SOBOLEV REGULARITY FOR RANK M WAVELETS

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July 1997

Abstract. This paper explores the Sobolev regularity of rank M wavelets and refinement schemes. We find that the regularity of orthogonal wavelets with maximal vanishing moments grows at most logarithmically with filter length when M is odd, but linearly for even M. When $M = 3$ and $M = 4$, we show that the regularity does achieve these upper bounds for asymptotic growth, complementing earlier results for $M = 2$. A new class of wavelet filters is introduced, by asserting zeros of the wavelet symbol at preperiodic points of the mapping $\tau : \omega \to M\omega \bmod 2\pi$. While this class includes the generalized Daubechies wavelets, numerical experiments demonstrate that the class also includes wavelet families with greater smoothness for a given filter length. Finally, members of the class of wavelets that have maximal Sobolev regularity for a given filter length are found as the solution to an optimization problem.

Key words. wavelets, refinement equations, Sobolev regularity, smoothness, filter design

AMS subject classifications. 42C15, 46E35, 39A10

1. Introduction.

This article presents work on the Sobolev regularity of rank M wavelets and refinement schemes. A rank M wavelet scaling function is the solution to the refinement scheme

$$\phi(x) = \sum_{k=0}^{L-1} a_k \phi(Mx - k) \tag{1.1}$$

with dilation factor $M \in \mathbf{Z}$, $M \geq 2$, and real coefficients a_k satisfying $\sum_k a_k = M$, in which case (1.1) has a unique solution $\phi(x)$ that is a distribution of compact support. Under mild additional conditions ([43], Theorem 4.1) ϕ belongs to $L^2(\mathbf{R})$. Consider the Fourier series

$$A(\omega) = \frac{1}{M} \sum_k a_k e^{ik\omega}$$

associated with the sequence $\{a_k\}$. Then the solution $\phi(x)$ to (1.1) has Fourier transform $\hat{\phi}$ satisfying

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} A(\xi/M^j) \ .$$

It is of particular interest to consider rank M orthogonal wavelets of compact support [14], [21], [47], [49], defined by a *rank M wavelet matrix* $a = (a_{r,k})$, which is an $M \times Mg$ matrix[1] satisfying the orthogonality condition

$$\sum_k a_{r,k} a_{r',k+Ml} = M\delta_{r,r'}\,\delta_{l,0} \tag{1.2}$$

and the sum condition

$$\sum_k a_{r,k} = M\delta_{r,0} \ . \tag{1.3}$$

[1] In this paper we consider only the case where the matrix entries $a_{r,k}$ are real; the more general case of complex entries is not substantially different.

The first row of the matrix is called the scaling sequence; it is a lowpass filter in signal processing terminology, and is the sequence used in the refinement equation (1.1). The remaining $M-1$ rows of the matrix are used in conjunction with the scaling function $\phi(x)$ to define a family of wavelets $\psi_{r,j,k}$ $(1 \le r \le M-1)$ according to

$$\psi_r(x) = \sum_k a_{r,k}\phi(Mx-k)$$

and

$$\psi_{r,j,k} = M^{-j/2}\psi_r(M^{-j}x-k) \; .$$

Daubechies' rank $M=2$ orthogonal wavelets [10] have been generalized to the rank M case in the following fashion: A further N-th order sum condition may be imposed on the scaling sequence:

$$\sum_k \zeta^k k^n a_{0,k} = 0 \quad \text{for } n = 0, 1, \ldots, N-1 \; , \tag{1.4}$$

where $\zeta = e^{2\pi i/M}$ is the primitive M-th root of unity. A wavelet system satisfying (1.4) is said to have *degree* N; this condition means that the lowpass filter and scaling function approximate polynomials of degree $\le N-1$; equivalently the first N wavelet moments vanish ([42], [18], [40]).

Lawton [29] proved that the collection $\{\psi_{r,j,k}(x)\}$ forms a tight frame for $L^2(\mathbf{R})$. Furthermore, the symbol $A(\omega)$ satisfies Cohen's condition (see section 3) and $\phi \in L^2(\mathbf{R})$, if and only if $\{\psi_{r,j,k}(x)\}$ forms a Riesz basis for $L^2(\mathbf{R})$. The present article addresses the Sobolev regularity of such wavelet bases and tight frames. Since the wavelets $\psi_{r,j,k}$ are linear combinations of dilates of the scaling function ϕ, this question reduces to the study of ϕ.

Given a real number s, the Sobolev space $\mathcal{H}^s$ is defined by

$$\mathcal{H}^s := \left\{ f \;:\; \|f\|^2_{\mathcal{H}^s} := \int_{\mathbf{R}} |\hat{f}(\xi)|^2 \left(1+|\xi|^2\right)^s d\xi < \infty \right\} \; .$$

Recall that for $f : \mathbf{R} \to \mathbf{C}$, $f \in \mathcal{H}^s \Rightarrow f \in \mathcal{C}^\alpha$, for $\alpha < s - \frac{1}{2}$. That is, if $s > n + \frac{1}{2}$, then f has n continuous derivatives. We define the critical Sobolev exponent (or Sobolev smoothness) of a function $\phi \in L^2(\mathbf{R})$ to be

$$s(\phi) := \sup\{s \;:\; \phi \in \mathcal{H}^s\} \; .$$

It has been shown [15] that a solution ϕ to the refinement equation (1.1) belongs to $\mathcal{H}^s$ if and only if $s < s(\phi)$. Our first question is the following: Given the scaling sequence $a_{0,k}$, what is the critical Sobolev exponent for $\phi(x)$? This work has been carried out in the rank 2 case by a number of authors ([10], [12], [43], , [6], [7]) and is generalized to arbitrary M in sections 2 and 3, where we obtain a formula for the critical Sobolev exponent of a wavelet scaling function in terms of the maximal eigenvalue of a finite-dimensional Perron-Frobenius operator.

A second question is: What is the asymptotic rate of increase in smoothness of scaling functions associated with a given family of scaling sequences, as the sequence length increases? For the $M=2$ Daubechies orthogonal wavelets, the rate of increase was found to be linear, with slope approximately $.2075N$, where N is the approximation degree of a sequence of length $2N$. In this paper we address this question for the rank M generalized Daubechies wavelets (wavelets with maximal vanishing moments), and find a striking dichotomy between the cases of even and odd M. As discussed in section 4, we demonstrate an upper bound for the asymptotic rate of increase in smoothness, and this upper bound is *linear* in N and the sequence length when the dilation factor M is even, and *logarithmic* in N and the sequence length when M is odd. Furthermore, we present proofs for the cases $M = 3$ and 4 that this upper bound is attained, i.e. that the asymptotic growth in smoothness is indeed linear with the given slope for $M = 4$ and logarithmic at the given rate for $M = 3$. We do so by generalizing the methods of Eirola [12], based on the study of positive operators on invariant cones. We also present numerical evidence that the upper bounds are attained for ranks $M > 4$ as well.

Based on the appearance in this study of the mapping

$$\tau : \omega \to M\omega \bmod 2\pi ,$$

we then introduce a new family of scaling functions which includes the maximal vanishing moments family as a special case. We call this family the PPZ wavelets (for PrePeriodic Zero), because members of the family are deffined by asserting zeros of specified orders at at specified *preperiodic* points of τ. Recall that a periodic point ω_P is a point which satisfies $\tau^k(\omega_P) = \omega_P$ for some $k \geq 0$, and a preperiodic point is a preimage under τ of a periodic point. The $M = 2$ Daubechies orthogonal wavelets and the arbitrary rank generalized Daubechies wavelets can all be described as PPZ wavelets. Several new PPZ wavelet families are introduced for the cases $M = 2$, $M = 3$, and $M = 4$, with the choice of zeros informed by our study of asymptotic regularity. In each case, numerical measurement of the critical Sobolev exponent for family members with sequence length up to 60 suggests that the smoothness of the new families grows at a rate faster than for the Daubechies case. In particular, for rank (dilation factor) 3, the asymptotic growth in smoothness appears to improve from $\log L$ to better than $\sqrt{L}$, where L is the filter length.

Finally, we examine the family of wavelets that are maximally smooth (in the Sobolev sense) for a given support length. This family is not *a priori* algebraically defined, and we employ optimization techniques to find elements of this family for filter lengths up through 30 in the rank 2 case. The resulting filters are smoother than the same-length elements of the previous wavelet families, and for the examples studied, the smoothness grows faster with filter length than for any of the PPZ families previously defined. An unresolved question is whether this family, defined by optimization procedure, can be expressed as a PPZ family (i.e. algebraically specified).

Rank M wavelets, their regularity, and the asymptotic growth in regularity are of concern to a variety of applications, from computer graphics and image processing to audio compression and telecommunications. The generalized Daubechies wavelets discussed here give rise to M-fold interpolation filters that are exact for polynomial functions [17], and the smoothness associated to the interpolators describes their suitability for use in image display [23]. Thus the asymptotic growth in smoothness can be used to determine the filter length required for a desired regularity. Rank M wavelets have also been used as a transform for lossy image compression by several authors [33], [20], [11], and the regularity of a wavelet system has been shown to directly affect the performance of a wavelet-transform-based image compression system [31], [36]. Smoothness has also been exploited as a constraint on image quantization, leading to higher quality compressed images [3]. In the domain of audio signal processing, tree structures of M-band wavelet filters have been employed to construct a transform matching the critical bands of human hearing [39], [26]. Any use of filter banks in tree structures raise questions of smoothness under iteration. Finally, rank M wavelets have been applied with great success to multicarrier modulation [38].

1.1. Explicit Formulae for Generalized Daubechies Wavelets. Since we are to consider the rank M generalizations of Daubechies wavelets, defined as the minimal length orthogonal wavelets satisfying N sum conditions (1.4), we will need explicit formulae for them, as derived in [18]. Examining the Fourier transforms

$$A_r(\omega) = \frac{1}{M} \sum_k a_{r,k} e^{ik\omega}$$

of the wavelet sequences, the conditions (1.2)-(1.4) can be restated as

$$\sum_{k,k'=0}^{M-1} A_r(\omega + 2\pi k/M)\, \overline{A_{r'}(\omega + 2\pi k'/M)} \equiv \frac{1}{M} \delta_{r,r'} \ , \tag{1.5}$$

$$A_r(0) = \delta_{r,0} \ , \tag{1.6}$$

and

$$A_0(\omega) = \left(\frac{1 - e^{iM\omega}}{1 - e^{i\omega}} \right)^N Q(e^{i\omega}) \ , \tag{1.7}$$

for some trigonometric polynomial $Q(e^{i\omega})$. Throughout the rest of this paper, since regularity is determined by the scaling sequence, we shall drop the subscript 0 and refer to the scaling sequence a_k and the symbol $A(\omega)$.

For our L^2 study of regularity, it will be useful to work with the trigonometric polynomial

$$P(\omega) = |A(\omega)|^2 \ .$$

If the wavelet system has degree N, we can factor P as

$$P(\omega) = \left[\frac{1 - \cos M\omega}{M(1 - \cos\omega)} \right]^N R(\omega) \ , \tag{1.8}$$

and since A satisfies the orthogonality condition (1.5),

$$P(\omega) + P\left(\omega + \frac{2\pi}{M}\right) + \cdots + P\left(\omega + \frac{\pi(M-1)}{M}\right) \equiv 1 \ . \tag{1.9}$$

When P is of the form (1.8) and satisfies (1.9), the general solution is given by a trigonometric polynomial R of the form

$$R(\omega) = R_N(\omega) + \tilde{R}(\omega) \ . \tag{1.10}$$

Here R_N is the minimal length solution, generalizing Daubechies dyadic (rank 2) wavelets. It is determined by the parameters M and N, and takes the form

$$R_N(\omega) = \sum_{n=0}^{N-1} r_n (\cos\omega - 1)^n \ . \tag{1.11}$$

When M is even,

$$r_n = \sum_{k_1+k_2+\cdots+k_{M/2}=n} \left\{ \prod_{m=1}^{M/2-1} \binom{2N+k_m-1}{2N-1} \left(1-\cos\frac{2\pi m}{M}\right)^{-k_m} \right\}$$

$$\times \binom{N+k_{M/2}-1}{N-1} (1-\cos\pi)^{-k_{M/2}} \ . \tag{1.12}$$

When M is odd,

$$r_n = \sum_{k_1+k_2+\cdots+k_{\frac{M-1}{2}}=n} \left\{ \prod_{m=1}^{\frac{M-1}{2}} \binom{2N+k_m-1}{2N-1} \left(1-\cos\frac{2\pi m}{M}\right)^{-k_m} \right\} . \tag{1.13}$$

The arbitrary-length degree N solutions are given by formula (1.10), where R_N is of the stated form and

$$\tilde{R}(\omega) = (1-\cos\omega)^N \sum_{n \neq Mk} \tilde{r}_n \cos n\omega \tag{1.14}$$

for some finite set of coefficients $\tilde{r}_n$. Given $R(\omega)$ as in (1.10) and P as in (1.8), the symbol A will be a spectral factor of P, of the form

$$A(\omega) = \left(\frac{1+e^{i\omega}+\cdots+e^{i(M-1)\omega}}{M} \right)^N Q(\omega) \, ,$$

where the trigonometric polynomial Q is a spectral factor of R, i.e. $|Q(\omega)|^2 = R(\omega)$. This factorization requires that $R_N(\omega) \geq 0$ for $\omega \in [0, 2\pi]$.

