = 9/128	
	a_5 = 0	
	a_6 = -1/256	
	a_{-8} = 15/16384	
	a_{-7} = 0	
	a_{-6} = -35/2048	
	a_{-5} = 0	
	a_{-4} = 345/4096	
	a_{-3} = -5/128	$\tilde{a}_{-3}$ = -5/128
	a_{-2} = -405/2048	$\tilde{a}_{-2}$ = 0
	a_{-1} = 15/32	$\tilde{a}_{-1}$ = 15/32
$N=4$	a_0 = 10317/8192	$\tilde{a}_0$ = 1
	a_1 = 45/64	$\tilde{a}_1$ = 45/64
	a_2 = -405/2048	$\tilde{a}_2$ = 0
	a_3 = -5/32	$\tilde{a}_3$ = -5/32
	a_4 = 345/4096	$\tilde{a}_4$ = 0
	a_5 = 3/128	$\tilde{a}_5$ = 3/128
	a_6 = -35/2048	
	a_7 = 0	
	a_8 = 15/16384	

3.3 Multiplication-free Discrete Wavelet Transform

As it can be seen in Table 1, the scaling vectors $\{a_k\}$ and $\{\tilde{a}_k\}$ are all dyadic rational, i.e., all the nonzero elements in the scaling vectors are of the form $(2p+1)/2^q$, for some integers p and q. Actually this assertion is true for scaling vectors of the minimum length biorthgonal Coifman wavelet systems of all degrees. Thus we have obtained a family of biorthogonal wavelet systems on which we can implement a very fast multiplication-free discrete wavelet transform [10], which consists of only addition and shift operations, on digital computers. This is one of the main advantage of biorthogonal Coifman wavelet systems over other widely used irrational wavelet systems, like the Cohen-Daubechies-Feauveau 9-7 biorthogonal filters (CDF-9-7) [7].

Theorem 5. *In a biorthogonal Coifman wavelet system of degree N, the scaling vectors are dyadic rationals, i.e., $\forall k \in \mathbf{Z}$, there must exist four integers*

p_1, p_2, q_1, q_2, *such that*

$$a_k = \frac{2p_1+1}{2^{q_1}}, \quad \tilde{a}_k = \frac{2p_2+1}{2^{q_2}},$$

whenever a_k *or* $\tilde{a}_k$ *are nonzero.*

Proof. It is sufficient to prove only $\tilde{a}_{2k+1}$ is dyadic rational since the addition, subtraction, and multiplication are closed operations in dyadic rationals.

1. If N is even, $N = 2n$, for nonzero $\tilde{a}_{2k+1}$, we have $-n \le k \le n$. When $k = -1, 0$, or n, the proof is trivial. Let's first look at $0 < k < n$. Since

$$\begin{aligned}\tilde{a}_{2k+1} &= \frac{(-1)^k}{2k+1}\begin{pmatrix} 2n-1 \\ n-1 \end{pmatrix}\begin{pmatrix} 2n \\ n+k \end{pmatrix}\frac{2n+1}{2^{4n-1}} \\ &= \frac{(-1)^k}{2^{4n-1}(2k+1)} \cdot \frac{(2n-1)!(2n+1)!}{(n-1)!n!(n+k)!(n-k)!},\end{aligned}$$

(a) If $2k+1$ is a prime number, from number theory, it suffices to show that

$$\left\lfloor \frac{2n-1}{2k+1} \right\rfloor + \left\lfloor \frac{2n+1}{2k+1} \right\rfloor > \left\lfloor \frac{n-1}{2k+1} \right\rfloor + \left\lfloor \frac{n}{2k+1} \right\rfloor + \left\lfloor \frac{n+k}{2k+1} \right\rfloor + \left\lfloor \frac{n-k}{2k+1} \right\rfloor \tag{3.1}$$

where $\lfloor \cdot \rfloor$ is the integer part of a real number. Set $n-k = t(2k+1)+r$, where $t, r \in \mathbf{Z}, t \ge 0, 0 \le r \le 2k$.

(i) If $r = 0$, then

$$\begin{aligned} n+k &= t(2k+1)+2k, \\ 2n+1 &= (2t+1)(2k+1). \end{aligned}$$

So

$$\left\lfloor \frac{2n+1}{2k+1} \right\rfloor = 2t+1 = \left\lfloor \frac{n-k}{2k+1} \right\rfloor + \left\lfloor \frac{n+k}{2k+1} \right\rfloor + 1.$$

We know that

$$\left\lfloor \frac{2n-1}{2k+1} \right\rfloor \ge \left\lfloor \frac{n-1}{2k+1} \right\rfloor + \left\lfloor \frac{n}{2k+1} \right\rfloor,$$

Thus (3.1) follows immediately.

