Dyadic Rational Biorthogonal Coifman Wavelet Systems*

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Abstract

We consider biorthogonal Coifman wavelet systems, a family of biorthogonal wavelet systems with the vanishing of moments equally distributed between the scaling functions and the wavelet functions. Using a time domain design method, exact formulas for the scaling vectors' coefficients are obtained. An attractive feature behind it is that all the coefficients are dyadic rational, which means that we can implement a very fast multiplication-free discrete wavelet transform, which consists of only addition and shift operations, on digital computers. Moreover these wavelet systems are compactly supported, provide fast wavelet sampling approximation with exponential decay, and converge to the Sinc wavelet system. And from biorthogonal Coifman wavelet systems, we propose a novel method to construct orthogonal Coifman wavelet systems (Coiflets).

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1 Introduction

The theory of wavelet analysis has grown explosively in the last decade. The terminology "wavelet" was first introduced, in the context of a mathematical transform, in 1984 by A. Grossmann and J. Morlet [14]. In 1988, I. Daubechies, in her celebrated paper [8], introduced a class of compactly supported orthonormal wavelet systems in general, as well as a family with growing smoothness for large support, the Daubechies wavelet systems. In 1989, S. G. Mallat [20] and Y. Meyer [21] presented the theory of multiresolution analysis. The spline family was studied by G. Battle [2], P. G. Lemarié [18], and C. K. Chui [5]. The necessary and sufficient conditions for an orthonormal wavelet system were given by A. Cohen [6] and W. M. Lawton [17]. Except for the Haar wavelet system, orthogonal wavelet systems can't be symmetric, though symmetry is highly desired, for example, in the applications in signal processing, where symmetry corresponds to linear phase. To obtain symmetry and keep the property of perfect reconstruction, A. Cohen, I. Daubechies, and J.-C. Feauveau [7] replaced the orthogonality condition with biorthogonality and thus established the theory of biorthogonal wavelet systems. At the same time, lots of pioneer work has been done by many scientists from mathematics, physics and engineering. For more details of wavelet theory, we refer to [4, 9, 21, 22, 27].

In this paper, we will present *biorthogonal Coifman wavelet systems*, a family of biorthogonal wavelet systems with very nice properties both in the theoretical sense and application sense. The original idea goes back to R. Coifman of Yale University. In the spring of 1989, he suggested that it might be worthwhile to construct orthogonal wavelet systems with vanishing moments not only for the wavelet functions (which is the hypothesis posed on the Daubechies wavelet systems), but also for the scaling functions. This turned out to be a big success. One key property of these orthogonal wavelet systems with vanishing moments equally distributed between scaling functions and wavelet functions (which are called *orthogonal Coifman wavelet systems*) is that they have very nice approximation properties with exponential decay. Biorthogonal Coifman wavelet systems, the biorthogonal counterparts of orthogonal Coifman wavelet systems, also have fast approximation properties with exponential decay. An attractive feature of biorthogonal Coifman wavelet systems is that all the scaling vectors are dyadic rational. Thus we have obtained a family of biorthogonal wavelet systems with multiplication-free discrete wavelet transform. Moreover they are compactly supported, and converge to the *Sinc wavelet system*. Half of the members (more precisely, those with odd degrees) of the family are symmetric, which indicates the potential promising of these biorthogonal Coifman wavelet systems in the applications of signal processing. Actually, based on a complete coding performance evaluation, biorthogonal Coifman wavelet systems are very useful for image transform coding and seem to be quite comparable to the wavelet systems used in the state-of-the-art compression systems.

This paper is organized as follows. We begin with a wavelet approximation theorem, which asserts that biorthogonal Coifman wavelet systems provide fast wavelet sampling approximation with exponential decay. The definition of biorthogonal Coifman wavelet systems is given in Section 2 and the problem of existence is considered. We introduce a time domain design method which is very straight forward. An attractive feature of biorthgonal Coifman wavelet systems is that all the scaling vectors are dyadic rational, and the proof is given in Section 3. Moreover, in Section 4, we will show that biorthogonal Coifman wavelet systems will converge to the Sinc wavelet system. Some other properties are studied in Section 5. The orthogonal counterpart, orthogonal Coifman wavelet systems, will be considered in Section 6. And we conclude this paper in Section 7.

2 Definition and Construction

First, we give a brief overview of wavelet analysis. For more details, we refer to [4, 9, 21, 22, 27]. An $L^2(\mathbf{R})$ function $\psi(x)$ induces an *orthonormal wavelet system* of $L^2(\mathbf{R})$ if the dilations and translations of $\psi(x)$ constitute an orthonormal basis of $L^2(\mathbf{R})$, i.e., $\{\psi_{j,k}(x) : j,k \in \mathbf{Z}\}$ is an orthonormal basis, where $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$. We call $\psi(x)$ the *wavelet function.* If $\psi(x)$ is compactly supported, P. G. Lemarié showed in [19] that this orthonormal wavelet system is associated with a *multiresolution analysis.* A multiresolution analysis consists of a sequence of embedded closed subspaces $V_j \subset L^2(\mathbf{R}), j \in \mathbf{Z}$, satisfying $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R})$ and $\bigcap_{j\in\mathbf{Z}} V_j = \{0\}$. Moreover, there should be an $L^2(\mathbf{R})$ function $\phi(x)$ such that $\{\phi_{j,k}(x) : k \in \mathbf{Z}\}$ is an orthonormal basis for V_j for all $j \in \mathbf{Z}$, where $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$. Since $\phi(x) \in V_0 \subset V_1$, we have

$$\phi(x) \;=\; \sum_{k\in\mathbf{Z}} a_k \phi(2x-k)\,, \tag{2.1}$$

then $\psi(x)$ and $\phi(x)$ are related by

$$\psi(x) \;=\; \sum_{k\in\mathbf{Z}} (-1)^k a_{-k+1}\phi(2x-k)\,. \tag{2.2}$$

We call $\{a_k : k \in \mathbf{Z}\}$ and $\phi(x)$ the *scaling vector* and the *scaling function* of the wavelet system, respectively. As one knows, it is straightforward to define an orthonormal wavelet system by (2.2) from a multiresolution analysis.

We are mainly interested in compactly supported wavelet systems since they are easy to implement and have great applications in practice. For this, let's assume the scaling vector has finite length, $a_k = 0$, for $k < N_1$, or $k > N_2$, where $N_1, N_2 \in \mathbf{Z}$, and $N_1 < N_2$. And we will assume that the scaling vector has finite length all through the rest of the paper. It then follows (see [9, 11]) that $\phi(x)$ has compact support. By (2.2), $\psi(x)$ also has compact support. We make another assumption through the rest of paper, that the scaling function is always normalized,

$$\int_{-\infty}^{\infty} \phi(x)\,dx \;=\; 1\,. \tag{2.3}$$

It is well-known that with vanishing moments posed on the wavelet function $\psi(x)$, we can achieve a better wavelet orthogonal projection. The *wavelet orthogonal projection* of an $L^2(\mathbf{R})$ function $f(x)$ at the level j is defined by

$$P^j(f) \;:=\; \sum_{k\in\mathbf{Z}} \left(\int_{-\infty}^{\infty} f(t)\phi_{j,k}(t)dt\right)\cdot \phi_{j,k}(x)\,,$$

which is exactly the orthogonal projection of $f(x)$ onto the subspace V_j in the multiresolution analysis. We state the result formally as the following lemma. The proof can be found, for example, in [13].

Lemma 2.1 *In an orthonormal wavelet system, if the vanishing moments of the wavelet function $\psi(x)$ is of degree N, i.e.,*

$$\mathrm{Mom}_p(\psi) \;:=\; \int_{-\infty}^{\infty} x^p \psi(x)\, dx \;=\; 0\,, \quad \text{for}\, p = 0, \cdots, N\,,$$

then for $f(x) \in C_0^{N,1}(\mathbf{R})$, we have

$$\left|\left| f(x) - P^j(f) \right|\right|_{L^2} \;\leq\; C\, 2^{-j(N+1)}\,,$$

where C is a constant, depending only on $f(x)$ and the scaling vector $\{a_k\}$.

So if we approximate an L^2 function $f(x)$ by its wavelet orthogonal projection, we would like to make $\psi(x)$ to have maximal vanishing moments, or in other words, to make N as large as possible, and the approximation error will converge to zero in exponential decay. That's one reason why Daubechies wavelet systems [8] put all freedom of parameters into the vanishing moments on the wavelet function.

Similarly, one can prove that (for example, see [22]) with vanishing moments posed on the scaling function $\phi(x)$, we can reduce the error between the wavelet orthogonal projection and the wavelet sampling approximation.

Lemma 2.2 *In an orthonormal wavelet system, if the vanishing moments of the scaling function $\phi(x)$ is of degree N, i.e.,*

$$\mathrm{Mom}_0(\phi) \;:=\; \int_{-\infty}^{\infty} \phi(x)\, dx \;=\; 1\,,$$

$$\mathrm{Mom}_p(\phi) \;:=\; \int_{-\infty}^{\infty} x^p \phi(x)\, dx \;=\; 0\,, \quad \text{for}\, p = 1, \cdots, N\,,$$

then for $f(x) \in C_0^{N,1}(\mathbf{R})$, we have

$$\left|\left| P^j(f) - S^j(f) \right|\right|_{L^2} \;\leq\; C\, 2^{-j(N+1)}\,,$$

where $S^j(f)$ *is the* wavelet sampling approximation *at the level* j*, defined by*

$$S^j(f) \ := \ 2^{-j/2} \sum_{k \in \mathbf{Z}} f\left(\frac{k}{2^j}\right) \phi_{j,k}(x) \,,$$

and C *is a constant, depending only on* $f(x)$ *and the scaling vector* $\{a_k\}$*.*

The wavelet sampling approximation is what is used in most applications of wavelets, as it is the easiest approximation to compute. (Simply let the sampling values of the given function to be the corresponding wavelet expansion coefficients.) In an orthonormal wavelet system, by definition,

$$\left|\left|f(x) - S^j(f)\right|\right|_{L^2}^2 \ = \ \left|\left|f(x) - P^j(f)\right|\right|_{L^2}^2 \ + \ \left|\left|P^j(f) - S^j(f)\right|\right|_{L^2}^2 \,,$$

to minimize the wavelet sampling approximation error (the left hand side), we need to minimize the two terms of the right hand side to the same degree. So based on Lemma 2.1 and 2.2, the optimal distribution of vanishing moments between scaling functions and wavelet functions will be the equal distribution. Also it follows that the vanishing moments of wavelet functions will reduce the error in the wavelet orthogonal projection, or the distance from the original L^2 function to the projection subspace, while the vanishing moments of scaling functions will reduce the difference between the wavelet orthogonal projection and the wavelet sampling approximation, where both of these two lie in the projection subspace.

Theorem 2.1 *In an orthonormal wavelet system, if both of the scaling function* $\phi(x)$ *and the wavelet function* $\psi(x)$ *have vanishing moments up to degree* N*,*

$$\mathrm{Mom}_0(\phi) \ = \ \int_{-\infty}^{\infty} \phi(x)\, dx \ = \ 1 \,,$$

$$\begin{aligned}
\mathrm{Mom}_p(\phi) &= \int_{-\infty}^{\infty} x^p \phi(x)\, dx \;=\; 0\,, \quad \text{for}\, p = 1, \cdots, N\,, \\
\mathrm{Mom}_p(\psi) &= \int_{-\infty}^{\infty} x^p \psi(x)\, dx \;=\; 0\,, \quad \text{for}\, p = 0, \cdots, N\,,
\end{aligned}$$

for $f(x) \in C_0^{N,1}(\mathbf{R}), j \in \mathbf{N}$, *define*

$$S^j(f)(x) \;:=\; 2^{-j/2} \sum_{k \in \mathbf{Z}} f\left(\frac{k}{2^j}\right) \phi_{j,k}(x)\;,$$

where $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$. *Then*

$$||f(x) - S^j(f)(x)||_{L^2} \;\le\; C 2^{-j(N+1)}\,,$$

where C *is a constant, independent of* j *and* N.