These general parametrizations [18] (see also [10], [48]) can be used to describe various families of orthogonal wavelets, including the smoother families of section 5.1, the "coiflets" of [10], and the smoother wavelets of [46].

2. The Transfer Operator T_P. The *transfer operator* associated with a scaling sequence will play a key role in our study of the smoothness of wavelet systems. The (rank M) transfer operator[2] associated with a periodic function $P(\omega)$ is defined in the frequency domain as the map taking a function u on $[0, 2\pi)$ to the function

$$T_P u\,(\omega) = P\left(\frac{\omega}{M}\right) u\left(\frac{\omega}{M}\right) + P\left(\frac{\omega+2\pi}{M}\right) u\left(\frac{\omega+2\pi}{M}\right) + \dots$$
$$\dots + P\left(\frac{\omega+2\pi(M-1)}{M}\right) u\left(\frac{\omega+2\pi(M-1)}{M}\right) \ ,$$

where we have extended the domain of definition of u by its natural periodization. The transfer operator is defined in the time domain as the operator taking a sequence u_n to

$$(\check{T}_P u)_n = M \sum_k p_k u_{Mn-k} \ . \tag{2.15}$$

[2] This operator has been referred to variously as the "transition operator," [8, 7], the "wavelet-Galerkin operator," [29, 30], and the "multiresolution operator" [22] in the literature. It seems to have first arisen in the ergodic theory literature as the "transfer operator of a spin system" [2, 37, 34], and so we use that terminology.

If $P(\omega)$ has the Fourier expansion $\sum_n p_n e^{in\omega}$, and $u(\omega) = \sum_n u_n e^{in\omega}$, then (2.15) expresses the relationship between the Fourier coefficients u_n of u and those of $\check{T_P} u$; i.e. $(\check{T_P} u)_n$ is the n-th Fourier coefficient of Tu. The transfer operator has a natural meaning in the context of multirate signal processing [9]: it represents convolution with the sequence $\{p_k\}$ followed by downsampling by a factor of M.

We use the notation T_P for the transfer operator defined by $P(\omega)$ (the rank M is implicit) and refer to P as the *symbol* of the operator T_P, in analogy with the theory of pseudodifferential operators. If P is a trigonometric polynomial (i.e. $\{p_n\}$ is a finite sequence), then T_P is a bounded linear operator on the Banach space $\mathcal{C}[0,\pi]$ of continuous functions on $[0,\pi]$ equipped with the sup norm. While the transfer operator T_P defined by $P = |A|^2$, is of interest, we will find greater use for the *reduced transfer operator* T_R formed from the remainder $R(\omega)$:

$$T_R u(\omega) = R\left(\frac{\omega}{M}\right) u\left(\frac{\omega}{M}\right) + R\left(\frac{\omega+2\pi}{M}\right) u\left(\frac{\omega+2\pi}{M}\right) + \dots$$
$$\dots + R\left(\frac{\omega+2\pi(M-1)}{M}\right) u\left(\frac{\omega+2\pi(M-1)}{M}\right) .$$

2.1. Invariant finite-dimensional subspaces for T_P and T_R. Given a scaling sequence $\{a_k\}$, the transfer operators T_P and T_R have interesting properties when restricted to certain finite-dimensional subspaces, which we now develop after Cohen-Daubechies [7]. Let the trigonometric polynomials P and R be expressed in the form

$$P(e^{i\omega}) = \sum_{j=-L}^{L} p_j e^{ij\omega} , \quad R(e^{i\omega}) = \sum_{j=-J}^{J} \rho_j e^{ij\omega} , \tag{2.16}$$

Moreover define the $2L+1$-dimensional space of trigonometric polynomials

$$\mathcal{E}_L := \left\{ u(\omega) = \sum_{j=-L}^{L} u_j e^{ij\omega} \right\} ,$$

the analogous $2J+1$-dimensional subspace $\mathcal{E}_J$, and the distinguished subspace

$$\mathcal{F}_{L,N} := \left\{ f \in \mathcal{E}_L : f(\omega) = (1-\cos\omega)^N g(\omega) \right\}$$

of $\mathcal{E}_L$. We now consider the action of the transfer operators T_P and T_R.

LEMMA 2.1. *Suppose that the wavelet system associated with the scaling sequence $\{a_k\}$ has degree N (the first N wavelet moments vanish). Then*

(i) *The subspace $\mathcal{E}_L$ of $\mathcal{C}[0,\pi]$ is an invariant subspace for the operator T_P, i.e.*

$$T_P : \mathcal{E}_L \to \mathcal{E}_L .$$

The operator T_P acting on $\mathcal{E}_L$ has the matrix representation $[T_P]_{j,k} = p_{Mj-k}$, with respect to the basis $\left\{e^{-iL\omega}, \dots, 1, \dots, e^{iL\omega}\right\}$, where $p_l = 0$ if $l < -L$ or $l > L$. This matrix is pseudo-circulant, obtained from shifts of the sequence p_l by M for each row. For example, if $L = Mg-1$ for some $g \in \mathbf{Z}$, as is the case when $\{a_k\}$ has length Mg, the matrix has the form

$$
T_P = \begin{bmatrix}
0 & 0 & & & \cdots & & & 0 & 0 \\
 & & & & \ddots & & & & \\
0 & 0 & & & \cdots & & & 0 & 0 \\
p_{L-M+2} & \cdots & p_L & 0 & 0 & & \cdots & 0 & 0 \\
p_{L-2M+2} & \cdots & p_{L-M} & p_{L-M+1} & \cdots & p_L & 0 & \cdots & 0 \\
 & & & \ddots & & & & & \\
p_{-L} & \cdots & p_{M-2-L} & p_{M-1-L} & p_{M-L} & & \cdots & & p_L \\
0 & \cdots & 0 & 0 & p_{-L} & \cdots & & & p_{L-M} \\
 & & & & \ddots & & & & \\
0 & 0 & & & \cdots & 0 & p_{-L} & \cdots & p_{M-2-L} \\
0 & 0 & & & \cdots & & & 0 & 0 \\
 & & & & & \ddots & & & \\
0 & 0 & & & \cdots & & & 0 & 0
\end{bmatrix} .
$$

(ii) *Similarly, the subspace $\mathcal{E}_J$ of $\mathcal{C}[0,\pi]$ is an invariant subspace for the operator T_R, i.e.*

$$
T_R : \mathcal{E}_J \to \mathcal{E}_J \, .
$$

We write $\widetilde{T}_R$ for the restriction of T_R to the subspace $\mathcal{E}_J$; it has the matrix representation $\left[\widetilde{T}_R\right]_{j,k} = \rho_{Mj-k}$, with respect to the basis $\left\{e^{-iJ\omega}, \ldots, 1, \ldots, e^{iJ\omega}\right\}$. Here $\rho_j = 0$ if $j < -J$ or $j > J$. This matrix has a pseudocirculant form similar to that of T_P above.

(iii) *$T_P|_{\mathcal{E}_L}$ includes among its eigenvalues the numbers M^{-n}, $n = 0, 1, \ldots, 2N-1$.*

(iv) *Suppose $f \in \mathcal{F}_{L,N}$, $f(\omega) = (1-\cos\omega)^N g(\omega)$; then*

$$
T_P f(\omega) = \frac{1}{M^{2N}} (1-\cos\omega)^N T_R \, g(\omega) \, . \tag{2.17}
$$

Hence $\mathcal{F}_{L,N}$ is an invariant subspace for T_P, and moreover if f is an eigenvector for T_P in $\mathcal{F}_{L,N}$ with eigenvalue μ, then

$$
T_R g(\omega) = M^{2N} \mu g(\omega) \, ,
$$

so that g is an eigenfunction for T_R with eigenvalue $M^{2N}\mu$. **Remark:** Part (iv) establishes a one-to-one correspondence between eigenfunctions of $T_P|_{\mathcal{F}_{L,N}}$ and eigenfunctions of $T_R|_{\mathcal{E}_{L-N}}$. This will enable us to reduce the study of the spectrum of T_P to that of $\widetilde{T}_R$.

Proof: To prove (i), given $u \in \mathcal{E}_L$ and P as in (2.16), extend $T_P u$ to be an even function on $[-\pi,\pi]$, and expand it in a Fourier series on this interval. The sum over M terms in the definition of T_P offsets the dilation by $\frac{1}{M}$ so that no fractional powers of sine or cosine appear. The orthogonality of the set $\{\sin k\omega, \cos k\omega\}$ ensures that all but a finite number of coefficients in the Fourier expansion of $T_P u$ vanish. In fact, $T_P u$ is a cosine polynomial of degree $\mathrm{ceil}(L/M) < L$, and $T_P u \in \mathcal{E}_L$.

The proof of (ii) is identical.

(iii) is a consequence of the factorization

$$P(\omega) = \left| \frac{1 - e^{iM\omega}}{M(1 - e^{i\omega})} \right|^{2N} R(\omega) , \tag{2.18}$$

which implies that the partial moments

$$\mathcal{M}_{k,n} = \sum_l (k + Ml)^n p_{k+Ml}$$

are equal to one number $\mathcal{M}_n$ independent of k, $k = 0, 1, \ldots, M-1$ for $n = 0, 1, \ldots, 2N-1$. Upon considering the polynomial sequence $e_n = (j^n)_{j=-L,\ldots,L} \in \mathcal{E}_L$, one finds that

$$e_n T = M^{-n} \sum_{k=0}^{n} \binom{n}{k} \mathcal{M}_{n-k} e_k .$$

Since $\mathcal{M}_0 = 1$ with our normalizations, each e_n is a generalized left eigenvector of T_P with eigenvalue M^{-n} for $n = 0, 1, \ldots, 2N-1$.

(iv) follows from the factorization (2.18) of P:

$$\begin{aligned} T_P f(\omega) &= T_P \left(\left| 1 - e^{i\omega} \right|^{2N} g(\omega) \right) \\ &= \left| \frac{1 - e^{i\omega}}{M(1 - e^{i\omega/M})} \right|^{2N} R\left(\frac{\omega}{M}\right) \left| 1 - e^{i\omega/M} \right|^{2N} g\left(\frac{\omega}{M}\right) + \cdots \\ &= \frac{1}{M^{2N}} \left| 1 - e^{i\omega} \right|^{2N} T_R\, g(\omega) . \end{aligned}$$

□

2.2. The Positive Operator T_R. When $R(\omega) > 0$, as is the case for the minimal length solutions $R_N(\omega)$, we can make use of the theory of positive operators to uncover further facts about the transfer operator T_R. In this section we show that the spectral radius of T_R as an operator on $\mathcal{C}[0,\pi]$ can in fact be read off as the largest positive eigenvalue of the matrix $\widetilde{T}_R$ defined in the previous section.