(ii) If $1 \le r \le k$, then

$$\begin{aligned} n &= t(2k+1)+(k+r), \\ n-1 &= t(2k+1)+(k+r-1), \\ 2n-1 &= (2t+1)(2k+1)+(2r-2). \end{aligned}$$

Thus

$$\left\lfloor \frac{2n-1}{2k+1} \right\rfloor = 2t+1 = \left\lfloor \frac{n-1}{2k+1} \right\rfloor + \left\lfloor \frac{n}{2k+1} \right\rfloor + 1.$$

(iii) If $k+1 \le r \le 2k$, then

$$n+k \;=\; (t+1)(2k+1)+(r-1)\,,$$
$$2n+1 \;=\; (2t+2)(2k+1)+(2r-2k-1)\,.$$

So

$$\left\lfloor \frac{2n+1}{2k+1} \right\rfloor \;=\; 2t+2 \;=\; \left\lfloor \frac{n-k}{2k+1} \right\rfloor + \left\lfloor \frac{n+k}{2k+1} \right\rfloor + 1\,.$$

(b) If $2k+1$ is not a prime number, assume $2a+1$ is a prime factor of $2k+1$, and $2k+1=(2a+1)(2b+1)$, i.e., $k=2ab+a+b$. Then it suffices to show that

$$\left\lfloor \frac{2n-1}{2a+1} \right\rfloor + \left\lfloor \frac{2n+1}{2a+1} \right\rfloor > \left\lfloor \frac{n-1}{2a+1} \right\rfloor + \left\lfloor \frac{n}{2a+1} \right\rfloor + \left\lfloor \frac{n+k}{2a+1} \right\rfloor + \left\lfloor \frac{n-k}{2a+1} \right\rfloor \tag{3.2}$$

Since

$$\begin{aligned}\left\lfloor \frac{n+k}{2a+1} \right\rfloor + \left\lfloor \frac{n-k}{2a+1} \right\rfloor \;&=\; \left\lfloor \frac{n+2ab+a+b}{2a+1} \right\rfloor + \left\lfloor \frac{n-2ab-a-b}{2a+1} \right\rfloor \\ &=\; \left\lfloor \frac{n+a}{2a+1} \right\rfloor + \left\lfloor \frac{n-a}{2a+1} \right\rfloor\end{aligned}$$

the inequality (3.2) is exactly (3.1) when replacing k with a.

Now suppose $-n \le k \le -2$. Set $l=-k, 2 \le l \le n$. We want to show that (as before, if $2l-1$ is not prime, it can be converted to the following inequality)

$$\left\lfloor \frac{2n-1}{2l-1} \right\rfloor + \left\lfloor \frac{2n+1}{2l-1} \right\rfloor \;>\; \left\lfloor \frac{n-1}{2l-1} \right\rfloor + \left\lfloor \frac{n}{2l-1} \right\rfloor + \left\lfloor \frac{n+l}{2l-1} \right\rfloor + \left\lfloor \frac{n-l}{2l-1} \right\rfloor \tag{3.3}$$

Set $n-l=t(2l-1)+r$, where $t,r \in \mathbf{Z}, t \ge 0, 0 \le r \le 2l-2$.

(a) If $0 \le r \le l-2$, then

$$n \;=\; t(2l-1)+(l+r)\,,$$
$$n-1 \;=\; t(2l-1)+(l+r-1)\,,$$
$$2n-1 \;=\; (2t+1)(2l-1)+2r\,.$$

So

$$\left\lfloor \frac{2n-1}{2l-1} \right\rfloor \;=\; 2t+1 \;=\; \left\lfloor \frac{n-1}{2l-1} \right\rfloor + \left\lfloor \frac{n}{2l-1} \right\rfloor + 1\,.$$

Since

$$\left\lfloor \frac{2n+1}{2l-1} \right\rfloor \;\ge\; \left\lfloor \frac{n+l}{2l-1} \right\rfloor + \left\lfloor \frac{n-l}{2l-1} \right\rfloor\,,$$

the inequality (3.3) follows.