Surprisingly, the orthogonality is not a necessary condition for the wavelet sampling approximation theorem, though it is used in the above argument. All we need is the vanishing moments conditions on the scaling function and the wavelet function. Recall that the vanishing moments conditions of the scaling function and the wavelet function are equivalent to the linear conditions on the scaling vectors, as we state in the next lemma.

Lemma 2.3 *Suppose* $\phi(x), \psi(x) \in L^2(\mathbf{R})$ *satisfying*

$$\begin{aligned}
\phi(x) &= \sum_{k \in \mathbf{Z}} a_k \phi(2x - k)\,, \\
\psi(x) &= \sum_{k \in \mathbf{Z}} (-1)^k a_{-k+1} \phi(2x - k)\,,
\end{aligned}$$

where $\{a_k\}$ *is a finite length sequence,* $\phi(x) \in L^1(\mathbf{R})$ *and it is normalized*

$$\int_{-\infty}^{\infty} \phi(x)\, dx \;=\; 1\,.$$

Then the following two conditions are equivalent,

1. *The vanishing moments of $\phi(x)$ and $\psi(x)$ are both of degree N, i.e.,*

$$\mathrm{Mom}_p(\phi) \;=\; \int_{-\infty}^{\infty} x^p \phi(x)\, dx \;=\; 0\,, \quad \text{for}\, p = 1, \cdots, N\,,$$

$$\mathrm{Mom}_p(\psi) \;=\; \int_{-\infty}^{\infty} x^p \psi(x)\, dx \;=\; 0\,, \quad \text{for}\, p = 0, \cdots, N\,.$$

2. *The sequence $\{a_k\}$ satisfying*

$$\sum_{k\in\mathbf{Z}} (2k)^p a_{2k} \;=\; \sum_{k\in\mathbf{Z}} (2k+1)^p a_{2k+1} \;=\; 0\,, \quad \text{for}\, p = 1, \cdots, N\,,$$

$$\sum_{k\in\mathbf{Z}} a_{2k} \;=\; \sum_{k\in\mathbf{Z}} a_{2k+1} \;=\; 1\,.$$

Thus we can present the wavelet approximation theorem with conditions on $\{a_k\}$ only.

Theorem 2.2 (Wavelet Approximation Theorem) *Suppose $\phi(x)$ is an $L^2(\mathbf{R})$ solution of the two-scale difference equation*

$$\phi(x) \;=\; \sum_{k\in\mathbf{Z}} a_k \phi(2x-k)\,,$$

and it is normalized

$$\int_{-\infty}^{\infty} \phi(x)\, dx \;=\; 1\,,$$

where $\{a_k\}$ is a finite length sequence, satisfying

$$\sum_{k\in\mathbf{Z}} (2k)^p a_{2k} \;=\; \sum_{k\in\mathbf{Z}} (2k+1)^p a_{2k+1} \;=\; 0\,, \quad \text{for}\, p = 1, \cdots, N\,,$$

$$\sum_{k\in\mathbf{Z}} a_{2k} \;=\; \sum_{k\in\mathbf{Z}} a_{2k+1} \;=\; 1\,.$$

For $f(x) \in C_0^{N,1}(\mathbf{R}), j \in \mathbf{N}$, *define*

$$S^j(f)(x) \;:=\; 2^{-j/2} \sum_{k \in \mathbf{Z}} f\left(\frac{k}{2^j}\right) \phi_{j,k}(x)\;,$$

where $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$. *Then*

$$||f(x) - S^j(f)(x)||_{L^2} \;\leq\; C2^{-j(N+1)}\,,$$

where C *is a constant, depending only on* f *and* $\{a_k\}$.
If in addition, $\phi(x) \in C^n(\mathbf{R})$*, where* $n \in \mathbf{Z}, 0 \leq n \leq N$*, then*

$$||f(x) - S^j(f)(x)||_{H^n} \;\leq\; C2^{-j(N+1-n)}\,,$$

where C *is a constant, depending only on* f *and* $\{a_k\}$.

A complete proof of Theorem 2.2 can be found in [22]. Since it doesn't require the orthogonality condition, we can apply Theorem 2.2 to biorthogonal wavelet systems as well. In a *biorthogonal wavelet system*, the decomposition function and the reconstruction function can be different. Thus we will have an expansion which looks like

$$f(x) \;=\; \lim_{J \to \infty} \sum_{j=-J}^{J} \sum_{k \in \mathbf{Z}} < f, \psi_{j,k} > \tilde{\psi}_{j,k}\,.$$

In a biorthogonal wavelet system, we have two scaling vectors, the *analysis scaling vector* $\{a_k, k \in \mathbf{Z}\}$ and the *synthesis scaling vector* $\{\tilde{a}_k, k \in \mathbf{Z}\}$. They satisfy the linear conditions

$$\sum_{k \in \mathbf{Z}} a_k \;=\; 2\,, \tag{2.4}$$

$$\sum_{k \in \mathbf{Z}} \tilde{a}_k \;=\; 2\,, \tag{2.5}$$

and the bilinear condition

$$\sum_{k\in\mathbf{Z}} a_k\tilde{a}_{k+2l} \;=\; 2\delta_{0,l} \;:=\; \begin{cases} 2 & l=0 \\ 0 & \text{otherwise} \end{cases} \quad , \quad \forall l \in \mathbf{Z}\,. \tag{2.6}$$

As before, assume both $\{a_k\}$ and $\{\tilde{a}_k\}$ have finite length, and suppose $\phi(x)$, $\tilde{\phi}(x)$ are L^2 solutions of the two-scale difference equations,

$$\begin{aligned} \phi(x) \;&=\; \sum_{k\in\mathbf{Z}} a_k\phi(2x-k)\,, \\ \tilde{\phi}(x) \;&=\; \sum_{k\in\mathbf{Z}} \tilde{a}_k\tilde{\phi}(2x-k)\,, \end{aligned}$$

and they are normalized,

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(x)\,dx \;&=\; 1\,, \\ \int_{-\infty}^{\infty} \tilde{\phi}(x)\,dx \;&=\; 1\,. \end{aligned}$$

Define $\psi(x)$ and $\tilde{\psi}(x)$ by

$$\begin{aligned} \psi(x) \;&=\; \sum_{k\in\mathbf{Z}} (-1)^k\tilde{a}_{-k+1}\phi(2x-k)\,, \\ \tilde{\psi}(x) \;&=\; \sum_{k\in\mathbf{Z}} (-1)^k a_{-k+1}\tilde{\phi}(2x-k)\,, \end{aligned}$$

then the $\psi_{j,k}$ and $\tilde{\psi}_{j,k}$ constitute a weak dual frame of $L^2(\mathbf{R})$, i.e., for any $f(x), g(x) \in L^2(\mathbf{R})$,

$$\lim_{J\to\infty} \sum_{j=-J}^{J} \sum_{k\in\mathbf{Z}} < f, \psi_{j,k} >< \tilde{\psi}_{j,k}, g > = < f, g >\,,$$

or, in a weak sense,

$$f(x) \;=\; \lim_{J\to\infty} \sum_{j=-J}^{J} \sum_{k\in\mathbf{Z}} < f, \psi_{j,k} > \tilde{\psi}_{j,k}\,, \tag{2.7}$$

where $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \tilde{\psi}_{j,k}(x) = 2^{j/2}\tilde{\psi}(2^j x - k)$. And we call $\phi(x)$, $\psi(x)$, $\tilde{\phi}(x)$, $\tilde{\psi}(x)$ the *analysis scaling function*, the *analysis wavelet function*, the *synthesis scaling function* and the *synthesis wavelet function*, respectively.

If the Fourier transforms of scaling functions satisfy

$$|\hat{\phi}(\xi)| \;\leq\; C(1+|\xi|)^{-1/2-\epsilon}, \quad |\hat{\tilde{\phi}}(\xi)| \;\leq\; C(1+|\xi|)^{-1/2-\epsilon},$$

for some constant C, then the limit in (2.7) converges strongly in $L^2(\mathbf{R})$.

In a biorthogonal system, we can't get the orthogonality, but we can impose some conditions to establish a dual Riesz bases. The sufficient and necessary condition of a dual Riesz bases for the scaling vectors $\{a_k\}$ and $\{\tilde{a}_k\}$ satisfying (2.4), (2.5), (2.6) and having finite length is similar to the orthogonal cases, which states

$$\int_{-\infty}^{\infty} \phi(x)\tilde{\phi}(x-k)\,dx \;=\; \delta_{0,k}\,, \quad \forall k \in \mathbf{Z}\,.$$

Similar to Lemma 2.3, in biorthogonal wavelet systems, the vanishing moments on scaling functions and wavelet functions are equivalent to the linear conditions on scaling vectors.

Lemma 2.4 *Suppose* $\phi(x), \psi(x), \tilde{\phi}(x), \tilde{\psi}(x) \in L^2(\mathbf{R})$ *satisfying*

$$\begin{aligned}
\phi(x) &= \sum_{k\in\mathbf{Z}} a_k \phi(2x-k)\,,\\
\psi(x) &= \sum_{k\in\mathbf{Z}} (-1)^k \tilde{a}_{-k+1}\phi(2x-k)\,,\\
\tilde{\phi}(x) &= \sum_{k\in\mathbf{Z}} \tilde{a}_k \tilde{\phi}(2x-k)\,,\\
\tilde{\psi}(x) &= \sum_{k\in\mathbf{Z}} (-1)^k a_{-k+1}\tilde{\phi}(2x-k)\,,
\end{aligned}$$

where $\{a_k\}$ *and* $\{\tilde{a}_k\}$ *are finite length sequences,* $\phi(x), \tilde{\phi}(x) \in L^1(\mathbf{R})$ *and they are normalized*

$$\int_{-\infty}^{\infty} \phi(x)\, dx \;=\; 1\,,$$

$$\int_{-\infty}^{\infty} \tilde{\phi}(x)\, dx \;=\; 1\,.$$

Then the following two conditions are equivalent,

1. *The vanishing moments of* $\phi(x), \psi(x), \tilde{\phi}(x), \tilde{\psi}(x)$ *are all of degree* N*, i.e.,*

$$\mathrm{Mom}_p(\phi) \;=\; \int_{-\infty}^{\infty} x^p \phi(x)\, dx \;=\; 0\,, \quad \text{for}\, p = 1, \cdots, N\,,$$

$$\mathrm{Mom}_p(\psi) \;=\; \int_{-\infty}^{\infty} x^p \psi(x)\, dx \;=\; 0\,, \quad \text{for}\, p = 0, \cdots, N\,,$$

$$\mathrm{Mom}_p(\tilde{\phi}) \;=\; \int_{-\infty}^{\infty} x^p \tilde{\phi}(x)\, dx \;=\; 0\,, \quad \text{for}\, p = 1, \cdots, N\,,$$

$$\mathrm{Mom}_p(\tilde{\psi}) \;=\; \int_{-\infty}^{\infty} x^p \tilde{\psi}(x)\, dx \;=\; 0\,, \quad \text{for}\, p = 0, \cdots, N\,,$$

2. *The sequence* $\{a_k\}$ *and* $\{\tilde{a}_k\}$ *satisfying*

$$\sum_{k \in \mathbf{Z}} (2k)^p a_{2k} \;=\; \sum_{k \in \mathbf{Z}} (2k+1)^p a_{2k+1} \;=\; 0\,, \quad \text{for } p = 1, \cdots, N\,,$$

$$\sum_{k \in \mathbf{Z}} a_{2k} \;=\; \sum_{k \in \mathbf{Z}} a_{2k+1} \;=\; 1\,,$$

$$\sum_{k \in \mathbf{Z}} (2k)^p \tilde{a}_{2k} \;=\; \sum_{k \in \mathbf{Z}} (2k+1)^p \tilde{a}_{2k+1} \;=\; 0\,, \quad \text{for}\, p = 1, \cdots, N\,,$$

$$\sum_{k \in \mathbf{Z}} \tilde{a}_{2k} \;=\; \sum_{k \in \mathbf{Z}} \tilde{a}_{2k+1} \;=\; 1\,.$$

For more details of biorthogonal wavelet systems theory, we refer to [7, 28].