We will use results on positive operators from [27], and follow the terminology of that monograph, as in [12]. The underlying Banach space will be taken to be $\mathcal{C}[0,\pi]$ equipped with the sup norm. A *cone* in $\mathcal{C}[0,\pi]$ is a closed subset which does not contain a one dimensional subspace, and which does contain halflines emanating from the origin. The subset K of $\mathcal{C}[0,\pi]$ consisting of nonnegative continuous functions on $[0,\pi]$ is a cone in $\mathcal{C}[0,\pi]$ and it is normal, solid, and reproducing in the language of [27]. A vector v in $\mathcal{C}[0,\pi]$ is *nonnegative* if $v \in K$. The cone K induces a partial ordering of the Banach space $\mathcal{C}[0,\pi]$:

$$u \succeq v , \text{ for } u, v \in \mathcal{C}[0,\pi] \text{ if } u - v \in K . \tag{2.19}$$

The interior of the cone K is the subset K^0 of elements $u \in K$ satisfying $u(\omega) > 0$ for all $\omega \in [0, \pi]$.

The set $K_J := K \cap \mathcal{E}_J$ is a cone in $\mathcal{E}_J$, where $\mathcal{E}_J$ is equipped also with the maximum norm as a sub-Banach space of $\mathcal{C}[0, \pi]$. We let $K_J^0 := K^0 \cap \mathcal{E}_J$ be the strictly positive functions in $\mathcal{E}_J$. A bounded linear operator $T : \mathcal{C}[0, \pi] \to \mathcal{C}[0, \pi]$ is *positive* (with respect to a given cone, which in our case will always be K), if $T(K) \subset K$. A linear operator is *strongly positive* if for any nonzero $u_0 \in K$, there exists $n_0 > 0$ such that $T^{n_0} u_0 \in K^0$ (and then $T^n u_0 \in K^0$ for all $n > n_0$). The next lemma demonstrates that for positive symbols R, the transfer operator T_R is strongly positive.

LEMMA 2.2. *Let $R(\omega) > 0 \,\forall\, \omega$ and $0 \neq u \in K$. Then there exists $n > 0$ such that $T_R^n u \in K^0$.*

Proof: T_R is made up of translations and dilations; define the translation operator

$$\tau_\alpha : [0, \pi] \to [0, M\pi], \quad \alpha \in \{0, 1, \ldots, M-1\}$$

by

$$\tau_\alpha(\omega) := \omega + 2\pi\alpha\,, \tag{2.20}$$

and the dilation operator

$$\sigma(\omega) := \frac{\omega}{M}\,. \tag{2.21}$$

Then we can write the transfer operator as

$$\begin{aligned} T_R(u)(\omega) &= \sum_\alpha Ru(\sigma(\tau_\alpha(\omega))) \quad \text{, or} \\ T_R u &= \sum_\alpha \tau_\alpha^* \sigma^*(Ru)\,. \end{aligned} \tag{2.22}$$

The cascaded operator can be expressed by

$$T_R^n u(\omega) = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_n} \tau_{\alpha_n}^* \sigma^* R \tau_{\alpha_{n-1}} \sigma^* \ldots \tau_{\alpha_1} \sigma^* Ru(\omega)\,,$$

which is an average of values of Ru sampled on intervals of width π/M^n.

Now consider the effect of σ and τ_α on the interval $[0, \pi]$. The composition of σ with τ_α is

$$\sigma \circ \tau_\alpha(\omega) = \frac{\omega + 2\pi\alpha}{M} \bmod 2\pi\,,$$

since we will always be working with 2π-periodic functions. Letting $I := [0, \pi]$,

$$\sigma \circ \tau_\alpha(I) = I_\alpha,$$

where I_α is a subinterval of I, of length $|I_\alpha| = \frac{\pi}{M}$. We can iterate the mappings; if we write

$$I_{\alpha_1 \cdots \alpha_n} := \sigma \circ \tau_{\alpha_1} \sigma \circ \tau_{\alpha_2} \cdots \sigma \circ \tau_{\alpha_n}(I)\,,$$

each $\alpha_i \in \{0, 1, \ldots, M-1\}$, then the collection $\{I_{\alpha_1 \cdots \alpha_n}\}$ is an M-ary closed covering of I with $|I_{\alpha_1 \cdots \alpha_n}| = \frac{\pi}{M^n}$, and consequently these coverings have arbitrarily small mesh size.

Now let S be any subset of I, and define

$$\widetilde{S} := \bigcup_{\alpha} (\sigma \circ \tau_\alpha)^{-1}(S),$$

the union of the inverse images of S under the mappings $\sigma \circ \tau_\alpha$. Suppose that $u(\omega) > 0$ on S, then we claim that $T_R u(\omega) > 0$ on $\widetilde{S}$. Let $\tilde{\omega} \in \widetilde{S}$, then $\tilde{\omega} = (\sigma \circ \tau_{\alpha_0})^{-1}(\omega)$ for some $\omega \in I$ and for some α_0. Thus

$$T_R u(\tilde{\omega}) = Ru(\sigma \circ \tau_{\alpha_0}(\tilde{\omega})) + \sum_{\alpha \neq \alpha_0} Ru(\sigma \circ \tau_\alpha(\tilde{\omega})), \tag{2.23}$$

but the first term in (2.23) is positive by hypothesis, and the remaining terms are nonnegative, so it follows that $T_R u(\tilde{\omega}) > 0$.

To conclude the proof of the lemma, since $u \neq 0$, $u(\omega) > 0$ at some point $\omega_0 \in I$, and hence, by continuity, $u > 0$ in some neighborhood of ω_0, which will in turn contain one of the closed intervals $I_{\alpha_1 \cdots \alpha_n}$, for a large enough n (a fine enough mesh) and a suitable choice of $\alpha_1, \ldots, \alpha_n$. Then $T_R u > 0$ on $I_{\alpha_1 \cdots \alpha_{n-1}}$, $T_R^2 u > 0$ on $I_{\alpha_1 \cdots \alpha_{n-2}}$, and by induction, $T_R^{n-1} u > 0$ on I_{α_1}. Thus $T_R^n u > 0$ on I, as desired. □

The following theorem, adapted from [12], shows that one can compute $\rho(T_R)$ in terms of the spectral radius of a finite-dimensional restriction. In essence, the strong positivity of the operator T_R leads to the existence of a nonnegative fixed point $\widetilde{u}$, and since the restriction $\widetilde{T}_R$ of T_R to $\mathcal{E}_J$ is also positive (with a fixed point by the classical Perron theorem), the two fixed points are identical, and we may find $\widetilde{u}$ and its associated eigenvalue μ by analyzing the finite-dimensional operator.

THEOREM 2.3. *Given $R \in K_J$ such that the associated transfer operator T_R is strongly positive, there exists a function $\widetilde{u} \in K_J^0$ and a real number $\mu > 0$ such that:*

(a) $\widetilde{T}_R \widetilde{u} = \mu \widetilde{u}$ *($\widetilde{u}$ is an eigenvector of $\widetilde{T}_R$ with eigenvalue μ).*

(b) $\rho(\widetilde{T}_R) = \mu$ *(the spectral radius of $\widetilde{T}_R$ is μ.)*

(c) $T_R(\widetilde{u}) = \mu \widetilde{u}$ *($\widetilde{u}$is an eigenvector of T_R with eigenvalue μ.*

(d) $\rho(T_R) = \mu$ *(the spectral radius of T_R is μ).*

(e) *Given $0 \neq u_0 \in K$,*

$$\lim_{n \to \infty} \|T_R^n u_0\|^{1/n} = \mu. \tag{2.24}$$

Proof: The restriction $\widetilde{T}_R$ is a positive operator on the finite-dimensional vector space $\mathcal{E}_J$ with respect to the cone K_J. By the classical Perron-Frobenius theory [25], $\widetilde{T}_R$ has a unique norm-1 positive eigenvector $\widetilde{u} \in K_J$ with a positive eigenvalue μ, and μ is the spectral radius of $\widetilde{T}_R$. Since $\widetilde{u} \in K_J$, it follows from Lemma 2.2 that there exists n_0 with

$$T_R^{n_0} \widetilde{u} = \widetilde{T}_R^{n_0} \widetilde{u} \in K_J^0 .$$

But $\widetilde{T}_R^{n_0} \widetilde{u} = \mu^{n_0} \widetilde{u}$. Thus $\widetilde{u} \in K_J^0$; $\widetilde{u}$ is a positive eigenvector for T_R with eigenvalue μ. By the strong positivity of T_R and Theorems 2.10 and 2.13 of [27], μ is a simple eigenvalue of T_R, greater than the absolute values of all other eigenvalues of T_R. Stetsenko has shown (Theorem 9.1 in [28]) that if K is a normal cone and $u_0 \in K^0$, then

$$\lim_{n\to\infty} \|T_R^n u_0\|^{1/n} = \rho(T_R), \tag{2.25}$$

where $\rho(T_R)$ is the spectral radius of T_R. Our cone K is normal in $\mathcal{C}[0,\pi]$, and by strong positivity $T_R^n u_0 \in K^0$ for all n greater than some n_0; thus Stetsenko's formula (2.25) holds for any $u_0 \in K$. Letting $u_0 = \tilde{u}$ in (2.25), we immediately obtain

$$\rho(T_R) = \mu \ .$$

Thus the spectral radius of the infinite-dimensional operator is computable in terms of the eigenvalue of the positive eigenvector of the finite-dimensional operator $\widetilde{T}_R$. This completes the proof of the theorem.

□

Remark: This theorem required strong positivity of the transfer operator T_R, guaranteed for $R(\omega) > 0$ by Lemma 2.2. However, Theorem 2.3 also holds for a subclass of those transfer operators associated with nonnegative symbols $R(\omega)$ that have a finite number of zeros on the unit circle, so long as $R(0) = 1$. Working with the notion of invariant compact sets used by Hervé ([24], section 2), it can be shown that if $R(0) = 1$, and $R(\omega)$ satisfies Cohen's condition, and $\phi \in L^2(\mathbf{R})$, then Theorem 2.3 still holds.

By the periodicity of functions on $[-\pi, \pi)$, we know that

$$\int_{-\pi}^{\pi} T_R f(\omega) g(\omega) d\omega = M \int_{-\pi}^{\pi} f(\omega) R(\omega) g(M\omega) d\omega \ .$$

As shown in [43], this leads to

LEMMA 2.4. *There exist c, C, > 0 such that for all $n \in \mathbf{N}$,*

$$c\rho(T_R)^n \le \int_{-\pi}^{\pi} \prod_{k=0}^{n-1} R(M^k\omega) d\omega \le C\rho(T_R)^n \ . \tag{2.26}$$

Here $\rho(T_R)$ is the spectral radius of T_R (and the largest eigenvalue of $\widetilde{T}_R$) as in the preceding theorem.

3. Sobolev Regularity via the Transfer Operator. We can now apply these tools to determine the differentiability of wavelet scaling functions. In this section we generalize the work of Eirola [12] and Villemoes [43] to the rank M setting, deriving a formula for the Sobolev regularity of scaling functions in terms of the spectral radius of the transfer operator defined above. Related work appears in [4, 6, 7, 8, 22].

We defined the scaling function ϕ as the unique solution in $L^2(\mathbf{R}) \bigcap L^1(\mathbf{R})$ of the dilation equation

$$\phi(x) = \sum_k a_k \phi(Mx - k) \ . \tag{3.27}$$

We now answer the question "Given a rank M scaling sequence, how differentiable is the corresponding ϕ?" by finding the critical Sobolev exponent for ϕ. Our goal in this section is to determine a formula relating the critical Sobolev exponent $s(\phi)$ to the spectrum of the transfer operator T_R, given a rank M scaling sequence $\{a_k\}$.