(b) If $l-1 \le r \le 2l-3$, then

$$n+l \;=\; (t+1)(2l-1)+(r+1)\,,$$

$$2n+1 \;=\; (2t+2)(2l-1)+(2r-2l+3)\,.$$

So

$$\left\lfloor \frac{2n+1}{2l-1} \right\rfloor \;=\; 2t+2 \;=\; \left\lfloor \frac{n+l}{2l-1} \right\rfloor + \left\lfloor \frac{n-l}{2l-1} \right\rfloor + 1\,.$$

(c) If $r = 2l-2$, then

$$n+l \;=\; (t+2)(2l-1)\,,$$

$$2n+1 \;=\; (2t+3)(2l-1)\,.$$

So

$$\left\lfloor \frac{2n+1}{2l-1} \right\rfloor \;=\; 2t+3 \;=\; \left\lfloor \frac{n+l}{2l-1} \right\rfloor + \left\lfloor \frac{n-l}{2l-1} \right\rfloor + 1\,.$$

2. If N is odd, $N = 2n-1$,

$$\tilde{a}_{2k+1} \;=\; \frac{(-1)^k}{2k+1} \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix} \begin{pmatrix} 2n-1 \\ n+k \end{pmatrix} \frac{2n-1}{2^{4n-3}}\,.$$

Since $\tilde{a}_{2k+1} = \tilde{a}_{-2k-1}$, it will be sufficient to only prove for the case $1 \le k \le n-2$. Let $n-k-1 = t(2k+1)+r$, where $t, r \in \mathbf{Z}, t \ge 0, 0 \le r \le 2k$.

(a) If $r = 0$, then

$$2n-1 \;=\; (2t+1)(2k+1)\,,$$

i.e.,

$$(2k+1)|(2n-1)\,.$$

(b) If $1 \le r \le 2k$, we will prove the inequality (again, if $2k+1$ is not prime, it can be reduced to the following inequality)

$$\left\lfloor \frac{2n-2}{2k+1} \right\rfloor + \left\lfloor \frac{2n-1}{2k+1} \right\rfloor > \left\lfloor \frac{n-1}{2k+1} \right\rfloor + \left\lfloor \frac{n-1}{2k+1} \right\rfloor + \left\lfloor \frac{n-k-1}{2k+1} \right\rfloor + \left\lfloor \frac{n+k}{2k+1} \right\rfloor$$

(i) If $1 \le r \le k$, then

$$n-1 \;=\; t(2k+1)+(k+r)\,,$$

$$2n-2 \;=\; (2t+1)(2k+1)+(2r-1)\,.$$

So

$$\left\lfloor \frac{2n-2}{2k+1} \right\rfloor \;=\; 2t+1 \;=\; \left\lfloor \frac{n-1}{2k+1} \right\rfloor + \left\lfloor \frac{n-1}{2k+1} \right\rfloor + 1\,.$$

(ii) If $k+1 \leq r \leq 2k$, then

$$
\begin{aligned}
n+k &= (t+1)(2k+1)+r\,, \\
2n-1 &= (2t+2)(2k+1)+(2r-2k-1)\,.
\end{aligned}
$$

So

$$
\left\lfloor \frac{2n-1}{2k+1} \right\rfloor = 2t+2 = \left\lfloor \frac{n-k-1}{2k+1} \right\rfloor + \left\lfloor \frac{n+k}{2k+1} \right\rfloor + 1\,.
$$

3.4 Convergence to Sinc Wavelet System

The *sinc wavelet system* is a basic wavelet system whose scaling vector $\{a_k^{sinc}, k \in \mathbf{Z}\}$ is defined by

$$
a_{2k}^{sinc} = \delta_{0,k}\,, \quad a_{2k+1}^{sinc} = \frac{(-1)^k 2}{(2k+1)\pi}\,.
$$

It had been a problem for some time to find a sequence of scaling functions with compact supports which approximates the function $\mathrm{sinc}(\pi x) = \frac{\sin \pi x}{\pi x}$, which is the scaling function of the sinc wavelet system. This problem is important because of the special relation of sinc function to signal processing applications. The family of biorthogonal Coifman wavelet systems just provides a very suitable candidate.

Theorem 6. *Suppose $\{\tilde{a}_k^N, k \in \mathbf{Z}\}$ to be the synthesis scaling vector of the biorthogonal Coifman wavelet system of degree N. Then*

$$
\lim_{N\to\infty} \left|\left|(\tilde{a}^N) - (a^{sinc})\right|\right|_{l^2} = \lim_{N\to\infty} \left(\sum_{k\in\mathbf{Z}} \left(\tilde{a}_k^N - a_k^{sinc}\right)^2 \right)^{1/2} = 0\,.
$$

Before proving Theorem 6, we first prove that $\{\tilde{a}^N\}$ converges to $\{a^{sinc}\}$ termwise.