Now back to Theorem 2.2. For a biorthogonal wavelet system, in the decomposition process, based on Theorem 2.2, we would like to set the analysis scaling vector to have vanishing moments up to some degree, then we can sample on dyadic rationals and take these values as the wavelet expansion coefficients. Especially we can take the sample values as the discrete wavelet transform coefficients at the starting level, apply the Mallat Algorithm [20] on these sample values, and analyze the data (compression, denoising, etc). In the reconstruction process, again, based on Theorem 2.2, if we set the synthesis scaling vector to have vanishing moments up to some degree, we can take the inverse discrete wavelet transform coefficients as the sample values on dyadic rationals and reconstruct the original data. Based on this observation, we introduce biorthogonal Coifman wavelet systems.

Definition 2.1 *A biorthogonal wavelet system with compact support is called a* biorthogonal Coifman wavelet system *(in short,* BCW*) of degree N if the following two conditions are satisfied,*

- *the vanishing moments of the scaling function $\tilde{\phi}(x)$ and the wavelet function $\tilde{\psi}(x)$ are both of degree N, i.e.,*

$$\mathrm{Mom}_0(\tilde{\phi}) \;=\; \int_{-\infty}^{\infty} \tilde{\phi}(x)\,dx \;=\; 1\,, \tag{2.8}$$

$$\mathrm{Mom}_p(\tilde{\phi}) \;=\; \int_{-\infty}^{\infty} x^p\tilde{\phi}(x)\,dx \;=\; 0\,, \quad \text{for}\, p = 1, \cdots, N, \tag{2.9}$$

$$\mathrm{Mom}_p(\tilde{\psi}) \;=\; \int_{-\infty}^{\infty} x^p\tilde{\psi}(x)\,dx \;=\; 0\,, \quad \text{for}\, p = 0, \cdots, N, \tag{2.10}$$

- *the vanishing moment of the wavelet function $\psi(x)$ is of degree N,*

$$\mathrm{Mom}_p(\psi) \;=\; \int_{\mathbf{R}} x^p \psi(x)\,dx \;=\; 0\,, \quad \text{for } p = 0, \cdots, N. \tag{2.11}$$

Note that in the definition of the biorthogonal Coifman wavelet system, although there is no vanishing moment requirement on the analysis scaling function $\phi(x)$, it turns out that $\phi(x)$ also has vanishing moments up to degree N, because of the perfect reconstruction condition (2.6).

Lemma 2.5 *For a biorthogonal Coifman wavelet system of degree N, the vanishing moments' degree of the analysis scaling function $\phi(x)$ is also N,*

$$\mathrm{Mom}_p(\phi) \;=\; \int_{-\infty}^{\infty} x^p \phi(x)\,dx \;=\; 0\,, \quad \text{for } p = 1, \cdots, N\,.$$

Proof: From Lemma 2.4, it is sufficient to prove that

$$\sum_{k \in \mathbf{Z}} a_k k^p \;=\; 0\,, \quad \text{for } p = 1, \cdots, N\,.$$

From the vanishing moment condition of $\psi(x)$ and $\tilde{\phi}(x)$, we have

$$\sum_{k \in \mathbf{Z}} \tilde{a}_{2k}(2k)^m \;=\; \sum_{k \in \mathbf{Z}} \tilde{a}_{2k+1}(2k+1)^m \;=\; \delta_{0,m}\,, \quad \text{for } 0 \le m \le p\,.$$

Thus

$$\begin{aligned}
\sum_{k,l \in \mathbf{Z}} k^{p-m}(k+2l)^m a_k \tilde{a}_{k+2l} &= \sum_{k \in \mathbf{Z}} k^{p-m} a_k \sum_{l \in \mathbf{Z}} (k+2l)^m \tilde{a}_{k+2l} \\
&= \sum_{k \in \mathbf{Z}} k^{p-m} a_k \cdot \delta_{0,m}
\end{aligned}$$

So

$$
\begin{aligned}
\sum_{k\in\mathbf{Z}} a_k k^p &= \sum_{k,l\in\mathbf{Z}} k^p a_k \tilde{a}_{k+2l} \\
&= \sum_{k,l\in\mathbf{Z}} k^p a_k \tilde{a}_{k+2l} \\
&\quad + \left(\sum_{m=1}^{p} \left(\sum_{k,l\in\mathbf{Z}} k^{p-m}(k+2l)^m a_k \tilde{a}_{k+2l} \right) \cdot (-1)^m \left(\begin{array}{c} p \\ m \end{array} \right) \right) \\
&= \sum_{m=0}^{p} \left((-1)^m \left(\begin{array}{c} p \\ m \end{array} \right) \sum_{k,l\in\mathbf{Z}} k^{p-m}(k+2l)^m a_k \tilde{a}_{k+2l} \right) \\
&= \sum_{k,l\in\mathbf{Z}} a_k \tilde{a}_{k+2l} \sum_{m=0}^{p} \left((-1)^m \left(\begin{array}{c} p \\ m \end{array} \right) k^{p-m}(k+2l)^m \right) \\
&= \sum_{k,l\in\mathbf{Z}} a_k \tilde{a}_{k+2l} \left(k-(k+2l)\right)^p \\
&= \sum_{k,l\in\mathbf{Z}} (-2l)^p a_k \tilde{a}_{k+2l} \\
&= \sum_{l\in\mathbf{Z}} (-2l)^p \sum_{k\in\mathbf{Z}} a_k \tilde{a}_{k+2l} \\
&= \sum_{l\in\mathbf{Z}} (-2l)^p \cdot 2\delta_{0,l} \\
&= 0
\end{aligned}
$$

□

Thus in a biorthogonal Coifman wavelet system, both the analysis pair $\{\phi, \psi\}$ and the synthesis pair $\{\tilde{\phi}, \tilde{\psi}\}$ have vanishing moments up to the same degree N. By Theorem 2.2, the wavelet sampling approximation with biorthogonal Coifman wavelet systems will converge to the original function in exponential decay. Now the natural question is whether biorthogonal Coifman wavelet systems exist. If yes, how can one design them? We will study the existence problem next.

Now let's see how many conditions are there imposed on biorthogonal Coifman wavelet systems? From the theory of biorthogonal wavelet systems, we know that the analysis scaling vector $\{a_k\}$ and the synthesis scaling vector $\{\tilde{a}_k\}$ must satisfy the linear condition

$$\sum_{k\in\mathbf{Z}} a_k \;=\; \sum_{k\in\mathbf{Z}} \tilde{a}_k \;=\; 2\,, \tag{2.12}$$

and the perfect construction condition

$$\sum_{k\in\mathbf{Z}} a_k\tilde{a}_{k+2l} \;=\; 2\delta_{0,l}\,, \quad \forall l\in\mathbf{Z}\,. \tag{2.13}$$

The vanishing moment conditions in the definition of the biorthogonal Coifman wavelet system are equivalent to

$$\sum_{k\in\mathbf{Z}}(2k)^p a_{2k} \;=\; \sum_{k\in\mathbf{Z}}(2k+1)^p a_{2k+1} \;=\; \delta_{0,p}\,, \quad \text{for } p=0,\cdots,N\,, \tag{2.14}$$

$$\sum_{k\in\mathbf{Z}}(2k)^p \tilde{a}_{2k} \;=\; \sum_{k\in\mathbf{Z}}(2k+1)^p \tilde{a}_{2k+1} \;=\; \delta_{0,p}\,, \quad \text{for } p=0,\cdots,N\,. \tag{2.15}$$

In practice, we prefer that the length of the synthesis scaling vector be shorter than the length of the analysis scaling vector. The reason is that for longer analysis scaling vectors, we will get more redundant information after the wavelet transform, which will be ideal for the further processing in the wavelet domain; and for shorter synthesis scaling vectors, the inverse discrete wavelet transform will provide a compact representation of the original data, which is very useful, for example, for data storage and transmission. Based on this thought, we will first look at the synthesis scaling vector $\{\tilde{a}_k\}$ and try to find the minimum length solution of it at each degree.

The linear condition (2.12) for $\{\tilde{a}_k\}$ is already included in (2.15) and we don't need to worry about the bilinear condition (2.13) right now. Thus there are totally $2N+2$ linear conditions on the scaling vector $\{\tilde{a}_k\}$. And the minimum length solution will have, of course, $2N+2$ elements in the finite length sequence $\{\tilde{a}_k\}$. For symmetry reasons, we will distribute these $2N+2$ elements in $\{\tilde{a}_k\}$ as symmetric as possible. More precisely, we are looking for a solution of the form $\{\tilde{a}_{-N}, \tilde{a}_{-N+1}, \cdots, \tilde{a}_{N+1}\}$. Let's have another look at the linear equation system (2.15). By its form, these $2N+2$ linear equations can be naturally divided into two parts, those on the even terms $\tilde{a}_{2k}$ and those on the odd terms $\tilde{a}_{2k+1}$. For those $N+1$ linear equations on the even terms, it is easy to see that the solution is exactly that all are zero except that $\tilde{a}_0 = 1$. For those $N+1$ linear equations on the odd terms, the coefficient matrix of these $N+1$ variables $\tilde{a}_{2k+1}$ is a Vandermonde's matrix. Thus the solution for these $N+1$ odd terms always exist and it is unique. Using the determinant formula of a Vandermonde's matrix, we get the exact formulas for these odd terms,

- if N is even, $N = 2n$,

$$\tilde{a}_{2k+1} = \frac{\prod_{j\neq k, j=-n}^{n}(2j+1)}{2^N \prod_{j\neq k, j=-n}^{n}(j-k)} = \frac{(-1)^k}{2k+1} \begin{pmatrix} 2n-1 \\ n-1 \end{pmatrix} \begin{pmatrix} 2n \\ n+k \end{pmatrix} \frac{2n+1}{2^{4n-1}}$$

- if N is odd, $N = 2n-1$,

$$\tilde{a}_{2k+1} = \frac{\prod_{j\neq k, j=-n}^{n-1}(2j+1)}{2^N \prod_{j\neq k, j=-n}^{n-1}(j-k)} = \frac{(-1)^k}{2k+1} \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix} \begin{pmatrix} 2n-1 \\ n+k \end{pmatrix} \frac{2n-1}{2^{4n-3}}$$

here we define $\begin{pmatrix} m \\ l \end{pmatrix} := 0$ if $l > m$ or $l < 0$, which is standard in combinatorial theory.