We now briefly recapitulate a series of arguments from [43], generalized to rank M, which enable us to characterize $s(\phi)$. The Fourier transform of the scaling relation (3.27) yields

$$\hat{\phi}(\omega) = \prod_{k=1}^{\infty} A(M^{-k}\omega) \ ,$$

and

$$|\hat{\phi}(\omega)|^2 = \prod_{k=1}^{\infty} P(M^{-k}\omega) \ .$$

Consider also the compactly supported distribution defined in a similar way from the remainder trigonometric polynomial $Q(\omega)$ (recall that $|Q(\omega)|^2 = R(\omega)$):

$$\hat{\phi}_Q(\omega) = \prod_{k=1}^{\infty} Q(M^{-k}\omega) \ , \quad |\hat{\phi}_Q(\omega)|^2 = \prod_{k=1}^{\infty} R(M^{-k}\omega) \ . \tag{3.28}$$

Then if $\mathbf{1}^N$ represents the smoothing term obtained by taking the N-fold convolution of the characteristic function of the unit interval (N-th order basic spline),

$$|\hat{\phi}(\omega)|^2 = |\widehat{\mathbf{1}^N}(\omega)|^2 |\hat{\phi}_Q(\omega)|^2 \ .$$

The Sobolev exponents of ϕ and ϕ_Q are linked:

$$s(\phi) = N + s(\phi_Q) \ .$$

Recall that a wavelet symbol Q is said to satisfy *Cohen's criterion* [5], [10] if there exists a compact set $K \subset \mathbf{R}$ containing a neighborhood of $\omega = 0$ and equal to a finite union of closed intervals, such that

(i) For all but a finite number of $\omega \in [-\pi, \pi]$, there is a unique $\eta \in K$ with $\eta - \omega \in 2\pi\mathbf{Z}$.

(i) $Q(\omega) \neq 0$ for $\omega \in \bigcup_{j=1}^{\infty} M^{-j}K$.

As shown in [5], this is equivalent to the Riesz basis property for the wavelet system associated with Q and A. If the symbol $Q(\omega)$ satisfies Cohen's criterion and $Q(\pi) > 0$ (and hence the same is true for $R(\omega)$), then the following argument holds for $t > 0$:

$$\begin{aligned}
\phi_Q \in H^{-t} &\Leftrightarrow \sum_n M^{-2tn} \int_{M^{n-1}\rho < |\omega| < M^n \rho} |\hat{\phi}_Q(\omega)|^2 d\omega < \infty \qquad \text{(Littlewood-Paley [41])} \\
&\Leftrightarrow \sum_n M^{-2tn} \int_{-\rho}^{\rho} \left(\prod_{k=0}^{n-1} R(M^k\omega) \right) |\hat{\phi}_Q(\omega)|^2 d\omega < \infty \\
&\Leftrightarrow \sum_n M^{-2tn} \int_{-\pi}^{\pi} \left(\prod_{k=0}^{n-1} R(M^k\omega) \right) |\hat{\phi}_Q(\omega)|^2 d\omega < \infty \\
&\Leftrightarrow \sum_n M^{-2tn} \int_{-\pi}^{\pi} \left(\prod_{k=0}^{n-1} R(M^k\omega) \right) d\omega < \infty \quad \text{(by Cohen's criterion)}
\end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow \sum_n M^{-2tn} \left[\rho(T_R)\right]^n < \infty \quad \text{(by Lemma 2.4)} \\
&\Leftrightarrow M^{-2t}\rho(T_R) < 1 \\
&\Leftrightarrow t < -\frac{\log \rho(T_R)}{2 \log M} \,.
\end{aligned}
$$

This demonstrates the following theorem, due in the rank 2 case to Eirola, Villemoes, and Cohen-Daubechies:

THEOREM 3.1. *Given a scaling sequence a with symbol A and the nonnegative trigonometric polynomial R as defined as above satisfying Cohen's criterion, with* $R(\pi) \neq 0$*, then the critical Sobolev exponent of* ϕ*,* $s(\phi)$*, can be expressed in terms of the spectral radius of the transfer operator* T_R *on the space* $\mathcal{C}[0,\pi]$*, which in turn can be measured as the maximum eigenvalue* μ *of* $\widetilde{T}_R = T_R|_{\mathcal{E}_J}$*:*

$$
s(\phi) \;=\; N - \frac{\log \rho(T_R)}{2 \log M} \;=\; N - \frac{\log \mu}{2 \log M} \,. \tag{3.29}
$$

This explicit formula immediately yields the information in Table 3.1 below, the critical Sobolev exponents of the minimal length rank M orthogonal scaling functions with N vanishing moments.

4. Asymptotic Behavior of the Sobolev Regularity for Large Genus.

We wish to compute the *asymptotic* smoothness of the generalized Daubechies (maximal vanishing moments) family of scaling functions, that is, the growth of the Sobolev exponent with support length. This has been done in the $M = 2$ case by Volkmer [45], [46], Cohen-Conze [6], and Eirola [12]. In this section we find analogous asymptotic estimates for other values of the dilation factor M, with a sharp dichotomy between the cases of even and odd M.

4.1. An Ergodic Map and its Periodic Points.

Underlying our investigation of asymptotic regularity is the ergodic map

$$
\tau : \omega \to M\omega \bmod 2\pi \,. \tag{4.30}
$$

The transfer operator T_R is nothing more than the Perron-Frobenius operator associated with this map [2], [37], [12]. The decay as ω approaches ∞ of the product $\prod_k R(M^{-k}\omega)$ is strongly affected by the behavior of $R(\omega)$ at those ω which form periodic points or invariant cycles for the map τ. These connections have been exploited by a number of authors: [5], [6], [10], [16], [44]. A periodic point ω_P is a point which satisfies $\tau^k(\omega_P) = \omega_P$ for some $k \geq 0$, and a preperiodic point is a preimage under τ of a periodic point. Also remember that in our application, $R(\omega)$ is always a cosine polynomial, so that $\omega \approx \pi - \omega$. One significant family of periodic points is given by

$$
\omega = \frac{2\pi m}{M^k - 1} \,.
$$

When M is even, we single out the periodic point with $k = 1$ that is closest to π:

$$
\omega_c = \frac{M\pi}{M+1} \,. \tag{4.31}
$$

N	M=2	M=3	M=4	M=5	M=6
1	0.5000	0.5000	0.5000	0.5000	0.5000
2	1.0000	0.9087	0.8538	0.8174	0.7915
3	1.4150	1.1599	1.0498	0.9835	0.9390
4	1.7756	1.2950	1.1758	1.0905	1.0344
5	2.0968	1.3665	1.2749	1.1696	1.1051
6	2.3884	1.4133	1.3614	1.2306	1.1614
7	2.6587	1.4499	1.4404	1.2780	1.2087
8	2.9147	1.4809	1.5141	1.3152	1.2502
9	3.1617	1.5081	1.5836	1.3447	1.2877
10	3.4027	1.5323	1.6497	1.3686	1.3225
11	3.6398	1.5542	1.7130	1.3883	1.3552
12	3.8740	1.5741	1.7740	1.4050	1.3863
13	4.1060	1.5925	1.8329	1.4196	1.4162
14	4.3363	1.6094	1.8900	1.4325	1.4450
15	4.5651	1.6252	1.9455	1.4442	1.4729
16	4.7927	1.6400	1.9996	1.4548	1.5000
17	5.0191	1.6539	2.0524	1.4647	1.5264
18	5.2446	1.6670	2.1041	1.4739	1.5522
19	5.4692	1.6793	2.1548	1.4825	1.5775
20	5.6930	1.6911	2.2046	1.4907	1.6023
21	5.9161	1.7022	2.2535	1.4984	1.6266
22	6.1385	1.7129	2.3017	1.5057	1.6505
23	6.3604	1.7230	2.3492	1.5127	1.6740
24	6.5817	1.7327	2.3960	1.5194	1.6972
25	6.8024	1.7421	2.4422	1.5258	1.7200

TABLE 3.1

Sobolev exponents for minimal support rank M scaling functions with N vanishing moments, $M = 2, 3, 4, 5, 6$ and $N = 1, 2, \ldots, 25$.

When M is odd, π itself is a periodic point, with $k = 1$ and $m = \frac{M-1}{2}$:

$$\omega_c = \pi \ . \tag{4.32}$$

The values of $R(\omega)$ at these periodic points and their preimages will prove critical in studying the asymptotic growth in smoothness of rank M scaling functions. The different choice of ω_c depending on whether M is even or odd will lead to dramatically different asymptotic behavior.

4.2. An Upper Bound on Sobolev Smoothness. We can rapidly obtain an upper bound on the Sobolev regularity of the maximal vanishing moments scaling functions, following the arguments that Villemoes [43] gave in the rank 2 case. Given the symbol $R(\omega)$, define

$$K_n = \left[\sup_{\omega \in [-\pi, \pi]} \prod_{k=0}^{n-1} R(M^k \omega) \right]^{1/n} . \tag{4.33}$$

The numbers K_n are related to the spectral radius $\rho(T_R)$. In particular,

$$T_R^n u(\xi) = \sum_{k=0}^{M^n-1} \prod_{j=1}^{n} R(M^{-j}(\xi + 2\pi k))\, u(M^{-n}(\xi + 2\pi k) \ .$$

Thus

$$\|T_R^n\| = \sup_{\|u\|=1} \|T_R^n u\| = \sup_{\xi} \sum_{k=0}^{M^n-1} \prod_{j=1}^{n} R(M^{-j}(\xi + 2\pi k)) \ ,$$

and

$$\|T_R^n\|^{1/n} \geq K_n \ ,$$

hence

$$\rho(T_R) \geq \limsup_n K_n \ .$$

Now the periodic points ω_c has the property that $R(M^k \omega_c) = R(\omega_c)$; thus

$$K_n \geq R(\omega_c)$$

and

$$\rho(T_R) \geq R(\omega_c) \ . \tag{4.34}$$

Hence we have

Proposition 4.1. *An upper bound for the critical Sobolev exponent of the scaling function ϕ is given by*

$$s(\phi) \leq N - \frac{\log R(\omega_c)}{2 \log M} \ . \tag{4.35}$$

In the case of the rank M generalized Daubechies wavelets, this bound demonstrates a significant difference in behavior, depending on whether M is even or odd, as we now show.

For even values of M,

$$R_N(\omega_c) = \sum_{n=0}^{N-1} \sum_{k_1+k_2+\cdots+k_{M/2}=n} \left\{ \prod_{m=1}^{M/2-1} \binom{2N+k_m-1}{2N-1} \left(1 - \cos\frac{2\pi m}{M}\right)^{-k_m} \right\}$$
$$\times \binom{N+k_{M/2}-1}{N-1} (1-\cos\pi)^{-k_{M/2}} \ (1-\cos\omega_c)^n \ .$$

The top term in this sum ($n = N-1$) will dominate, and applying Stirling's formula, $n! \sim \sqrt{2\pi} e^{-n-1}(n+1)^{n+\frac{1}{2}}$ to it we find

$$R_N(\omega_c) \geq M^{2N} \left(\frac{1 - \cos\frac{M\pi}{M+1}}{2} \right)^N \frac{1}{\sqrt{N}} \ . \tag{4.36}$$

When M is odd,

$$R_N(\pi) = \sum_{n=0}^{N-1} \sum_{k_1+k_2+\cdots+k_{\frac{M-1}{2}}=n} \left\{ \prod_{m=1}^{\frac{M-1}{2}} \binom{2N+k_m-1}{2N-1} \left(1-\cos\frac{2\pi m}{M}\right)^{-k_m} \right\} 2^n .$$

Again this sum is dominated by its top term ($n = N-1$), and using Stirling's formula we find that

$$R_N(\pi) \geq M^{2N} \frac{1}{\sqrt{N}} . \tag{4.37}$$

Combining these results with Proposition 4.1, we have the following generalization of the earlier work for $M = 2$ ([12], [43], [45], [6]):

THEOREM 4.2. *Consider the minimal length trigonometric polynomials yielding N-th order approximation (orthogonal wavelets with maximal vanishing moments), with remainders R_N defined in (1.11), (1.12) and (1.13). For even M, the asymptotic value of $R_N(\omega_c)$ as N grows large is bounded below by (4.36). Thus, if ϕ_N is the associated scaling function,*

$$s(\phi_N) \leq N \left(\frac{\log \frac{2}{1-\cos\frac{M\pi}{M+1}}}{2\log M} \right) + o(N) .$$

The growth in smoothness of ϕ_N is at most linear with N, with slope

$$\frac{\log \frac{2}{1+\cos\frac{\pi}{M+1}}}{2\log M} \sim \frac{\pi^2}{8(M+1)^2\log M} .$$

This slope has the value $1-\frac{\log 3}{\log 4} \sim .2075$ when $M=2$, $\frac{3}{4}-\frac{\log 5+\sqrt{5}}{4\log 2} \sim 0.0362$ when $M=4$, and approaches zero quite rapidly as M increases.