Lemma 7. *Assume the same condition as in Theorem 6, then*

$$
\lim_{N\to\infty} \tilde{a}_k^N = a_k^{sinc}\,.
$$

Proof. First let's look at the case when N is even, $N = 2n$. Assume $|k| < n$ (otherwise we can choose a larger N), we have

$$
\begin{aligned}
\tilde{a}_{2k+1}^N &= \frac{(-1)^k}{2k+1} \binom{2n-1}{n-1} \binom{2n}{n+k} \frac{2n+1}{2^{4n-1}} \\
&= \frac{(-1)^k}{2k+1} \cdot \frac{(2n-1)!(2n+1)!}{2^{4n-1}(n-1)!n!(n-k)!(n+k)!}\,.
\end{aligned}
$$

We want to show that

$$\lim_{n\to\infty} \frac{(2n-1)!(2n+1)!}{2^{4n-1}(n-1)!n!(n-k)!(n+k)!} \;=\; \frac{2}{\pi}\,. \tag{3.4}$$

Recall that Stirling's formula states

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi}n^{n+1/2}e^{-n}} \;=\; 1\,,$$

so

$$\begin{aligned}
&\lim_{n\to\infty} \frac{(2n-1)!(2n+1)!}{2^{4n-1}(n-1)!n!(n-k)!(n+k)!} \\
&= \lim_{n\to\infty} \frac{(2n-1)^{2n-1/2}(2n+1)^{2n+3/2}e^{-1}}{2^{4n}\pi(n-1)^{n-1/2}n^{n+1/2}(n-k)^{n-k+1/2}(n+k)^{n+k+1/2}} \\
&= \lim_{n\to\infty} \frac{e^{-1}}{2^{4n}\pi}\left(\frac{2n-1}{n-1}\right)^{n-1/2}\left(\frac{2n+1}{n}\right)^{n+1/2}\left(\frac{2n-1}{n-k}\right)^{n} \\
&\quad \cdot\left(\frac{2n+1}{n+k}\right)^{n+1}\left(\frac{n-k}{n+k}\right)^{k-1/2} \\
&= \lim_{n\to\infty} \frac{e^{-1}}{2^{4n}\pi}\cdot 2^{n-1/2}e^{1/2}\cdot 2^{n+1/2}e^{1/2}\cdot 2^{n}e^{1/2}\cdot 2^{n+1}e^{-1/2}\cdot 1 \\
&= \frac{2}{\pi}\,.
\end{aligned}$$

When N is odd, $N = 2n-1$, the ratio

$$\frac{\tilde{a}^N_{2k+1}}{\tilde{a}^{N-1}_{2k+1}} \;=\; \frac{2n-1}{2n+2k} \;\longrightarrow\; 1 \quad \text{as } N\to\infty\,.$$

Thus the result also holds when N is odd.

Proof of Theorem 6. Based on Lemma 7, to apply Lebesgue's Dominated Convergence Theorem, we just need to find an $l^2(\mathbf{R})$ dominating sequence $\{d_k, k\in\mathbf{Z}\}$ such that

$$|\tilde{a}^N_k - a^{\mathrm{sinc}}_k| \;\le\; d_k \;\text{ for all }\; k\,.$$

1. If N is even, $N = 2n$, assume $|k|\le n$. Note that

$$(n-k)!(n+k)! \;\ge\; (n!)^2$$

then

$$\begin{aligned}
|\tilde{a}^N_{2k+1}| \;&=\; \left|\frac{(2n-1)!(2n+1)!}{(2k+1)2^{4n-1}(n-1)!n!(n-k)!(n+k)!}\right| \\
&\le\; \left|\frac{(2n-1)!(2n+1)!}{(2k+1)2^{4n-1}(n-1)!(n!)^3}\right|
\end{aligned}$$

Set $k = 0$ in (3.4),

$$\lim_{n\to\infty} \frac{(2n-1)!(2n+1)!}{2^{4n-1}(n-1)!(n!)^3} \;=\; \frac{2}{\pi}$$