Thus we have obtained the formula for the minimum length synthesis scaling vector $\{\tilde{a}_k\}$ for every degree N. As it can be seen in the above discussion, the minimum length synthesis scaling vector is unique. We notice that the scaling vectors $\tilde{a}_k$ when N is odd are exactly the *Deslauriers-Dubuc filters* [11, 12]. The Deslauriers-Dubuc filters are also known in different forms in the literature [1, 24, 23].

The next theorem tells us how to construct the analysis scaling vector $\{a_k\}$ from the synthesis scaling vector $\{\tilde{a}_k\}$.

Theorem 2.3 *Assume $\tilde{a}_k$ to be defined as above, depending on whether N is even or odd. Set*

$$a_{2k+1} = \tilde{a}_{2k+1}\,, \tag{2.16}$$

and

$$a_{2k} = 2\delta_{0,k} - \sum_{l\in\mathbf{Z}} \tilde{a}_{2l+1}\tilde{a}_{2l+1-2k}\,. \tag{2.17}$$

Then the resulting $\{a_k\}$ and $\{\tilde{a}_k\}$ will constitute a biorthogonal Coifman wavelet system of degree N.

Proof: We need to check (2.13) and (2.14).

First,

$$\sum_{k\in\mathbf{Z}} a_k\tilde{a}_{k+2l} = \sum_{k\in\mathbf{Z}} a_{2k}\tilde{a}_{2k+2l} + \sum_{k\in\mathbf{Z}} a_{2k+1}\tilde{a}_{2k+1+2l}$$

$$
\begin{aligned}
&= a_{-2l} + \sum_{k\in\mathbf{Z}} \tilde{a}_{2k+1}\tilde{a}_{2k+1+2l} \\
&= 2\delta_{0,-l} - \sum_{k\in\mathbf{Z}} \tilde{a}_{2k+1}\tilde{a}_{2k+1+2l} + \sum_{k\in\mathbf{Z}} \tilde{a}_{2k+1}\tilde{a}_{2k+1+2l} \\
&= 2\delta_{0,-l} \\
&= 2\delta_{0,l}
\end{aligned}
$$

All remaining is to show that

$$
\sum_{k\in\mathbf{Z}} (2k)^p a_{2k} \ = \ \delta_{0,p}\,, \quad \text{for } p = 0, \cdots, N\,.
$$

We have

$$
\begin{aligned}
&\sum_{k\in\mathbf{Z}} (2k)^p a_{2k} \\
&= \sum_{k\in\mathbf{Z}} \left((2k)^p \left(2\delta_{0,k} - \sum_{n\in\mathbf{Z}} \tilde{a}_{2n+1}\tilde{a}_{2n+1-2k} \right) \right) \\
&= 2\delta_{0,p} - \sum_{k,n\in\mathbf{Z}} (2k)^p \tilde{a}_{2n+1}\tilde{a}_{2n+1-2k} \\
&= 2\delta_{0,p} - \sum_{m,n\in\mathbf{Z}} (2n-2m)^p \tilde{a}_{2n+1}\tilde{a}_{2m+1} \\
&= 2\delta_{0,p} - \sum_{m,n\in\mathbf{Z}} ((2n+1)-(2m+1))^p \tilde{a}_{2n+1}\tilde{a}_{2m+1} \\
&= 2\delta_{0,p} - \sum_{m,n\in\mathbf{Z}} \sum_{l=0}^{p} \left((-1)^l \begin{pmatrix} p \\ l \end{pmatrix} (2n+1)^{p-l}(2m+1)^l \tilde{a}_{2n+1}\tilde{a}_{2m+1} \right) \\
&= 2\delta_{0,p} - \sum_{l=0}^{p} (-1)^l \left(\begin{pmatrix} p \\ l \end{pmatrix} \sum_{m\in\mathbf{Z}} (2m+1)^l \tilde{a}_{2m+1} \sum_{n\in\mathbf{Z}} (2n+1)^{p-l} \tilde{a}_{2n+1} \right) \\
&= 2\delta_{0,p} - \sum_{l=0}^{p} (-1)^l \left(\begin{pmatrix} p \\ l \end{pmatrix} \sum_{m\in\mathbf{Z}} (2m+1)^l \tilde{a}_{2m+1}\delta_{0,p-l} \right) \\
&= 2\delta_{0,p} - (-1)^p \sum_{m\in\mathbf{Z}} (2m+1)^p \tilde{a}_{2m+1}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\delta_{0,p} - (-1)^p \delta_{0,p} \\
&= 2\delta_{0,p} - \delta_{0,p} \\
&= \delta_{0,p}
\end{aligned}
$$

$\square$

Note that the analysis scaling vector given by Theorem 2.3 is the minimum length solution for the analysis scaling vector. Because the bilinear condition (2.13) is exactly

$$a_{2k} = 2\delta_{0,k} - \sum_{l \in \mathbf{Z}} a_{2l+1}\tilde{a}_{2l+1-2k} ,$$

the minimum length analysis scaling vector must be the analysis scaling vector having the minimum number of nonzero odd terms. From (2.14) and (2.15), such an analysis scaling vector will have the same odd terms as the minimum length synthesis scaling vector. Thus we obtain the biorthogonal Coifman wavelet system with minimum length.

In the remainder of this paper, a biorthogonal Coifman wavelet system of degree N will always be refered to as biorthogonal Coifman wavelet system of degree N with minimum length, which is given by in Theorem 2.3, unless it is otherwise stated. For convenience, we will call it *BCW-N*. The scaling vectors of the minimum length biorthogonal Coifman wavelet systems with degrees $N = 0, 1, 2, 3$, and 4 are listed in Table 1. The biorthogonal Coifman wavelet system with degree 0 (BCW-0) is exactly the Haar wavelet system, which is also orthogonal. The biorthogonal Coifman wavelet system with degree 1 (BCW-1) is a spline system. We include the illustrations for degree 1 and 3 (BCW-1, and BCW-3) in Figure 1 and 2.

Table 1: The Coefficients of Biorthogonal Coifman Wavelet Systems

N	a_k	$\tilde{a}_k$
$N = 0$	$a_0 = 1$	$\tilde{a}_0 = 1$
	$a_1 = 1$	$\tilde{a}_1 = 1$
$N = 1$	$a_{-2} =$ -1/4	
	$a_{-1} = 1/2$	$\tilde{a}_{-1} = 1/2$
	$a_0 = 3/2$	$\tilde{a}_0 = 1$
	$a_1 = 1/2$	$\tilde{a}_1 = 1/2$
	$a_2 =$ -1/4	
$N = 2$	$a_{-4} = 3/64$	
	$a_{-3} = 0$	
	$a_{-2} =$ -3/16	
	$a_{-1} = 3/8$	$\tilde{a}_{-1} = 3/8$
	$a_0 = 41/32$	$\tilde{a}_0 = 1$
	$a_1 = 3/4$	$\tilde{a}_1 = 3/4$
	$a_2 =$ -3/16	$\tilde{a}_2 = 0$
	$a_3 =$ -1/8	$\tilde{a}_3 =$ -1/8
	$a_4 = 3/64$	
$N = 3$	$a_{-6} =$ -1/256	
	$a_{-5} = 0$	
	$a_{-4} = 9/128$	
	$a_{-3} =$ -1/16	$\tilde{a}_{-3} =$ -1/16
	$a_{-2} =$ -63/256	$\tilde{a}_{-2} = 0$
	$a_{-1} = 9/16$	$\tilde{a}_{-1} = 9/16$
	$a_0 = 87/64$	$\tilde{a}_0 = 1$

N	a_k	$\tilde{a}_k$
$N = 3$	$a_1 = 9/16$	$\tilde{a}_1 = 9/16$
	$a_2 =$ -63/256	$\tilde{a}_2 = 0$
	$a_3 =$ -1/16	$\tilde{a}_3 =$ -1/16
	$a_4 = 9/128$	
	$a_5 = 0$	
	$a_6 =$ -1/256	
$N = 4$	$a_{-8} = 15/16384$	
	$a_{-7} = 0$	
	$a_{-6} =$ -35/2048	
	$a_{-5} = 0$	
	$a_{-4} = 345/4096$	
	$a_{-3} =$ -5/128	$\tilde{a}_{-3} =$ -5/128
	$a_{-2} =$ -405/2048	$\tilde{a}_{-2} = 0$
	$a_{-1} = 15/32$	$\tilde{a}_{-1} = 15/32$
	$a_0 = 10317/8192$	$\tilde{a}_0 = 1$
	$a_1 = 45/64$	$\tilde{a}_1 = 45/64$
	$a_2 =$ -405/2048	$\tilde{a}_2 = 0$
	$a_3 =$ -5/32	$\tilde{a}_3 =$ -5/32
	$a_4 = 345/4096$	$\tilde{a}_4 = 0$
	$a_5 = 3/128$	$\tilde{a}_5 = 3/128$
	$a_6 =$ -35/2048	
	$a_7 = 0$	
	$a_8 = 15/16384$	