For odd M, the asymptotic growth with N of $R_N(\omega_c)$ is bounded below by (4.37); thus the critical Sobolev exponent cannot grow any faster than

$$s(\phi_N) \leq \frac{\log N}{4\log M} + o(N) .$$

The growth in smoothness with increasing filter length and approximation order is at best logarithmic for odd M.

The logarithmic upper bound for odd M is is due to the fact that we evaluate $R(\omega)$ at the critical point π rather than at $\frac{M\pi}{M+1}$. In the following subsections we prove that the asymptotic upper bounds of Theorem 4.2 are also lower bounds for $M=3$ and $M=4$.

4.3. Lower Bounds via Algebraically Defined Invariant Cones. We will now use the notion of T_R as a positive operator mapping a special N-dimensional cone K of positive functions to itself to develop asymptotic estimates for the spectral radius of T_R as $N \to \infty$, as was done for the $M=2$ case in [12]. These asymptotic estimates lend themselves to both upper and lower bounds on the spectral radius, showing that the asymptotic smoothness of the maximal vanishing moments wavelets does indeed grow linearly for even M and logarithmically for odd M. We give proofs here for the cases $M=3$ and $M=4$.

4.3.1. Invariant Cones for Even Rank.

When M is even, define the special cone

$$K^J = \left\{ u \in \mathcal{C}[0,\pi] \, : \, u(\omega) = \sum_{j=0}^{J} u_j (1-\cos\omega)^j \, (1+\cos\omega)^{J-j} \, , \; u_j \geq 0 \right\} .$$

Observe that $0 \neq u \in K^J$ implies $u(\omega) > 0$ for $\omega \in (0,\pi)$. It is shown in [12] that

$$\begin{aligned} &(i) \;\; u \in K^J, v \in K^L \Rightarrow uv \in K^{J+L} \\ &(ii) \;\; \text{If } J \leq L, \; K^J \subset K^L, \; \text{ and } K^J + K^L \subset K^L \; . \end{aligned}$$

In particular, $(1-x)^k(1+x)^l \in K^J$ for all $J \geq k+l$. The key fact about K^J is its invariance under the transfer operator:

LEMMA 4.3. *When $M = 2$ and $M = 4$, given $R \in K^J$,*

$$T_R : K^J \to K^J \; .$$

Proof: By the above facts, $u \in K^J \Rightarrow R(\omega)u(\omega) \in K^{2J}$, so that the critical issue here is to show that if $u \in K^J$, $(\downarrow M)u \in K^{2J/M}$. Here we use $(\downarrow M)$ to denote the decimation operation:

$$(\downarrow M)u(\omega) = u\left(\frac{\omega}{M}\right) + u\left(\frac{\omega + 2\pi}{M}\right) + \ldots + u\left(\frac{\omega + 2(M-1)\pi}{M}\right) \; .$$

Since $u(\omega)$ is of the form $\sum_{k=0}^{J} u_k (1-\cos\omega)^k \, (1+\cos\omega)^{J-k}$, what we must show is that for each j, the term

$$P_{J,k}(\omega) = (1-\cos\omega)^k \, (1+\cos\omega)^{J-k}$$

satisfies

$$(\downarrow M) P_{J,k}(\omega) \in K^{2J/M} \; .$$

When $M = 2$, the proof was given in [12].

When $M = 4$, write $x = \cos\omega$, $y = \cos\frac{\omega}{4}$ and $y_1 = \sqrt{1-y^2}$, as well as $z = \cos\frac{\omega}{2}$. Then the following hold:

$$y^2 = \frac{1+z}{2} \, , \tag{4.38}$$

$$1 - y^2 = \frac{1-z}{2} \, , \tag{4.39}$$

$$z^2 = \frac{1+x}{2} \, , \tag{4.40}$$

$$y^2(1-y^2) = \frac{1-z^2}{4} = \frac{1-x}{8} \tag{4.41}$$

Then

$$(\downarrow 4)P_{J,k}(\omega) = (1+y)^k(1-y)^{J-k} + (1+y_1)^k(1-y_1)^{J-k} + (1-y)^k(1+y)^{J-k} + (1-y_1)^k(1+y_1)^{J-k}$$

Taking $2k \leq J$ for the moment,

$$(\downarrow 4)P_{J,k}(\omega) = (1-y^2)^k \left[(1-y)^{J-2k} + (1+y)^{J-2k}\right] + (1-y_1^2)^k \left[(1-y_1)^{J-2k} + (1+y_1)^{J-2k}\right] .$$

The first term in brackets can be reduced using the binomial theorem to

$$2 \sum_{l=0}^{\lfloor \frac{J}{2} \rfloor - k} \binom{J-2k}{2l} y^{2l}$$

and analogously in y_1 for the second bracketed term. This yields

$$(\downarrow 4)P_{J,k}(\omega) = \sum_{l=0}^{\lfloor \frac{J}{2} \rfloor - k} 2 \binom{J-2k}{2l} \left[(1-y^2)^k y^{2l} + (1-y_1^2)^k y_1^{2l}\right] .$$

Using the definition of y_1 and (4.41), and assuming $2k \geq \lfloor \frac{J}{2} \rfloor$,

$$(\downarrow 4)P_{J,k}(\omega) = \sum_{l=0}^{\lfloor \frac{J}{2} \rfloor - k} 2 \binom{J-2k}{2l} \left(\frac{1-x}{8}\right)^l \left[(1-y^2)^{k-l} + y^{2(k-l)}\right] .$$

Now applying (4.38) and (4.39), followed by a binomial expansion and then use of (4.40),

$$\begin{aligned}
(\downarrow 4)P_{J,k}(\omega) &= \sum_{l=0}^{\lfloor \frac{J}{2} \rfloor - k} 2^{1-3l} \binom{J-2k}{2l} (1-x)^l \left[\left(\frac{1-z}{2}\right)^{k-l} + \left(\frac{1+z}{2}\right)^{k-l}\right] \\
&= \sum_{l=0}^{\lfloor \frac{J}{2} \rfloor - k} 2^{1-2l-k} \binom{J-2k}{2l} (1-x)^l \sum_{j=0}^{\lfloor \frac{k-l}{2} \rfloor} \binom{k-l}{2j} z^{2j} \\
&= \sum_{l=0}^{\lfloor \frac{J}{2} \rfloor - k} \sum_{j=0}^{\lfloor \frac{k-l}{2} \rfloor} 2^{1-2l-k-j} \binom{J-2k}{2l} \binom{k-l}{2j} (1-x)^l (1+x)^j
\end{aligned}$$

which is a sum of terms, each belonging to $K^{\lfloor \frac{J}{2} \rfloor - k + \lfloor \frac{k-l}{2} \rfloor} \subset K^{J/2}$. Lifting the assumptions $2k \leq J$ and $2k \geq \lfloor \frac{J}{2} \rfloor$ requires only some reindexing of these sums, and again leads to a sum of terms belonging to $K^{J/2}$. Thus $T_R : K^J \to K^J$ for $M = 4$.

□

4.3.2. Invariant Cones for Odd Rank. When M is odd, the value of the symbol R at $\omega_c = \pi$ will prove important, and so the cone defined above will not produce useful estimates; functions in that cone can vanish at π. Instead, define

$$K^J = \left\{ u \in \mathcal{C}[0,\pi] \, : \, u(\omega) = \sum_{j=0}^{J} u_j (1-\cos\omega)^j \, , \; u_j \geq 0 \right\} .$$

This cone has the same product, sum, and inclusion relations as the cone K^J defined for even M above, trivially. As before, we must now show

Lemma 4.4. *When $M = 3$, given $R \in K^J$,*

$$T_R : K^J \to K^J \ .$$

Proof: As in the previous lemma, since elements of the cone have the given form, what we need to show is that for a "basic element"

$$P_l(\omega) = (1 - \cos\omega)^l \in K^J \ ,$$

downsampling maps it to a lower-order cone:

$$(\downarrow 3)P_l(\omega) \in K^{J/M} \ .$$

Elementary trigonometry tells us that a cosine polynomial of degree J is reduced to one of degree at most $J/3$ by the downsampling operation; what is at issue is whether the result is of the form $\sum_{k \le k_0} u_k(1 - \cos\omega)^k$ with nonnegative coefficients u_k. Write

$$(\downarrow 3)P_l(\omega) = Q_l(y) = \sum_{k=0}^{\lfloor J/3 \rfloor} q_{l,k} y^k$$

where $y = 1 - \cos\omega$. Now

$$\left(1 - \cos\frac{\omega}{3}\right)^3 = \frac{1}{4}(1 - \cos\omega) + 3\left(1 - \cos\frac{\omega}{3}\right)^2 - \frac{9}{4}\left(1 - \cos\frac{\omega}{3}\right) \ .$$

Thus

$$\left(1 - \cos\frac{\omega}{3}\right)^l = \frac{1}{4}(1 - \cos\omega)\left(1 - \cos\frac{\omega}{3}\right)^{l-3} + 3\left(1 - \cos\frac{\omega}{3}\right)^{l-1} - \frac{9}{4}\left(1 - \cos\frac{\omega}{3}\right)^{l-2} \ ,$$

or in terms of y,

$$Q_l(y) = \frac{1}{4} y Q_{l-3}(y) - \frac{9}{4} Q_{l-2}(y) + 3Q_{l-1}(y) \ .$$

This recursion relation implies that

$$q_{l,0} = 3q_{l-1,0} - \frac{9}{4} q_{l-2,0} \ , \tag{4.42}$$

and

$$q_{l,k} = 3q_{l-1,k} - \frac{9}{4} q_{l-2,k} + \frac{1}{4} q_{l-3,k-1} \ , \ \text{for } k > 0 \ . \tag{4.43}$$

Checking the first few values of $q_{l,0}$ and using (4.42), one finds that

$$q_{l,0} = 3\left(\frac{3}{2}\right)^{l-1} \ .$$

Now we show inductively that $q_{l,k} \ge \frac{3}{2} q_{l-1,k}$ and $q_{l,k} \ge 0$. This can be verified for $l = 0, 1, 2$. Assume that

$$q_{j,k} \geq \frac{3}{2} q_{j-1,k}\ , \quad \text{for } j \leq l-1\ ,$$
$$q_{j,k} \geq 0\ , \quad \text{for } j \leq l-1\ .$$

Then (4.43) can be broken up as

$$q_{l,k} = \frac{3}{2} q_{l-1,k} + \frac{3}{2}\left(q_{l-1,k} - \frac{3}{2} q_{l-2,k}\right) + \frac{1}{4} q_{l-3,k-1}\ .$$

Each of the last two terms on the right is nonnegative, and so

$$q_{l,k} \geq \frac{3}{2} q_{l-1,k} \geq 0\ .$$

This completes the induction argument, with the consequence that each of the coefficients $q_{l,k}$ is nonnegative, and so $(\downarrow 3)P_l(\omega) \in K^{J/M}$. Hence the cone K^J is invariant under the transfer operator T_R.

□

Remark: Numerical computations indicate that Lemmas 4.3 and 4.4 are valid for all M for even and odd M respectively. We will make this assumption below in our numerical asymptotic projections when we consider cases where $M > 4$.

4.3.3. Lower Bounds for Asymptotic Smoothness when $M = 3, 4$. Now define the dual cone K^{J^*} and consider the evaluation functionals $\eta_\theta \in K^{J^*}$ defined by

$$\langle \eta_\theta, u \rangle = u(\theta)\ .$$

Define the dual operator T_R^* by $\langle \eta_\theta, T_R u\rangle = \langle T_R^* \eta_\theta, u\rangle$; observe that

$$T_R^* \eta_\theta = R\left(\tfrac{\theta}{M}\right)\eta_{\theta/M} + R\left(\tfrac{\theta+2\pi}{M}\right)\eta_{(\theta+2\pi)/M} + \ldots R\left(\tfrac{\theta+2\pi(M-1)}{M}\right)\eta_{(\theta+2\pi(M-1))/M}\ .$$

One can prove [12]:
LEMMA 4.5. *If $R \in K^J$, and there exists $\lambda > 0$ and $\eta \in K^{J^*}$ such that*

$$T_R^*\eta - \lambda\eta \in K^{J^*}\ ,$$

then $\mu \geq \lambda$, and conversely, if

$$\lambda\eta - T_R^*\eta \in K^{J^*}\ ,$$

then $\mu \leq \lambda$.