So for N large enough,

$$\frac{(2n-1)!(2n+1)!}{2^{4n-1}(n-1)!(n!)^3} \;\leq\; 1$$

$$|\tilde{a}^N_{2k+1}| \;\leq\; \left|\frac{1}{2k+1}\right|$$

2. If N is odd, $N = 2n-1$, since

$$(n+k)!(n-1-k)! \;\geq\; (n-1)!n!\,,$$

we have

$$\begin{aligned}
|\tilde{a}^N_{2k+1}| &= \left|\frac{1}{2k+1}\begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix}\begin{pmatrix} 2n-1 \\ n+k \end{pmatrix}\frac{2n-1}{2^{4n-3}}\right| \\
&= \left|\frac{((2n-1)!)^2}{2^{4n-3}(2k+1)((n-1)!)^2(n+k)!(n-1-k)!}\right| \\
&\leq \left|\frac{((2n-1)!)^2}{2^{4n-3}(2k+1)((n-1)!)^2(n-1)!n!}\right| \\
&= \left|\frac{1}{2k+1}\tilde{a}^N_1\right|
\end{aligned}$$

Because

$$\lim_{N\to\infty} \tilde{a}^N_1 \;=\; a^{\text{sinc}}_1 \;=\; \frac{2}{\pi}\,,$$

it follows that

$$\tilde{a}^N_1 \;\leq\; 1 \;\text{ for } N \text{ large enough}\,.$$

Then

$$|\tilde{a}^N_{2k+1}| \;\leq\; \left|\frac{1}{2k+1}\right|\,.$$

Set

$$d_{2k+1} \;=\; \left|\frac{1}{2k+1}\right| + |a^{\text{sinc}}_{2k+1}| \;\leq\; \left|\frac{2}{2k+1}\right|\,, \quad d_{2k} \;=\; 0$$

we have $(d_k)_{k\in\mathbf{Z}} \in l^2(\mathbf{R})$ and

$$|\tilde{a}^N_k - a^{\text{sinc}}_k| \;\leq\; d_k\,,$$

By Lebesgue's Dominated Convergence Theorem and Lemma 7,

$$\lim_{N\to\infty} \left|\left|\tilde{\alpha}^N - \alpha^{\text{sinc}}\right|\right|_{l^2} \;=\; 0\,.$$

The theorem follows.

4 Conclusions

In this paper we present a new family of wavelet systems, the biorthogonal Coifman wavelet systems, which have very nice properties both in the theoretical sense and the application sense. The vanishing moments conditions have played an important role in such systems. These conditions not only give growing smoothness of wavelet systems, improve symmetry, but also provide fast wavelet sampling approximation. Another attractive feature of biorthogonal Coifman wavelet systems is that all the coefficients are dyadic rational. Thus we can have a very fast multiplication-free discrete wavelet transform implemented on digital computers.

Bibliography

1. Daubechies, I. (1992). *Ten lectures on wavelets.* CBMS-NSF regional conference series in applied mathematics. SIAM, Philadelphia, Pennsylvania.

2. Daubechies, I. (1988). Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, **XLI**:906–966.

3. Daubechies, I. (1993). Orthonormal bases of compactly supported wavelets II. Variations on a theme. *SIAM J. Math. Anal.*, **24**(2):499–519.

4. Tian, J. and Wells, Jr., R. O. (1996). A remark on vanishing moments. In *Proc. 30th Asilomar Conf. on Signals, Systems, and Computers*, Pacific Grove, CA.

5. Steffen, P. and Heller, P. N. and Gopinath, R. A. and Burrus, C. S. (Dec. 1993). Theory of regular m-band wavelet bases. *IEEE Trans. on Signal Processing*, **41**(12).

6. Heller, P. N. (1995). Rank m wavelet matrices with n vanishing moments. *SIAM J. Matr. Anal.*, **16**:502–518.

7. Cohen, A. and Daubechies, I. and Feauveau, J.-C. (1992). Biorthogonal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, **XLV**:485–560.

8. Vetterli, M. and Herley, C. (1992). Wavelets and filter banks: Theory and design. *IEEE Trans. Acoust. Speech Signal Process.*, **40**:2207–2232.

9. Tian, J. and Wells, Jr., R. O. (1996). Dyadic rational biorthogonal Coifman wavelet systems. Technical Report CML TR 96-07, Rice University. (ftp://cml.rice.edu/pub/reports/CML9607.ps.Z).

10. Mallat, S. G. (Sept. 1989). Multiresolution approximation and wavelet orthonormal bases of $L^2(\mathbf{R})$. *Trans. AMS*, **315**(1):69–87.