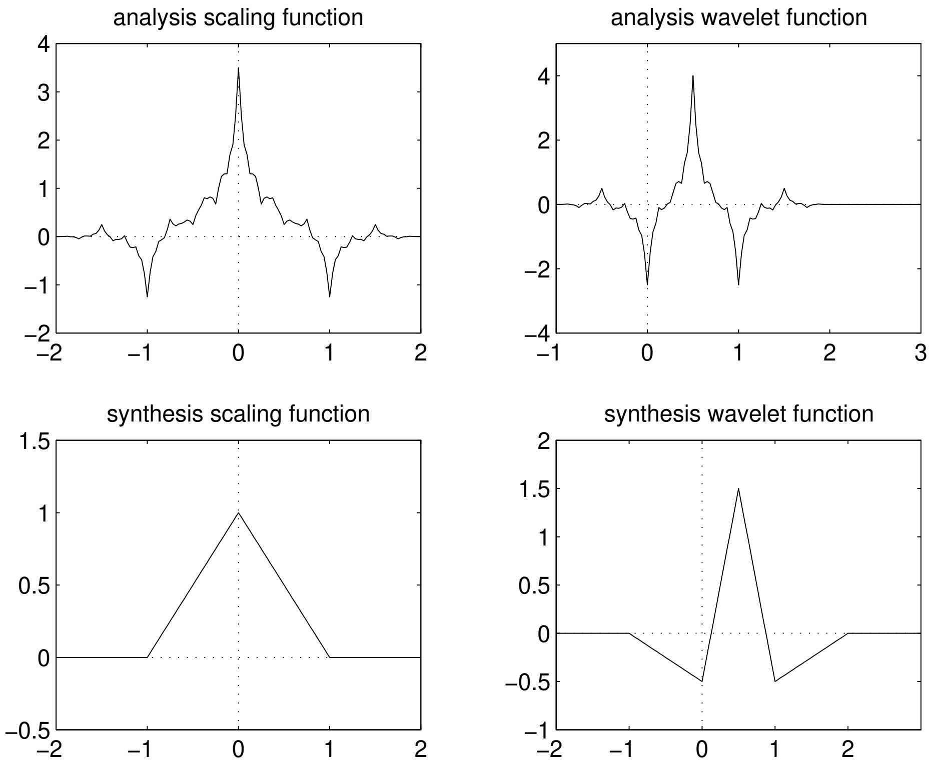

Figure 1: The Biorthogonal Coifman Wavelet System of Degree 1

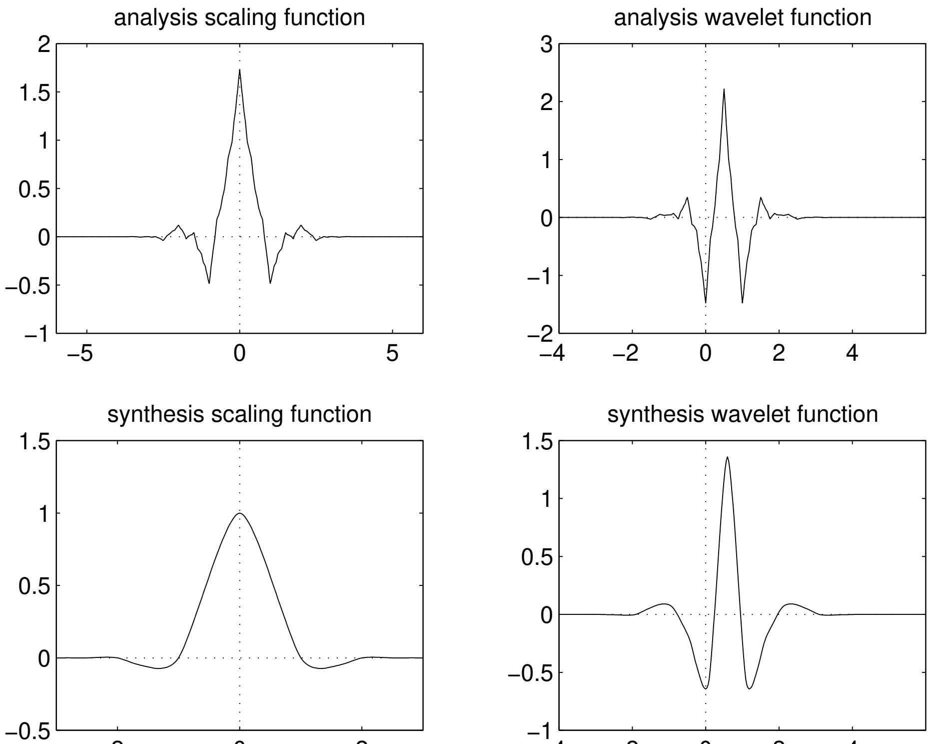

Figure 2: The Biorthogonal Coifman Wavelet System of Degree 3

We are informed that some work by other researchers [26, 25, 3] are related to biorthogonal Coifman wavelet systems. In [26] the odd degrees ($N = 1, 3, 5, 7, \cdots,$) biorthogonal Coifman wavelet systems (which are called *biorthogonal Coiflets* in [26]) were constructed using the lifting scheme. An algorithm was proposed in [3] to solve the frequency response of *pseudo-coiflets*, which are biorthogonal.

3 Multiplication-Free Discrete Wavelet Transform

Though we didn't expect it, the scaling vectors of biorthogonal Coifman wavelet systems are all dyadic rationals, i.e., all the elements in the scaling vectors are of the form $(2p+1)/2^q$, for some $p, q \in \mathbf{Z}$, as it can be seen in Table 1 for $N = 0, 1, 2, 3, 4$. This is a really attractive feature since we can therefore implement a very fast multiplication-free discrete wavelet transform on digital computers. It is one of the main advantage of biorthogonal Coifman wavelet systems over other widely used irrational biorthogonal wavelet systems.

Theorem 3.1 *In a biorthogonal Coifman wavelet system of degree N, the scaling vectors are dyadic rationals, i.e., $\forall k \in \mathbf{Z}$, there must exist four integers p_1, p_2, q_1, q_2, such that*

$$a_k = \frac{2p_1+1}{2^{q_1}}, \quad \tilde{a}_k = \frac{2p_2+1}{2^{q_2}},$$

whenever a_k or $\tilde{a}_k$ are nonzero.

Proof: It is clear that the addition, subtraction, or multiplication between two dyadic rationals is still a dyadic rational. Or we can say these three operations are close in dyadic rationals. So all we need to prove is that $\tilde{a}_{2k+1}$ is a dyadic rational, for all nonzero $\tilde{a}_{2k+1}$. There are two possible cases.

1. if N is even, $N = 2n$,

$$\tilde{a}_{2k+1} = \frac{(-1)^k}{2k+1} \begin{pmatrix} 2n-1 \\ n-1 \end{pmatrix} \begin{pmatrix} 2n \\ n+k \end{pmatrix} \frac{2n+1}{2^{4n-1}} .$$

Both $\begin{pmatrix} 2n-1 \\ n-1 \end{pmatrix}$ and $\begin{pmatrix} 2n \\ n+k \end{pmatrix}$ are integers. Thus it is sufficient to prove that $(2k+1)$ can divide $\begin{pmatrix} 2n-1 \\ n-1 \end{pmatrix} \begin{pmatrix} 2n \\ n+k \end{pmatrix} (2n+1)$. All possible choices for $\tilde{a}_{2k+1}$ being nonzero are $-n \le k \le n$. When $k = -1, 0$, or n, the proof is trivial. So first let's look at $0 < k < n$. Assume for now that $2k+1$ is a prime number. Recall that if p is prime, then for a positive integer m, the factorization of $m!$ has a multiplicity $d(m,p)$ to the p-factor, i.e.,

$$p^{d(m,p)} \mid m! ,$$

$$p^{d(m,p)+1} \not| \, m! ,$$

where the function $d(m,p)$ is defined by

$$d(m,p) := \sum_{l=1}^{\infty} \lfloor \frac{m}{p^l} \rfloor = \lfloor \frac{m}{p} \rfloor + \lfloor \frac{m}{p^2} \rfloor + \cdots + \lfloor \frac{m}{p^l} \rfloor + \cdots ,$$

where $\lfloor \cdot \rfloor$ is the integer part of a real number. Since

$$\begin{pmatrix} 2n-1 \\ n-1 \end{pmatrix} \begin{pmatrix} 2n \\ n+k \end{pmatrix} (2n+1) = \frac{(2n-1)!(2n+1)!}{(n-1)!n!(n+k)!(n-k)!} ,$$

it suffices to show that

$$\lfloor\frac{2n-1}{2k+1}\rfloor+\lfloor\frac{2n+1}{2k+1}\rfloor \;\geq\; \lfloor\frac{n-1}{2k+1}\rfloor+\lfloor\frac{n}{2k+1}\rfloor+\lfloor\frac{n+k}{2k+1}\rfloor+\lfloor\frac{n-k}{2k+1}\rfloor+1\,. \tag{3.18}$$

Let $n-k=t(2k+1)+r$, where $t,r\in\mathbf{Z}, t\geq 0, 0\leq r\leq 2k$.

- if $r=0$, then

$$n+k \;=\; t(2k+1)+2k\,,$$

$$2n+1 \;=\; (2t+1)(2k+1)\,.$$

So

$$\lfloor\frac{2n+1}{2k+1}\rfloor \;=\; 2t+1 \;=\; \lfloor\frac{n-k}{2k+1}\rfloor+\lfloor\frac{n+k}{2k+1}\rfloor+1\,.$$

We know that

$$\lfloor\frac{2n-1}{2k+1}\rfloor \;\geq\; \lfloor\frac{n-1}{2k+1}\rfloor+\lfloor\frac{n}{2k+1}\rfloor\,,$$

Thus (3.18) follows immediately.

- if $1\leq r\leq k$, then

$$n \;=\; t(2k+1)+(k+r)\,,$$

$$n-1 \;=\; t(2k+1)+(k+r-1)\,,$$

$$2n-1 \;=\; (2t+1)(2k+1)+(2r-2)\,.$$

Thus

$$\lfloor\frac{2n-1}{2k+1}\rfloor \;=\; 2t+1 \;=\; \lfloor\frac{n-1}{2k+1}\rfloor+\lfloor\frac{n}{2k+1}\rfloor+1\,.$$

Since

$$\lfloor\frac{2n+1}{2k+1}\rfloor \;\geq\; \lfloor\frac{n+k}{2k+1}\rfloor+\lfloor\frac{n-k}{2k+1}\rfloor\,,$$

the above two imply (3.18).

- if $k+1 \le r \le 2k$, then

$$n+k \;=\; (t+1)(2k+1)+(r-1)\,,$$
$$2n+1 \;=\; (2t+2)(2k+1)+(2r-2k-1)\,.$$

So

$$\lfloor\frac{2n+1}{2k+1}\rfloor \;=\; 2t+2 \;=\; \lfloor\frac{n-k}{2k+1}\rfloor+\lfloor\frac{n+k}{2k+1}\rfloor+1\,.$$

Since

$$\lfloor\frac{2n-1}{2k+1}\rfloor \;\ge\; \lfloor\frac{n-1}{2k+1}\rfloor+\lfloor\frac{n}{2k+1}\rfloor\,,$$

Again, we get (3.18).

If $2k+1$ is not a prime number, assume $2a+1$ is a prime factor of $2k+1$, and $2k+1=(2a+1)(2b+1)$, i.e., $k=2ab+a+b$. Then it suffices to show that

$$\lfloor\frac{2n-1}{2a+1}\rfloor+\lfloor\frac{2n+1}{2a+1}\rfloor \;\ge\; \lfloor\frac{n-1}{2a+1}\rfloor+\lfloor\frac{n}{2a+1}\rfloor+\lfloor\frac{n+k}{2a+1}\rfloor+\lfloor\frac{n-k}{2a+1}\rfloor+1\,. \tag{3.19}$$

Since

$$\begin{aligned}\lfloor\frac{n+k}{2a+1}\rfloor+\lfloor\frac{n-k}{2a+1}\rfloor \;&=\; \lfloor\frac{n+2ab+a+b}{2a+1}\rfloor+\lfloor\frac{n-2ab-a-b}{2a+1}\rfloor\\ &=\; \lfloor\frac{n+a}{2a+1}\rfloor+\lfloor\frac{n-a}{2a+1}\rfloor\end{aligned}$$

the inequality (3.19) is exactly (3.18) when replacing k with a.

Now suppose $-n \le k \le -2$. Set $l=-k, 2 \le l \le n$. We want to show that (as before, if $2l-1$ is not prime, it can be converted to the following inequality)

$$\lfloor\frac{2n-1}{2l-1}\rfloor+\lfloor\frac{2n+1}{2l-1}\rfloor \;\ge\; \lfloor\frac{n-1}{2l-1}\rfloor+\lfloor\frac{n}{2l-1}\rfloor+\lfloor\frac{n+l}{2l-1}\rfloor+\lfloor\frac{n-l}{2l-1}\rfloor+1\,. \tag{3.20}$$

Set $n-l=t(2l-1)+r$, where $t,r\in\mathbf{Z}, t\geq 0, 0\leq r\leq 2l-2$.

- if $0\leq r\leq l-2$, then

$$\begin{aligned} n &= t(2l-1)+(l+r)\,, \\ n-1 &= t(2l-1)+(l+r-1)\,, \\ 2n-1 &= (2t+1)(2l-1)+2r\,. \end{aligned}$$

So
$$\lfloor\frac{2n-1}{2l-1}\rfloor \;=\; 2t+1 \;=\; \lfloor\frac{n-1}{2l-1}\rfloor+\lfloor\frac{n}{2l-1}\rfloor+1\,.$$
Since
$$\lfloor\frac{2n+1}{2l-1}\rfloor \;\geq\; \lfloor\frac{n+l}{2l-1}\rfloor+\lfloor\frac{n-l}{2l-1}\rfloor\,,$$
the inequality (3.20) follows.