This lemma easily yields the lower bound on the spectral radius $\mu(T_R)$ of (4.34) by taking $\eta = \eta_{\omega_c}$.

We now work further to obtain upper bounds on $\mu(T_R)$, and lower bounds for the Sobolev regularity. In the case $M = 2$, Eirola applied Lemma 4.5 with $\eta = \eta_{\omega_c} + c\eta_{\pi/3}$, c a positive constant. In other words, η_{ω_c} provides a first approximation to an eigenvector for T_R^*, and we correct by adding a scalar multiple of evaluation at the preimage of ω_c under τ. Let us step through the argument with some care for $M = 2$; this will clarify the proofs for other values of M.

Take $\eta = \eta_{2\pi/3} + c\eta_{\pi/3}$; we want to find c such that $\lambda\eta - T_R^*\eta \in K^{J^*}$. Now

$$\lambda\eta - T_R^*\eta = \left[\lambda - R\left(\frac{2\pi}{3}\right)\right]\eta_{2\pi/3} + \left[c\lambda - R\left(\frac{\pi}{3}\right)\right]\eta_{\pi/3} - c\left[R\left(\frac{\pi}{6}\right)\eta_{\pi/6} + R\left(\frac{5\pi}{6}\right)\eta_{5\pi/6}\right]\ .$$

But the definition of the special cone enables us to state that for $u \in K^{N-1}$,

$$u\left(\frac{5\pi}{6}\right) \leq \left(\frac{1-\cos\frac{5\pi}{6}}{1-\cos\frac{2\pi}{3}}\right)^{N-1} u\left(\frac{2\pi}{3}\right) = \left(\frac{2+\sqrt{3}}{3}\right)^{N-1} u\left(\frac{2\pi}{3}\right) .$$

Thus, as functionals on K^{N-1}, with $\rho = \left(\frac{2+\sqrt{3}}{3}\right)^{N-1}$,

$$\rho\eta_{2\pi/3} - \eta_{5\pi/6} \in (K^{N-1})^* .$$

A similar relation holds between $\eta_{\pi/6}$ and $\eta_{\pi/3}$. We can then write

$$\lambda\eta - T_R^*\eta = \left[\lambda - R\left(\frac{2\pi}{3}\right) - c\rho R\left(\frac{5\pi}{6}\right)\right]\eta_{2\pi/3} + \left[c\lambda - R\left(\frac{\pi}{3}\right) - c\rho R\left(\frac{\pi}{6}\right)\right]\eta_{\pi/3} + \zeta ,$$

where $\zeta \in (K^{N-1})^*$. We will have $\lambda\eta - T_R^*\eta \in (K^{N-1})^*$ and be able to apply Lemma 4.5 if the contents of the two brackets are each zero. This leads to two equations:

$$\lambda - R\left(\frac{2\pi}{3}\right) - c\rho R\left(\frac{5\pi}{6}\right) = 0 , \tag{4.44}$$

$$c^2\rho R\left(\frac{5\pi}{6}\right) + c\left[R\left(\frac{2\pi}{3}\right) - \rho R\left(\frac{\pi}{6}\right)\right] - R\left(\frac{\pi}{3}\right) = 0 . \tag{4.45}$$

The quadratic equation for c has one positive real solution for the symbols $R(\omega)$ we will be considering. The corresponding value for λ is

$$\lambda = R\left(\frac{2\pi}{3}\right) + \frac{\rho R\left(\frac{\pi}{6}\right) - R\left(\frac{2\pi}{3}\right) + \sqrt{\left[R(\frac{2\pi}{3}) - \rho R(\frac{\pi}{6})\right]^2 + 4\rho R(\frac{\pi}{3})R(\frac{5\pi}{6})}}{2} .$$

With

$$\mathcal{E}_R = \frac{\rho R\left(\frac{\pi}{6}\right) - R\left(\frac{2\pi}{3}\right) + \sqrt{\left[R(\frac{2\pi}{3}) - \rho R(\frac{\pi}{6})\right]^2 + 4\rho R(\frac{\pi}{3})R(\frac{5\pi}{6})}}{2R\left(\frac{2\pi}{3}\right)} , \tag{4.46}$$

we can write

$$\lambda = R\left(\frac{2\pi}{3}\right)[1 + \mathcal{E}_R] .$$

This gives us an upper bound on the spectral radius μ, via lemma 4.5:

$$\mu(T_R) \leq R\left(\frac{2\pi}{3}\right)[1 + \mathcal{E}_R] .$$

Together with the lower bound from before, we obtain a set of lower and upper bounds for the Sobolev smoothness of the corresponding scaling function[3] ϕ:

$$N - \frac{1}{2\log 2}\log R\left(\frac{2\pi}{3}\right)[1 + \mathcal{E}_R] \leq s(\phi) \leq N - \frac{1}{2\log 2}\log R\left(\frac{2\pi}{3}\right) .$$

[3] Recall that $R(\omega)$ is the remainder term in $|A(\omega)|^2 = \left(\frac{1-\cos M\omega}{1-\cos\omega}\right)^N R(\omega)$; the filter with symbol $A(\omega)$ gives the coefficients for the dilation equation for ϕ.

A similar strategy works for larger values of M. When $M = 3$, take $\eta = \eta_\pi + c\eta_{\pi/3}$. Then form

$$\lambda\eta - T_R^*\eta = [\lambda - R(\pi)]\,\eta_\pi \; - \; c\left[R\left(\frac{5\pi}{9}\right)\eta_{5\pi/9} + R\left(\frac{7\pi}{9}\right)\eta_{7\pi/9}\right] \; + \left[c\lambda - 2R\left(\frac{\pi}{3}\right)\right]\eta_{\pi/3} \; - \; cR\left(\frac{\pi}{9}\right)\eta_{\pi/9} \; .$$

But on the special cone K^{N-1} we have defined for odd M, $\eta_{5\pi/9}$ and $\eta_{7\pi/9}$ are bounded above by η_π, while $\eta_{\pi/9}$ is bounded above by $\eta_{\pi/3}$. Thus we can write

$$\lambda\eta - T_R^*\eta = \left\{\lambda - R(\pi) - c\left[R\left(\frac{5\pi}{9}\right) + R\left(\frac{7\pi}{9}\right)\right]\right\}\eta_\pi \; + \left[c\lambda - 2R\left(\frac{\pi}{3}\right) - cR\left(\frac{\pi}{9}\right)\right]\eta_{\pi/3} \; + \; \zeta \; .$$

In order to apply Lemma 4.5, we are led to the two equations

$$\lambda - R(\pi) - c\left[R\left(\frac{5\pi}{9}\right) + R\left(\frac{7\pi}{9}\right)\right] = 0 \; ,$$

$$\lambda c - 2R\left(\frac{\pi}{3}\right) - cR\left(\frac{\pi}{9}\right) = 0 \; ,$$

with optimal solution

$$\lambda = R(\pi) \; + \; \frac{R(\pi/9) - R(\pi) + \sqrt{[R(\pi) - R(\pi/9)]^2 + 8R(\pi/3)\,[R(5\pi/9) + R(7\pi/9)]}}{2} \; . \tag{4.47}$$

Defining the error term

$$\mathcal{E}_R \; = \; \frac{R(\pi/9) - R(\pi) + \sqrt{[R(\pi) - R(\pi/9)]^2 + 8R(\pi/3)\,[R(5\pi/9) + R(7\pi/9)]}}{2R(\pi)} \; , \tag{4.48}$$

we can rewrite (4.47) as

$$\lambda = R(\pi)\,[1 + \mathcal{E}_R] \; .$$

Again, this λ is an upper bound for $\mu(T_R)$.

When $M = 4$, take $\eta = \eta_{4\pi/5} + c\eta_{7\pi/10}$. Notice that here we must be careful to take for the second evaluation point, the preimage of ω_c that is *closest* to π. With this choice of η,

$$\begin{aligned}\lambda\eta - T_R^*\eta = &\left[\lambda \; - \; R\left(\frac{4\pi}{5}\right) \; - \; c\left(\frac{1 - \cos\frac{33\pi}{40}}{1 - \cos\frac{4\pi}{5}}\right)^{N-1} R\left(\frac{33\pi}{40}\right)\right]\eta_{4\pi/5} \; + \\ &\left\{c\lambda \; - \; R\left(\frac{7\pi}{10}\right) \; - \; \left(\frac{1 - \cos\frac{\pi}{5}}{1 - \cos\frac{7\pi}{10}}\right)^{N-1} R\left(\frac{\pi}{5}\right) \; - \; \left(\frac{1 - \cos\frac{3\pi}{10}}{1 - \cos\frac{7\pi}{10}}\right)^{N-1} R\left(\frac{3\pi}{10}\right) \; - \right. \\ &c\left[\left(\frac{1 - \cos\frac{27\pi}{40}}{1 - \cos\frac{7\pi}{10}}\right)^{N-1} R\left(\frac{27\pi}{40}\right) \; + \; \left(\frac{1 - \cos\frac{13\pi}{40}}{1 - \cos\frac{7\pi}{10}}\right)^{N-1} R\left(\frac{13\pi}{40}\right) \; + \right. \\ &\left.\left.\left(\frac{1 - \cos\frac{7\pi}{40}}{1 - \cos\frac{7\pi}{10}}\right)^{N-1} R\left(\frac{7\pi}{40}\right)\right]\right\}\eta_{7\pi/10} \; + \; \zeta \; .\end{aligned}$$

Here we have used the properties of the cone K^{N-1}; e.g.

$$u\left(\frac{33\pi}{40}\right) \leq \left(\frac{1-\cos\frac{33\pi}{40}}{1-\cos\frac{4\pi}{5}}\right)^{N-1} u\left(\frac{4\pi}{5}\right)$$

to substitute $\eta_{4\pi/5}$ and $\eta_{7\pi/10}$ for other evaluation functionals. Seeking the best λ, c with $\lambda\eta - T_R^*\eta \in K^{N-1}$ leads to

$$\lambda = R\left(\frac{4\pi}{5}\right) + \frac{S_1 - R\left(\frac{4\pi}{5}\right) + \sqrt{\left[R\left(\frac{4\pi}{5}\right) - S_1\right]^2 + 4S_2\left(\frac{1-\cos\frac{33\pi}{40}}{1-\cos\frac{4\pi}{5}}\right)^{N-1} R\left(\frac{33\pi}{40}\right)}}{2},$$

where

$$S_1 = \left(\frac{1-\cos\frac{27\pi}{40}}{1-\cos\frac{7\pi}{10}}\right)^{N-1} R\left(\frac{27\pi}{40}\right) + \left(\frac{1-\cos\frac{13\pi}{40}}{1-\cos\frac{7\pi}{10}}\right)^{N-1} R\left(\frac{13\pi}{40}\right) + \left(\frac{1-\cos\frac{7\pi}{40}}{1-\cos\frac{7\pi}{10}}\right)^{N-1} R\left(\frac{7\pi}{40}\right),$$

and

$$S_2 = R\left(\frac{7\pi}{10}\right) + \left(\frac{1-\cos\frac{\pi}{5}}{1-\cos\frac{7\pi}{10}}\right)^{N-1} R\left(\frac{\pi}{5}\right) + \left(\frac{1-\cos\frac{3\pi}{10}}{1-\cos\frac{7\pi}{10}}\right)^{N-1} R\left(\frac{3\pi}{10}\right).$$

Defining

$$(4.49)\qquad \mathcal{E}_R = \frac{S_1 - R\left(\frac{4\pi}{5}\right) + \sqrt{\left[R\left(\frac{4\pi}{5}\right) - S_1\right]^2 + 4S_2\left(\frac{1-\cos\frac{33\pi}{40}}{1-\cos\frac{4\pi}{5}}\right)^{N-1} R\left(\frac{33\pi}{40}\right)}}{2R\left(\frac{4\pi}{5}\right)},$$

we have an estimate of the form

$$\mu(T_R) \leq R\left(\frac{4\pi}{5}\right)[1+\mathcal{E}_R].$$

For each M, we obtain a corresponding lower bound on the Sobolev exponent of ϕ. Combining these results with the Sobolev upper bounds of Proposition 4.1, we have the following

LEMMA 4.6. *When $M = 2, 3, 4$, the Sobolev exponent $s(\phi)$ can be bounded by values of the corresponding "reduced" symbol $R(\omega)$ at the critical periodic point, modulo an error term:*

$$(4.50)\qquad N - \frac{1}{2\log M}\log R(\omega_c)[1+\mathcal{E}_R] \leq s(\phi) \leq N - \frac{1}{2\log M}\log R(\omega_c).$$

$\mathcal{E}_R$ is given in equations (4.46), (4.48), (4.49) above.