- if $l-1\leq r\leq 2l-3$, then

$$\begin{aligned} n+l &= (t+1)(2l-1)+(r+1)\,, \\ 2n+1 &= (2t+2)(2l-1)+(2r-2l+3)\,. \end{aligned}$$

So
$$\lfloor\frac{2n+1}{2l-1}\rfloor \;=\; 2t+2 \;=\; \lfloor\frac{n+l}{2l-1}\rfloor+\lfloor\frac{n-l}{2l-1}\rfloor+1\,.$$
Since
$$\lfloor\frac{2n-1}{2l-1}\rfloor \;\geq\; \lfloor\frac{n-1}{2l-1}\rfloor+\lfloor\frac{n}{2l-1}\rfloor\,,$$
we get (3.20).

- if $r = 2l - 2$, then

$$n + l = (t+2)(2l-1)\,,$$

$$2n + 1 = (2t+3)(2l-1)\,.$$

So

$$\lfloor \frac{2n+1}{2l-1} \rfloor = 2t + 3 = \lfloor \frac{n+l}{2l-1} \rfloor + \lfloor \frac{n-l}{2l-1} \rfloor + 1\,.$$

Since

$$\lfloor \frac{2n-1}{2l-1} \rfloor \geq \lfloor \frac{n-1}{2l-1} \rfloor + \lfloor \frac{n}{2l-1} \rfloor\,,$$

we get (3.20).

2. if N is odd, $N = 2n - 1$,

$$\tilde{a}_{2k+1} = \frac{(-1)^k}{2k+1} \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix} \begin{pmatrix} 2n-1 \\ n+k \end{pmatrix} \frac{2n-1}{2^{4n-3}}\,.$$

Since $\tilde{a}_{2k+1} = \tilde{a}_{-2k-1}$, it will be sufficient to only prove for the case $1 \leq k \leq n-2$. Let $n - k - 1 = t(2k+1) + r$, where $t, r \in \mathbf{Z}, t \geq 0, 0 \leq r \leq 2k$.

- if $r = 0$, then

$$n = t(2k+1) + (k+1)\,,$$

$$2n - 1 = (2t+1)(2k+1)\,,$$

i.e.,

$$(2k+1)|(2n-1)\,.$$

- if $1 \le r \le 2k$, since

$$\binom{2n-2}{n-1}\binom{2n-1}{n+k} = \frac{(2n-2)!(2n-1)!}{(n-1)!(n-1)!(n+k)!(n-k-1)!},$$

we will prove the inequality (again, if $2k+1$ is not prime, it can be reduced to the following inequality)

$$\lfloor\frac{2n-2}{2k+1}\rfloor + \lfloor\frac{2n-1}{2k+1}\rfloor \ge$$
$$\lfloor\frac{n-1}{2k+1}\rfloor + \lfloor\frac{n-1}{2k+1}\rfloor + \lfloor\frac{n-k-1}{2k+1}\rfloor + \lfloor\frac{n+k}{2k+1}\rfloor + 1\,.$$

i. $1 \le r \le k$, then

$$n-1 = t(2k+1)+(k+r)\,,$$

$$2n-2 = (2t+1)(2k+1)+(2r-1)\,.$$

So

$$\lfloor\frac{2n-2}{2k+1}\rfloor = 2t+1 = \lfloor\frac{n-1}{2k+1}\rfloor + \lfloor\frac{n-1}{2k+1}\rfloor + 1\,.$$

ii. $k+1 \le r \le 2k$, then

$$n+k = (t+1)(2k+1)+r\,,$$

$$2n-1 = (2t+2)(2k+1)+(2r-2k-1)\,.$$

So

$$\lfloor\frac{2n-1}{2k+1}\rfloor = 2t+2 = \lfloor\frac{n-k-1}{2k+1}\rfloor + \lfloor\frac{n+k}{2k+1}\rfloor + 1\,.$$

□

4 Convergence to Sinc Wavelet System

The *Sinc wavelet system* is a basic wavelet system whose scaling vector $\{a_k^{sinc}, k \in \mathbf{Z}\}$ is defined by

$$a_{2k}^{sinc} = \delta_{0,k}, \quad a_{2k+1}^{sinc} = \frac{(-1)^k 2}{(2k+1)\pi}.$$

It had been a problem for some time to find a sequence of scaling functions with compact supports which approximates the function $\mathrm{sinc}(\pi x) = \frac{\sin \pi x}{\pi x}$, which is the scaling function of the Sinc wavelet system. This problem is important because of the special relation of the sinc function to signal processing applications. As observed by H. L. Resnikoff, the family of biorthogonal Coifman wavelet systems just provides a very suitable candidate.

Theorem 4.1 *Suppose $\{\tilde{a}_k^N, k \in \mathbf{Z}\}$ to be the synthesis scaling vector of the biorthogonal Coifman wavelet system of degree N. Then*

$$\lim_{N\to\infty} \left\| \left(\tilde{a}^N\right) - \left(a^{sinc}\right) \right\|_{l^2} = \lim_{N\to\infty} \left(\sum_{k\in\mathbf{Z}} \left(\tilde{a}_k^N - a_k^{sinc}\right)^2 \right)^{1/2} = 0.$$

Before proving Theorem 4.1, we first prove that $\{\tilde{a}^N\}$ converges to $\{a^{sinc}\}$ termwise.

Lemma 4.1 *Assume the same condition as in Theorem 4.1, then*

$$\lim_{N\to\infty} \tilde{a}_k^N = a_k^{sinc}.$$

Proof: From the definition, we only need to check

$$\lim_{N\to\infty} \tilde{a}_{2k+1}^N = a_{2k+1}^{\text{sinc}}. \tag{4.21}$$

First let's look at the case when N is even, $N = 2n$. Assume $|k| < n$ (otherwise we can choose a larger N), we have

$$\begin{aligned} \tilde{a}_{2k+1}^N &= \frac{(-1)^k}{2k+1} \begin{pmatrix} 2n-1 \\ n-1 \end{pmatrix} \begin{pmatrix} 2n \\ n+k \end{pmatrix} \frac{2n+1}{2^{4n-1}} \\ &= \frac{(-1)^k}{2k+1} \cdot \frac{(2n-1)!(2n+1)!}{2^{4n-1}(n-1)!n!(n-k)!(n+k)!} \end{aligned}$$

We want to show that

$$\lim_{n\to\infty} \frac{(2n-1)!(2n+1)!}{2^{4n-1}(n-1)!n!(n-k)!(n+k)!} = \frac{2}{\pi}. \tag{4.22}$$

Recall that Stirling's formula states

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi}n^{n+1/2}e^{-n}} = 1$$

So

$$\begin{aligned} &\lim_{n\to\infty} \frac{(2n-1)!(2n+1)!}{2^{4n-1}(n-1)!n!(n-k)!(n+k)!} \\ &= \lim_{n\to\infty} \frac{(2n-1)^{2n-1/2}(2n+1)^{2n+3/2}e^{-1}}{2^{4n}\pi(n-1)^{n-1/2}n^{n+1/2}(n-k)^{n-k+1/2}(n+k)^{n+k+1/2}} \\ &= \lim_{n\to\infty} \frac{e^{-1}}{2^{4n}\pi} \left(\frac{2n-1}{n-1}\right)^{n-1/2} \left(\frac{2n+1}{n}\right)^{n+1/2} \left(\frac{2n-1}{n-k}\right)^{n} \\ &\quad \cdot \left(\frac{2n+1}{n+k}\right)^{n+1} \left(\frac{n-k}{n+k}\right)^{k-1/2} \\ &= \lim_{n\to\infty} \frac{e^{-1}}{2^{4n}\pi} \cdot 2^{n-1/2}e^{1/2} \cdot 2^{n+1/2}e^{1/2} \cdot 2^n e^{1/2} \cdot 2^{n+1}e^{-1/2} \cdot 1 \\ &= \frac{2}{\pi} \end{aligned}$$

We have used

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e\,, \quad \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = e^{-1}$$

in the third step. Then (4.22) is proved.

When N is odd, $N = 2n-1$, the ratio

$$\frac{\tilde{a}^N_{2k+1}}{\tilde{a}^{N-1}_{2k+1}} = \frac{2n-1}{2n+2k} \longrightarrow 1 \quad \text{as } N \to \infty$$

Then (4.21) also holds when N is odd. $\square$

Proof of Theorem 4.1 Since

$$\lim_{N\to\infty}(\tilde{a}^N_k - a^{\text{sinc}}_k) = 0 \text{ for all } k\,,$$

to apply Lebesgue's Dominated Convergence Theorem, we just need to find an $l^2(\mathbf{R})$ dominating sequence $\{d_k, k \in \mathbf{Z}\}$ such that

$$|\tilde{a}^N_k - a^{\text{sinc}}_k| \le d_k \text{ for all } k\,.$$

If N is even, $N = 2n$, we can assume $|k| \le n$ since otherwise $\tilde{a}^N_{2k+1} = 0$ is always bounded. Note that

$$(n-k)!(n+k)! \ge (n!)^2$$

then

$$\begin{aligned}
|\tilde{a}^N_{2k+1}| &= \left|\frac{(2n-1)!(2n+1)!}{(2k+1)2^{4n-1}(n-1)!n!(n-k)!(n+k)!}\right| \\
&\le \left|\frac{(2n-1)!(2n+1)!}{(2k+1)2^{4n-1}(n-1)!(n!)^3}\right|
\end{aligned}$$

Set $k = 0$ in (4.22),

$$\lim_{n\to\infty} \frac{(2n-1)!(2n+1)!}{2^{4n-1}(n-1)!(n!)^3} \;=\; \frac{2}{\pi}$$

So for N large enough,

$$\frac{(2n-1)!(2n+1)!}{2^{4n-1}(n-1)!(n!)^3} \;\le\; 1$$

$$|\tilde{a}^N_{2k+1}| \;\le\; \left|\frac{1}{2k+1}\right|$$

If N is odd, $N = 2n-1$, since

$$(n+k)!(n-1-k)! \;\ge\; (n-1)!n!\,,$$

we have

$$\begin{aligned}
|\tilde{a}^N_{2k+1}| &= \left|\frac{1}{2k+1}\begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix}\begin{pmatrix} 2n-1 \\ n+k \end{pmatrix}\frac{2n-1}{2^{4n-3}}\right| \\
&= \left|\frac{((2n-1)!)^2}{2^{4n-3}(2k+1)((n-1)!)^2(n+k)!(n-1-k)!}\right| \\
&\le \left|\frac{((2n-1)!)^2}{2^{4n-3}(2k+1)((n-1)!)^2(n-1)!n!}\right| \\
&= \left|\frac{1}{2k+1}\tilde{a}^N_1\right|
\end{aligned}$$

Because

$$\lim_{N\to\infty} \tilde{a}^N_1 \;=\; a^{\mathrm{sinc}}_1 \;=\; \frac{2}{\pi}\,,$$

it follows that

$$\tilde{a}^N_1 \;\le\; 1 \;\text{ for } N \text{ large enough}\,.$$

Then

$$|\tilde{a}^N_{2k+1}| \;\le\; \left|\frac{1}{2k+1}\right|.$$

Set

$$d_{2k+1} \;=\; \left|\frac{1}{2k+1}\right| + |a^{\rm sinc}_{2k+1}| \;\leq\; \left|\frac{2}{2k+1}\right| , \quad d_{2k} \;=\; 0$$

we have $(d_k)_{k\in\mathbf{Z}} \in l^2(\mathbf{R})$ and

$$|\tilde{a}^N_k - a^{\rm sinc}_k| \;\leq\; d_k \,,$$

By Lebesgue's Dominated Convergence Theorem and Lemma 4.1,

$$\lim_{N\to\infty} \left|\left|\tilde{\alpha}^N - \alpha^{\rm sinc}\right|\right|_{l^2} \;=\; 0\,.$$

The theorem follows. $\Box$

The Mallat Algorithm only involves scaling vectors. Using the Cauchy Inequality, one can show that the discrete wavelet transforms of biorthogonal Coifman wavelet systems converge to the discrete wavelet transform of the Sinc wavelet system.