When $R(\omega)$ is taken to be the minimal length solution $R_N(\omega)$ (the generalization of Daubechies orthogonal wavelets to arbitrary M), we can make use of the explicit formulae (1.11), (1.12) and (1.13) to evaluate the estimates (4.50) and determine the asymptotic growth in smoothness of the scaling functions. We will treat this for $M = 3$ and $M = 4$.

As observed previously, when $M = 4$,

$$R_N(\omega_c) = R_N\left(\frac{4\pi}{5}\right) = \sum_{n=0}^{N-1}\left\{\sum_{k=0}^{n}\binom{2N+k-1}{2N-1}\binom{N+n-k-1}{N-1}2^{k-n}\right\}(1-\cos\omega_c)^n . \tag{4.51}$$

Again using Stirling's formula to approximate the factorials, this sum satisfies

$$R_N\left(\frac{4\pi}{5}\right) \approx \frac{1}{\sqrt{N}}\left[8\left(1-\cos\frac{4\pi}{5}\right)\right]^N . \tag{4.52}$$

Consider the error term $\mathcal{E}_{R_N}$ defined in (4.49). $R_N(\omega)$ increases monotonically with ω, and the only term in $\mathcal{E}_{R_N}$ that is greater than $R(\omega_c)$ is $R\left(\frac{33\pi}{40}\right)$; however, it appears in a product with the much smaller terms of the expression S_2. Thus it can be seen that $\mathcal{E}_{R_N}$ is $o(N)$, i.e. vanishes as N approaches infinity. As a result, when $M = 4$, the Sobolev regularity of ϕ_N, the minimal length scaling function with N-th order approximation, grows linearly with the support length:

$$s(\phi_N) \approx N - \frac{\log\left[8\left(1-\cos\frac{4\pi}{5}\right)\right]^N}{2\log 4} = \left(\frac{3}{4} - \frac{\log 5+\sqrt{5}}{4\log 2}\right)N \approx 0.0362N . \tag{4.53}$$

When $M = 3$,

$$R_N(\omega_c) = R_N(\pi) = \sum_{n=0}^{N-1}\frac{(2N-1+n)!}{(2N-1)!\,n!}\left(\frac{4}{3}\right)^n .$$

We find that

$$R_N(\pi) \approx \frac{C}{\sqrt{N}}3^{2N} .$$

As before, the error term $\mathcal{E}_{R_N}$ defined in (4.48) vanishes as N grows large. Thus for $M = 3$, the Sobolev regularity of the minimal length scaling function with N-th order approximation, grows as

$$s(\phi_N) \approx N - \frac{\log\left(\frac{1}{\sqrt{N}}3^{2N}\right)}{2\log 3} = \frac{\log N}{4\log 3} . \tag{4.54}$$

Combining these results with the earlier work for $M = 2$ ([12], [43], [45], [6]), we have the following

THEOREM 4.7. *Consider the minimal length orthogonal wavelets with N-th order approximation (N vanishing moments), and their associated scaling functions ϕ_N. For $M = 2$ and $M = 4$, the smoothness of ϕ_N does grow asymptotically linearly as N increases:*

$$s(\phi_N) \sim N\frac{\log\frac{2}{1-\cos\frac{M\pi}{M+1}}}{2\log M} .$$

For $M = 3$, the asymptotic growth in smoothness is logarithmic:

$$s(\phi_N) \sim \frac{\log N}{4\log M} .$$

Thus the upper bounds of Theorem 4.2 are attained for $M = 2, 3$, and 4. We conjecture that the asymptotic upper bounds of Theorem 4.2 are attained for all M. Numerical measurements of the Sobolev exponents confirm this for sequence lengths up to 100, and dilation factors $M \leq 6$.

The asymptotic growth described in the theorem is shown in Figure 4.1, which compares the true Sobolev exponents of the minimal length scaling functions (as computed at the end of section 3) with their predicted linear and logarithmic asymptotes.

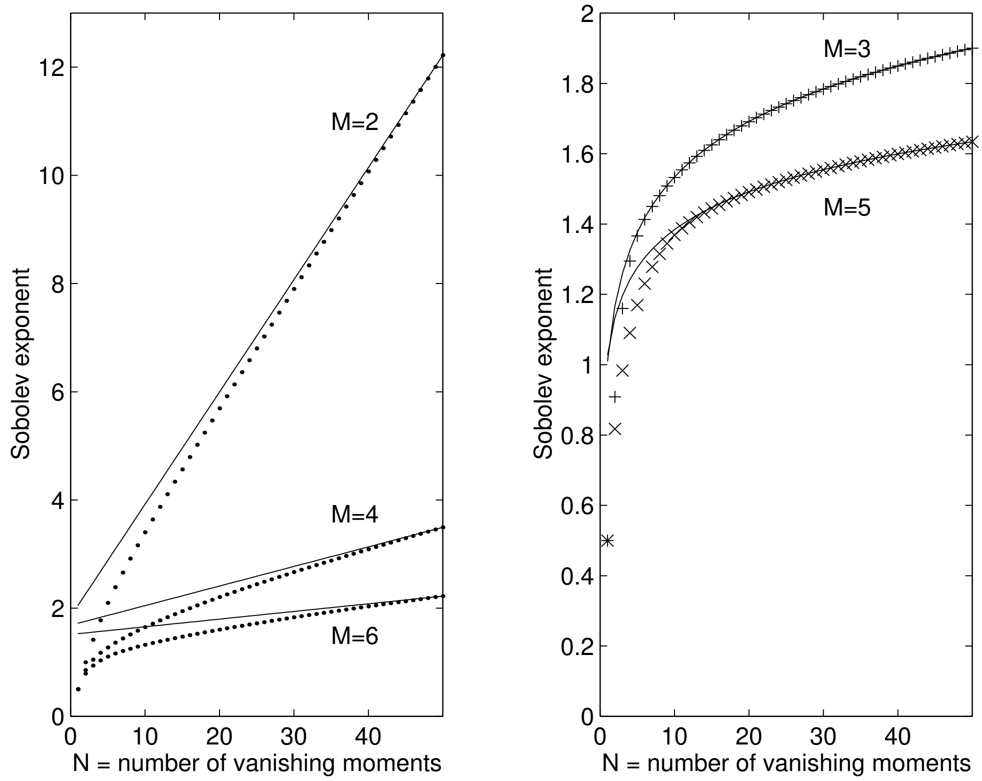

FIG. 4.1. *Asymptotic growth in Sobolev smoothness for minimal length degree N scaling functions, $N = 1, 2, \ldots, 50$. The left graph shows the actual Sobolev exponents and their linear asymptotes for even rank ($M = 2$, $M = 4$, $M = 6$), while the right graph shows the Sobolev exponents and their logarithmic asymptotes for odd rank ($M = 3$, $M = 5$).*

The difference between the linear growth of the cases $M = 2$, $M = 4$ and the case $M = 3$ is highlighted in Figure 4.2 which compares the graphs of scaling functions for these three dilation factors, with $N = 2$ and $N = 12$.

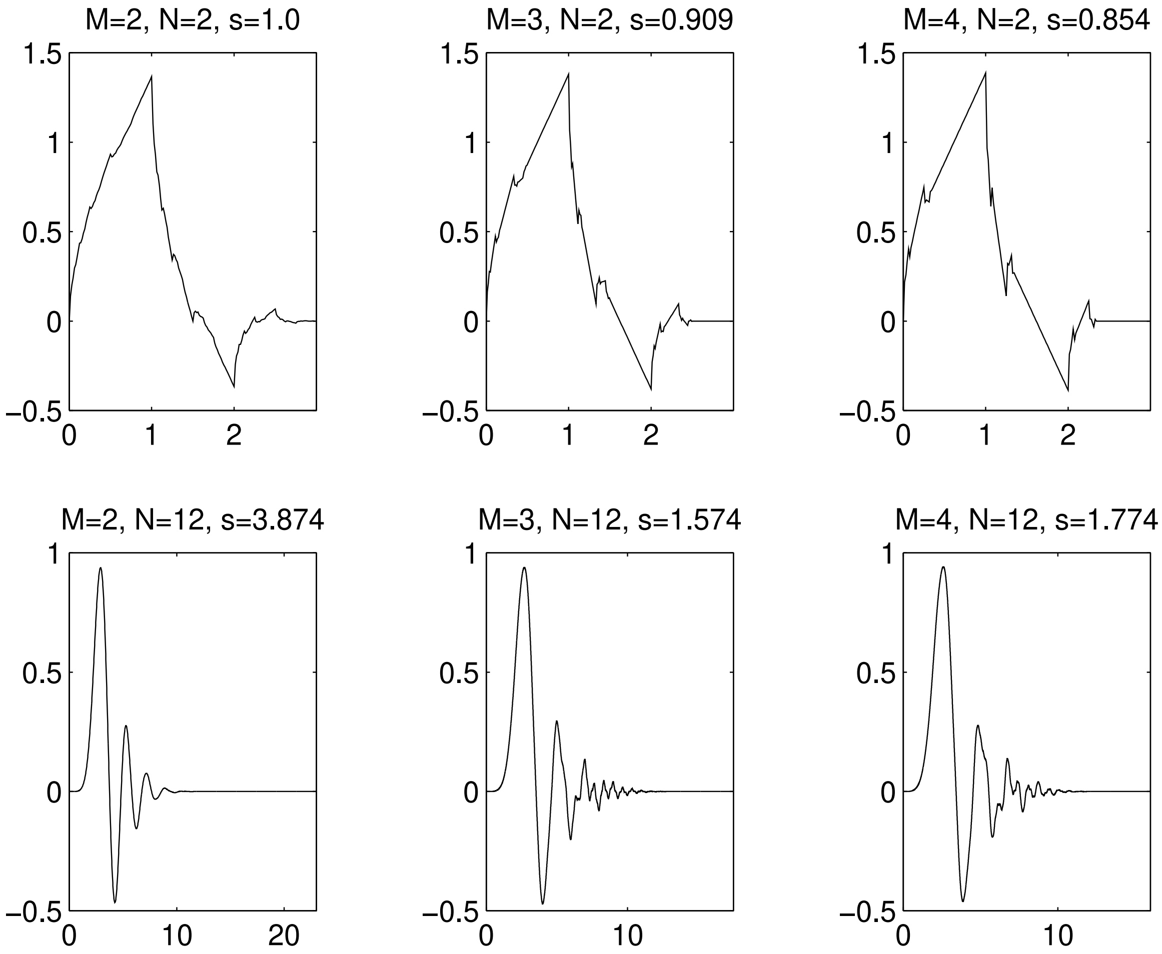

FIG. 4.2. *Scaling functions for $M = 2, 3, 4$ and $N = 2$ (top row) and $M = 2, 3, 4$ and $N = 12$ (bottom row).*

5. Additional Numerical Results. In this section we discuss two new families of orthonormal wavelets that have regularity greater than that of the Daubechies wavelets and their rank M generalizations, at least for short support lengths.[4] First we discuss PPZ wavelet familes based on zeros of the wavelet symbol (Fourier transform) at specified preperiodic points of the map τ, and then we discuss the family of wavelets that are maximally smooth for a given sequence length. For each family, we employ the formula (3.29) to determine the critical Sobolev exponent for example members of this family, producing numerical evidence towards the asymptotic behavior of the wavelet families as the sequence length increases.