Corollary 4.1 *If $\{s_k, k\in\mathbf{Z}\}$ is an $l^2(\mathbf{R})$ sequence, $(\sum_k s_k^2)^{1/2} < \infty$, then*

$$\lim_{N\to\infty} DWT(BCW\text{-}N, \{s_k\}) \;=\; DWT(sinc, \{s_k\})\,.$$

5 Some Properties

The wavelet approximation theorem is the starting point of biorthogonal Coifman wavelet systems. From Definition 2.1, the following theorem is just stating the wavelet approximation theorem in the language of biorthogonal wavelet systems.

Theorem 5.1 *For a biorthogonal Coifman wavelet system of degree N with the analysis scaling function $\phi(x)$ and the synthesis scaling function $\tilde{\phi}(x)$, if $f(x) \in C_0^{N,1}(\mathbf{R})$, define, for $j \in \mathbf{N}$,*

$$f^j(x) \;:=\; 2^{-j/2} \sum_{k \in \mathbf{Z}} f\left(\frac{k}{2^j}\right) \tilde{\phi}_{j,k}(x)\,,$$

where $\tilde{\phi}_{j,k}(x) = 2^{j/2}\tilde{\phi}(2^j x - k)$. Then

$$\left|\left|f(x) - f^j(x)\right|\right|_{L^2} \;\leq\; C 2^{-j(N+1)}\,,$$

where C depends only on f and $\tilde{\phi}$.
If in addition $\tilde{\phi} \in C^n(\mathbf{R})$, where $n \in \mathbf{Z}, 0 \leq n \leq N$, then

$$\left|\left|f(x) - f^j(x)\right|\right|_{H^n} \;\leq\; C 2^{-j(N+1-n)}\,,$$

where C depends only on f and $\tilde{\phi}$.
The same results hold when replacing $\tilde{\phi}$ with ϕ.

As we know, in a wavelet system, the symmetry of the scaling vector corresponds to the symmetry of the scaling function. Thus we call a wavelet system is *symmetric* is the scaling vector is symmetric. One big advantage of biorthogonal wavelet systems over orthogonal wavelet systems is that biorthogonal wavelet systems can be symmetric, while orthogonal ones can't, except for the Haar wavelet system. Symmetry is always pursued whenever possible in applications. For example, in image coding, if we have a symmetric biorthogonal wavelet system, then the image data can be reflectively extended to reduce the edge effect.

The even terms of biorthogonal Coifman wavelet systems are always symmetric, which is apparent from Theorem 2.3. Since the analysis scaling vector and the synthesis scaling vector have the same odd terms, the symmetry of a biorthogonal Coifman wavelet system will depend solely on the symmetry of the odd terms of scaling vectors. It is very easy to verify the following lemma.

Lemma 5.1 *If N is odd, then in a biorthogonal Coifman wavelet system of degree N,*

$$a_{2k+1} \; = \; a_{-2k-1} \, , \quad \tilde{a}_{2k+1} \; = \; \tilde{a}_{-2k-1} \, .$$

Thus all odd degrees biorthogonal Coifman wavelet systems are symmetric.

Theorem 5.2 *If N is odd, then the biorthogonal Coifman wavelet system of degree N is symmetric, i.e.,*

$$a_k \; = \; a_{-k} \, , \quad \tilde{a}_k \; = \; \tilde{a}_{-k} \, .$$

This theorem can also be tested for BCW-0, BCW-1, BCW-2, BCW-3, and BCW-4 in Table 1.

As we know, in a biorthogonal wavelet system with the analysis wavelet function $\psi(x)$ and the synthesis wavelet function $\tilde{\psi}(x)$, the wavelet expansion

$$f \; = \; \lim_{J \to \infty} \sum_{j=-J}^{J} \sum_{k \in \mathbf{Z}} < f, \psi_{j,k} > \tilde{\psi}_{j,k}$$

holds only in the weak L^2 sense. Without any other assumption, the $\psi_{j,k}$ or $\tilde{\psi}_{j,k}$ may even fail to constitute frames. Though we can't derive orthonormal bases from symmetric biorthogonal wavelet systems, we do desire to have some reasonable bases, such as unconditional bases where we can interchange the order of the summation in the wavelet expansion. In a Hilbert space, an *unconditional basis* is also called a *Riesz basis*. The $\psi_{j,k}$, $\tilde{\psi}_{j,k}$ constitute two dual Riesz bases if and only if

$$\int_{-\infty}^{\infty} \phi(x)\tilde{\phi}(x-k)\,dx \;=\; \delta_{0,k} \quad \forall k \in \mathbf{Z}\,. \tag{5.23}$$

We will employ Lawton's Lemma to verify unconditional bases.

Lemma 5.2 (Lawton) *In a compactly supported biorthogonal wavelet system with the analysis pair* $\{\phi(x), \psi(x)\}$ *and the synthesis pair* $\{\tilde{\phi}(x), \tilde{\psi}(x)\}$*, define a multiresolution matrix* $T = (t_{k,l})$*,*

$$t_{k,l} \;=\; \frac{1}{2} \sum_{m\in\mathbf{Z}} a_m \tilde{a}_{m+l-2k}\,,$$

where $\{a_k\}$ *and* $\{\tilde{a}_k\}$ *are the scaling vectors with finite length. If*

$$\int_{-\infty}^{\infty} \phi(x)\,dx \;=\; \int_{-\infty}^{\infty} \tilde{\phi}(x)\,dx \;=\; 1\,,$$

$$\int_{-\infty}^{\infty} \psi(x)\,dx \;=\; \int_{-\infty}^{\infty} \tilde{\psi}(x)\,dx \;=\; 0\,,$$

and T *has 1 as a nondegenerate eigenvalue, then*

$$\int_{-\infty}^{\infty} \phi(x)\tilde{\phi}(x-k)\,dx \;=\; \delta_{0,k} \quad \forall k \in \mathbf{Z}\,.$$

The proof of Lemma 5.2 is exactly the same of the one for orthonormal wavelet systems given by W. M. Lawton [17]. Based on this lemma, it is easy to check that for BCW-0, BCW-1, BCW-2, BCW-3, and BCW-4, $\psi(x)$ and $\tilde{\psi}(x)$ all constitute dual Riesz bases.

A complete transform coding evaluation of biorthogonal Coifman wavelet system was presented in [29]. It turns out that biorthogonal Coifman wavelet systems (especially BCW-3) are very useful for image transform coding and seem to be quite comparable to the wavelet systems used in the state-of-the-art compression systems. For the detailed results, we refer to [29].

6 Orthogonal Coifman Wavelet Systems

As we know, orthogonal wavelet systems not only provide orthonormal bases of $L^2(\mathbf{R})$, but also provide unconditional bases of $L^p(\mathbf{R})$, for $1 < p < \infty$, while biorthogonal ones can't. And orthogonal wavelet transform will preserve the L^2 norm. Thus we can have exact error estimates in the wavelet decomposition domain.

Orthogonal Coifman wavelet systems were first studied by I. Daubechies in [10] (she called these wavelet systems *Coiflets*).

Definition 6.1 *An orthogonal wavelet system with compact support is called an* orthogonal Coifman wavelet system *(in short,* OCW*) of degree N if the vanishing moments of the scaling function $\phi(x)$ and the wavelet function $\psi(x)$ are both of degree N, i.e.,*

$$\begin{aligned}
\mathrm{Mom}_p(\phi) &= \int_{-\infty}^{\infty} x^p \phi(x)\, dx = \delta_{0,p}, \quad \text{for}\, p = 0, \cdots, N\,, \\
\mathrm{Mom}_p(\psi) &= \int_{-\infty}^{\infty} x^p \psi(x)\, dx = 0, \quad \text{for}\, p = 0, \cdots, N\,.
\end{aligned}$$

Orthogonal Coifman wavelet systems seem more "symmetric" and more smooth than the Daubechies wavelet systems. In [10] a method to construct orthogonal Coifman wavelet systems was proposed but the general existence problem is still open. Here we propose a numerical method starting with biorthogonal Coifman wavelet systems.

Biorthogonal Coifman wavelet systems and orthogonal Coifman wavelet systems are connected via the same vanishing moments imposed on scaling functions and wavelet functions. The only difference between these two is that the quadratic condition in orthogonal Coifman wavelet systems

$$\sum_{k\in\mathbf{Z}} a_k a_{k+2l} \;=\; 2\delta_{0,l}\,, \quad \forall l \in \mathbf{Z}$$

is replaced by the bilinear condition

$$\sum_{k\in\mathbf{Z}} a_k \tilde{a}_{k+2l} \;=\; 2\delta_{0,l}\,, \quad \forall l \in \mathbf{Z}\,.$$

How to construct orthogonal Coifman wavelet systems from biorthogonal Coifman wavelet systems is the problem we are tackling. One of the most powerful method to approximate a solution of some system with a known starting point is the well-known Newton's method. One drawback of Newton's method is that even though numerically Newton's iterates might converge to some point, we still can't assert the limit will be exactly a solution. By utilizing a fundamental result due to L. Kantorovich, the existence for some orthogonal Coifman wavelet system can be proved theoretically.

Theorem 6.1 (Newton-Kantorovich Theorem) *Assume D is a bounded, open subset of $\mathbf{R}^n$, $f : D \to \mathbf{R}^n$ is C^1 on a convex set $D_0 \subset D$ such that*

$$||f'(x) - f'(y)|| \;\le\; \gamma ||x-y||\,, \quad \forall x, y \in D_0\,.$$

Suppose that there exists an $x_0 \in D_0$ *such that* $||\,(f'(x_0))^{-1}\,|| \leq \beta$ *and* $\alpha = \beta\gamma\eta \leq 1/2$*, where* $\eta \geq ||\,(f'(x_0))^{-1}\,f(x_0)||$*. Set*

$$t^* \;=\; (\beta\gamma)^{-1}[1-(1-2\alpha)^{1/2}]\,, \quad t^{**} \;=\; (\beta\gamma)^{-1}[1+(1-2\alpha)^{1/2}]\,,$$

and assume that $\overline{B}(x_0, t^*) \subset D_0$*. Then the Newton iterates*

$$x_{k+1} \;=\; x_k - (f'(x_k))^{-1}\,f(x_k)\,, \quad k = 0, 1, \cdots,$$

are well-defined, remain in $\overline{B}(x_0, t^*)$ *and converge to a solution* x^* *of* $f(x) = 0$ *which is unique in* $B(x_0, t^{**}) \cap D_0$*. Here* $B(x_0, r)$ *denotes the open ball of radius* r *about the point* x_0*, and* $\overline{B}(x_0, r)$ *is the close ball.*

The basic idea of the proof is to construct a majorizing sequence for x_k. For a complete proof and some applications of the Newton-Kantorovich theorem, see [16, 15].

The Newton-Kantorovich Theorem is just one example that one can get theoretical results from numerical computation. The significance of this theorem is that even if we don't know the existence of the solution of $f(x) = 0$, we can still apply the Newton's iteration method. If the numerical results are good enough (i.e., the conditions of the Newton-Kantorovich Theorem are satisfied), then it follows that $f(x) = 0$ has a solution and the numerical results will give a fast approximation.

Thus we propose our Newton's method algorithm to get orthogonal Coifman wavelet systems, starting from biorthognal Coifman wavelet systems. Note that since the vanishing moment conditions are just linear equations on the scaling vector $\{a_k\}$, every Newton iterate x_k will always satisfy the vanishing moment conditions.

Newton's method algorithm:

1. Take the synthesis scaling vector of the biorthogonal Coifman wavelet system of degree N as the initial point x_0.

2. Compute Newton iterates starting from x_0.

3. If for some x_{k_0}, the Newton-Kantorovich condition

$$\gamma \cdot ||(f'(x_{k_0}))^{-1}|| \cdot ||(f'(x_{k_0}))^{-1} f(x_{k_0})|| \leq 1/2$$

   is satisfied, where γ is the Lipschitz constant of $f'(x)$, then define $y_0 = x_{k_0}$.

4. An error estimate is needed if we are trying to prove the above inequality from numerical results.

5. From the Newton-Kantorovich Theorem, the Newton iterates starting from y_0 will converge to a solution of $f(x) = 0$.

6. Check that 1 is a nondegenerate eigenvalue of the corresponding multiresolution operator T. Then the iterate y_k we choose will be a good approximation to the scaling vector of the orthogonal Coifman wavelet system of degree N.

Based on the above algorithm, here we give the scaling vectors of orthogonal Coifman wavelet systems of degree through 0 to 9.

$$
\begin{aligned}
a^0 &= \{a_0 = 1 \quad a_1 = 1\} \\
a^1 &= \left\{ \begin{array}{cc}
a_{-3} = -2.214054305846309e-2 & a_{-2} = -1.028594569415369e-1 \\
a_{-1} = 5.442810861169262e-1 & a_0 = 1.205718913883074e+0 \\
a_1 = 4.778594569415369e-1 & a_2 = -1.028594569415369e-1
\end{array} \right\} \\
a^2 &= \left\{ \begin{array}{cc}
a_{-4} = 4.687500000000001e-2 & a_{-3} = -2.116013576461578e-2 \\
a_{-2} = -1.406250000000000e-1 & a-1 = 4.384804072938474e-1 \\
a_0 = 1.140625000000000e+0 & a_1 = 6.865195927061527e-1 \\
a_2 = -4.687500000000001e-2 & a_3 = -1.038398642353842e-1
\end{array} \right\} \\
a^3 &= \left\{ \begin{array}{cc}
a_{-7} = -1.019010798215288e-3 & a_{-6} = -2.578406712281320e-3 \\
a_{-5} = 7.935767225924024e-3 & a_{-4} = 3.348882032655902e-2 \\
a_{-3} = -8.405296092154323e-2 & a_{-2} = -1.081712141834229e-1 \\
a_{-1} = 5.897343873912384e-1 & a_0 = 1.149364787713728e+0 \\
a_1 = 5.460420930695333e-1 & a_2 = -9.527918062201626e-2 \\
a_3 = -5.864027596693714e-2 & a_4 = 2.317519347743374e-2
\end{array} \right\} \\
a^4 &= \left\{ \begin{array}{cc}
a_{-8} = 1.100684225065227e-3 & a_{-7} = -7.406275252609123e-4 \\
a_{-6} = -1.708984375000000e-2 & a_{-5} = 9.921465811937182e-3 \\
a_{-4} = 6.893895537402157e-2 & a_{-3} = -7.756041618077222e-2 \\
a_{-2} = -1.268710684973909e-1 & a_{-1} = 5.383395571089353e-1 \\
a_0 = 1.121367647372065e+0 & a_1 = 6.372385805173691e-1 \\
a_2 = -5.903279734843452e-2 & a_3 = -1.244177315465760e-1 \\
a_4 = 1.158642262467386e-2 & a_5 = 1.721917181436737e-2
\end{array} \right\} \\
a^5 &= \left\{ \begin{array}{cc}
a_{-11} = -4.893146851061624e-5 & a_{-10} = -1.003855491065533e-4 \\
a_{-9} = 6.593303475863657e-4 & a_{-8} = 1.580410201915188e-3 \\
a_{-7} = -3.640917831132484e-3 & a_{-6} = -1.273920202209764e-2 \\
a_{-5} = 2.245848192407572e-2 & a_{-4} = 4.886818864233392e-2 \\
a_{-3} = -1.163925015231710e-1 & a_{-2} = -1.015402815097777e-1 \\
a_{-1} = 6.059671435456483e-1 & a_0 = 1.122570513740660e+0 \\
a_1 = 5.730066705472950e-1 & a_2 = -8.644152712042383e-2 \\
a_3 = -9.301552895745398e-2 & a_4 = 3.316712095834069e-2 \\
a_5 = 1.100625341566277e-2 & a_6 = -5.364837341844136e-3
\end{array} \right\} \\
a^6 &= \left\{ \begin{array}{cc}
a_{-12} = 4.606896825041038e-5 & a_{-11} = -3.823393326605808e-5 \\
a_{-10} = -7.400929411026428e-4 & a_{-9} = 5.445459664138592e-4 \\
a_{-8} = 6.648554460215593e-3 & a_{-7} = -4.301103695860125e-3 \\
a_{-6} = -2.968707220366963e-2 & a_{-5} = 2.490802670578170e-2 \\
a_{-4} = 7.414330425968660e-2 & a_{-3} = -1.121458311492803e-1 \\
a_{-2} = -1.121080284733010e-1 & a_{-1} = 5.777841424573339e-1 \\
a_0 = 1.105616520630899e+0 & a_1 = 6.229169131781114e-1 \\
a_2 = -6.088387476537906e-2 & a_3 = -1.361653712212063e-1 \\
a_4 = 1.972245504790986e-2 & a_5 = 2.981989212955940e-2 \\
a_6 = -2.757834983508583e-3 & a_7 = -3.322980437587264e-3
\end{array} \right\}
\end{aligned}
$$

$$a^7 = \left\{ \begin{array}{cc} a_{-15} = -2.524358359989246e-6 & a_{-14} = -4.609838325353006e-6 \\ a_{-13} = 4.416569382461668e-5 & a_{-12} = 8.816045323194410e-5 \\ a_{-11} = -3.676592334272509e-4 & a_{-10} = -8.330003901883414e-4 \\ a_{-9} = 1.791187855390569e-3 & a_{-8} = 5.305329827060999e-3 \\ a_{-7} = -8.002021689901074e-3 & a_{-6} = -2.151263231017449e-2 \\ a_{-5} = 3.547166284540619e-2 & a_{-4} = 5.562727391693902e-2 \\ a_{-3} = -1.360762293560412e-1 & a_{-2} = -9.422547504779143e-2 \\ a_{-1} = 6.143146193357714e-1 & a_{0} = 1.106252910079098e+0 \\ a_{1} = 5.873348100322010e-1 & a_{2} = -7.930530592489823e-2 \\ a_{3} = -1.149284838038542e-1 & a_{4} = 3.773447713912610e-2 \\ a_{5} = 2.272492296652961e-2 & a_{6} = -1.038905032694064e-2 \\ a_{7} = -2.304450287539944e-3 & a_{8} = 1.261922422861893e-3 \end{array} \right\}$$

$$a^8 = \left\{ \begin{array}{cc} a_{-16} = 2.253190780495695e-6 & a_{-15} = -2.152143904068564e-6 \\ a_{-14} = -4.295842247192477e-5 & a_{-13} = 3.740552476712784e-5 \\ a_{-12} = 4.112285429205518e-4 & a_{-11} = -3.211387783790904e-4 \\ a_{-10} = -2.797279987693231e-3 & a_{-9} = 1.941769422898988e-3 \\ a_{-8} = 1.264304572933647e-2 & a_{-7} = -9.504571148718980e-3 \\ a_{-6} = -3.739795911061904e-2 & a_{-5} = 3.842472941974922e-2 \\ a_{-4} = 7.415880120667900e-2 & a_{-3} = -1.343151858130729e-1 \\ a_{-2} = -1.007162770426715e-1 & a_{-1} = 5.980087017878338e-1 \\ a_{0} = 1.094274969346707e+0 & a_{1} = 6.175739132097274e-1 \\ a_{2} = -5.997859151294635e-2 & a_{3} = -1.444589178644569e-1 \\ a_{4} = 2.482936945667842e-2 & a_{5} = 3.941500144032918e-2 \\ a_{6} = -6.044177030563314e-3 & a_{7} = -7.488005003312453e-3 \\ a_{8} = 6.575756338630860e-4 & a_{9} = 6.884499465383870e-4 \end{array} \right\}$$

$$a^9 = \left\{ \begin{array}{cc} a_{-19} = -1.358212135464472e-7 & a_{-18} = -2.296399299805941e-7 \\ a_{-17} = 2.915005842710325e-6 & a_{-16} = 5.233619320026196e-6 \\ a_{-15} = -3.008063596395375e-5 & a_{-14} = -5.829368777244230e-5 \\ a_{-13} = 1.984938227974709e-4 & a_{-12} = 4.268915950172564e-4 \\ a_{-11} = -9.016444801392472e-4 & a_{-10} = -2.349895868827066e-3 \\ a_{-9} = 3.438766968771003e-3 & a_{-8} = 9.562233598261282e-3 \\ a_{-7} = -1.295349950301171e-2 & a_{-6} = -2.794258537276394e-2 \\ a_{-5} = 4.620914455413364e-2 & a_{-4} = 5.838938555056139e-2 \\ a_{-3} = -1.492888402656793e-1 & a_{-2} = -8.773462965647225e-2 \\ a_{-1} = 6.194005181568405e-1 & a_{0} = 1.095016542708073e+0 \\ a_{1} = 5.961918029174383e-1 & a_{2} = -7.360510694893753e-2 \\ a_{3} = -1.299967565094465e-1 & a_{4} = 3.983803439596872e-2 \\ a_{5} = 3.310436661298578e-2 & a_{6} = -1.432822469882016e-2 \\ a_{7} = -5.882156328071385e-3 & a_{8} = 3.080573451990462e-3 \\ a_{9} = 5.071055047160665e-4 & a_{10} = -2.999290456691924e-4 \end{array} \right\}$$

Actually, using the Newton-Kantorovich Theorem, the existence problem of orthogonal Coifman wavelet systems of arbitrary degree can be reduced to the proof of nonsingularity of a specific Jacobian matrix. We will report on this in a forthcoming paper.

7 Conclusions

In this paper, we have studied biorthogonal Coifman wavelet systems, a family of biorthogonal wavelet systems whose scaling vectors being dyadic rational. Moreover, these wavelet systems converge to the Sinc wavelet system. The image coding evaluation of biorthogonal Coifman wavelet system is very positive and indicates the high promising of these biorthogonal systems in image transform coding.

In [29], a generalization of biorthogonal Coifman wavelet systems is discussed. In it, the condition that the analysis wavelet function and the synthesis wavelet function must have the same degree of vanishing moments is relaxed. Interested readers may refer to that paper.

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