[4] In fact, we concern ourselves here solely with the construction of the scaling sequence, or lowpass filter, since it determines the Sobolev regularity of the entire wavelet system. When $M = 2$, the wavelet sequence is determined as the "alternating flip" of the scaling sequence. For larger values of M, techniques for constructing a family of wavelets from a given scaling sequence appear in [18].

5.1. PrePeriodic Zero (PPZ) Wavelet Families. The asymptotic estimates of the previous section were expressed in terms of the values of the remainder trigonometric polynomial $R(\omega)$ at the periodic point ω_c of the map τ and its preimages. Notice also that the Daubechies and generalized Daubechies scaling filters are specified by zeros of $A(\omega)$ at the preperiodic points $\omega = 2\pi k/M,\ 1 \le k \le M-1$ (when $M = 2$, this reduces to an N-th order zero at π). Given these examples, we are led to define a new class of scaling functions and corresponding wavelet systems that includes the maximal vanishing moments family as a special case. The new class is defined to be those wavelet scaling filters (i.e. sequences satsifying (1.2) and (1.3)) whose Fourier transform has specified zeros of specified orders at preperiodic points of the map τ. We call these PPZ wavelet filters (for PrePeriodic Zero) and they are determined by the choice of preperiodic zeros at specified locations ω_i of order N_i; this collection of constraints is designated generically by $(\omega_1;\ N_1,\ \omega_2,\ N_2;\ ...;\ \omega_L,\ N_L)$. Note that the sum condition (1.3) simply amounts to requiring a zero of order 1 at $2\pi k/M,\ 1 \le k \le M-1$ for each PPZ wavelet filter. We now investigate several PPZ wavelet families for the cases of $M = 2,\ 3,\ 4$ and see how they can be used to improve the Sobolev regularity of a filter with a given length.

For $M = 2$, the dyadic wavelet case, we consider three PPZ wavelet families with filter length $2N$ and zeros of the specified locations and orders:

I_N : $(\pi,\ N)$ (Daubechies),

II_N : $(\pi,\ N-2;\ 2\pi/3,\ 1;\ 4\pi/3,\ 1)$

III_N : $(\pi,\ N-4;\ 2\pi/3,\ 1;\ 4\pi/3,\ 1;\ 5\pi/6,\ 1;\ 7\pi/6,\ 1)$

The definitions of families II_N and III_N were motivated by the dominant role of $R(\frac{2\pi}{3})$ and $R(\frac{5\pi}{6})$ in the Sobolev estimates of section 4. Notice that $\frac{5\pi}{6}$ is a preimage under τ of $\frac{\pi}{3}$ which is in turn a preimage of $\frac{2\pi}{3}$, a periodic point. The two families II_N and III_N can be constructed explicitly; instead of using the minimal length $R_N(\omega)$ for N vanishing moments, one uses the more general expression [10, 18]

$$R(\omega) = R_N(\omega) + (1-\cos\omega)^N \sum_{n \neq Mk} \tilde{r}_n \cos n\omega\ , \tag{5.55}$$

and choose the free parameters $\tilde{r}_n$ to assert additional zeros at the points ω_i. For example, the representative for family II_N with filter length $2N$ is determined by

$$R(\omega) = R_{N-1}(\omega) + (1-\cos\omega)^{N-1}\left(\tilde{r}_1 \cos\omega + \tilde{r}_3 \cos 3\omega\right)\ ,$$

where $\tilde{r}_1$ and $\tilde{r}_3$ are determined by the conditions

$$R\left(\frac{2\pi}{3}\right) = 0\ ; \quad R'\left(\frac{2\pi}{3}\right) = 0\ .$$

$R(\omega)$ must have a double zero at $\omega_c = \frac{2\pi}{3}$ because it is associated with the magnitude squared of the Fourier transform. The coefficients of $R_N(\omega)$ are predetermined by formulae (1.11)-(1.12), and we can solve for $\tilde{r}_1$ and $\tilde{r}_3$ to find

$$\begin{aligned} \tilde{r}_1 &= \rho_0\left[(4N+12)\cos^3\omega_c - 12\cos^2\omega_c - (3N+3)\cos\omega_c + 3\right] \\ &\qquad + \rho_1\left[-4\cos^3\omega_c + 3\cos\omega_c\right] \\ \tilde{r}_3 &= \rho_0\left[1-(N+1)\cos\omega_c\right] + \rho_1\cos\omega_c\ , \end{aligned}$$

where

$$\rho_0 = \frac{1}{8\cos^3\omega_c\,(1-\cos\omega_c)^{N+1}} \sum_{n=0}^{N-1} \binom{N-1+n}{N-1} (1-\cos\omega_c)^n \,,$$

$$\rho_1 = \frac{1}{8\cos^3\omega_c\,(1-\cos\omega_c)^{N}} \sum_{n=1}^{N-1} \binom{N-1+n}{N-1} n\,(1-\cos\omega_c)^{n-1} \,.$$

For each family, we used the formula (3.29) to determine the critical Sobolev exponent for sequences of length up to 60, producing numerical evidence for the asymptotic growth in regularity. Both families II_N and III_N appear to have Sobolev regularity growing faster with N than the $.2075N$ growth of the Daubechies orthogonal wavelets, and the Sobolev regularity of the members of family III_N grew faster than for members of II_N, as shown in Figure 5.3. Notice that family II_N does not yield a smoother scaling function than I_N (Daubechies) until the sequence length reaches 20, and family III_N does not do so until the sequence length reaches 18. Figure 5.4 shows the scaling functions and their fourth derivatives for representatives of each of these families corresponding to length-30 scaling sequences. The Sobolev exponents of the three scaling functions are 4.565, 5.410, and 6.291, respectively. While the scaling functions appear quite similar, the difference shows up in the graph of the fourth derivative. Table 5.2 lists representative scaling sequences (lowpass filters) from families II_N and III_N for the first values of N that yield smoother scaling functions than the Daubechies orthonormal case.

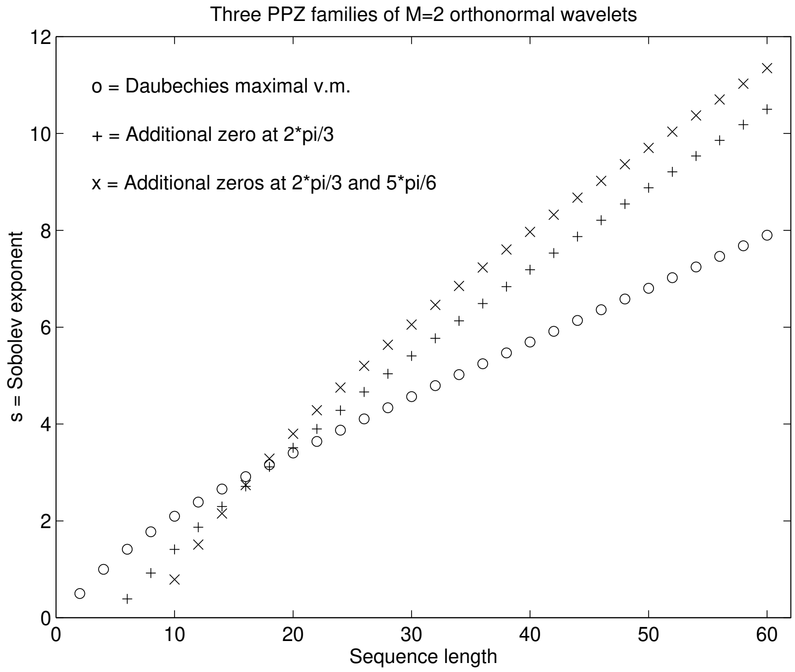

FIG. 5.3. *Sobolev exponents for the PPZ families I_N, II_N, and III_N of $M = 2$ orthonormal wavelets.*

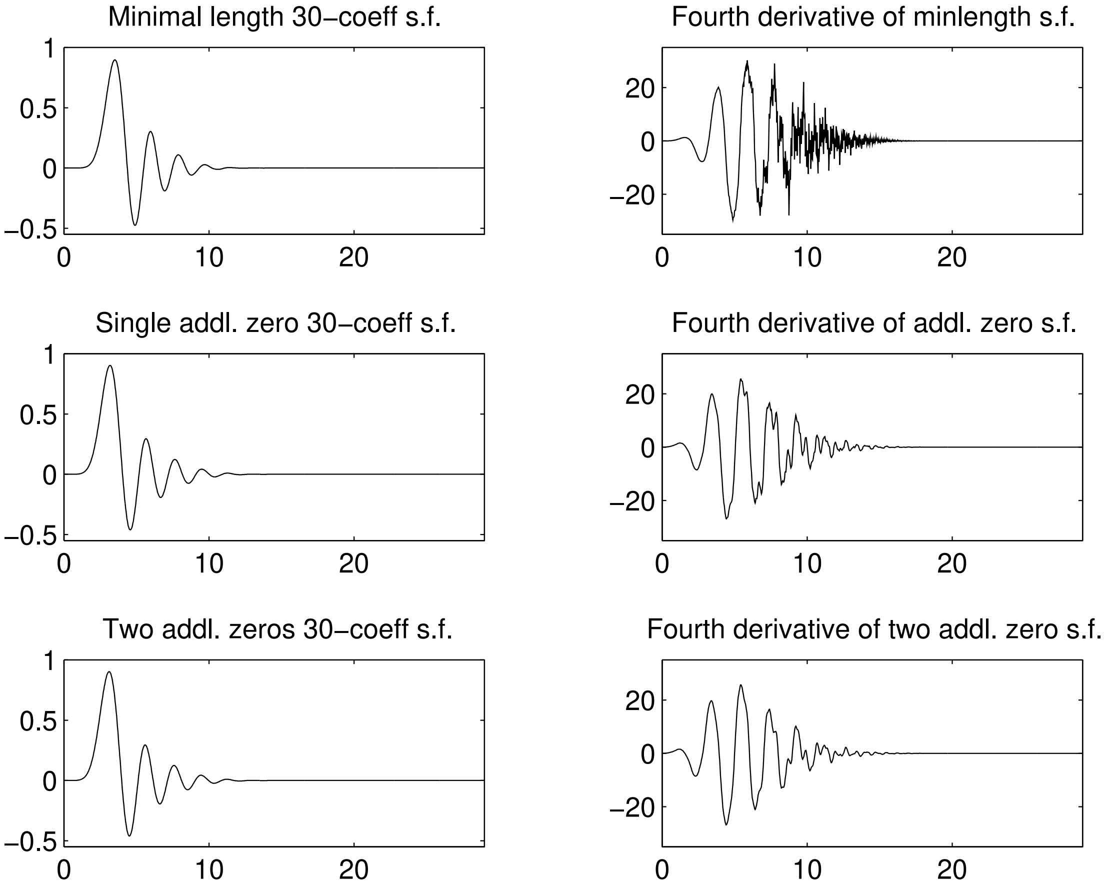

FIG. 5.4. *Scaling functions and their fourth derivatives for 30-coefficient representatives of three families. From top to bottom: Daubechies minimum length, single additional zero at $2\pi/3$, and two additional zeros at $2\pi/3$ and $5\pi/6$.*

$N=9$		$N=10$	
	III_N	II_N	III_N
	1.1883722e-01	7.2584548e-02	8.3467960e-02
	5.5558500e-01	4.1122321e-01	4.4422494e-01
	9.8474067e-01	9.0043268e-01	9.1959202e-01
	6.8956527e-01	8.5656699e-01	8.2310206e-01
	-1.1120155e-01	1.0698565e-01	6.0374447e-02
	-3.8516021e-01	-4.0736463e-01	-4.1198211e-01
	6.9702115e-03	-1.3764537e-01	-1.1069819e-01
	2.2305371e-01	2.2315537e-01	2.3155622e-01
	-6.5279338e-03	8.7630402e-02	7.0914601e-02
	-1.2772284e-01	-1.3594394e-01	-1.3981326e-01
	2.0087591e-02	-3.6210833e-02	-2.8000499e-02
	6.1066959e-02	7.8782544e-02	8.0164525e-02
	-2.2314465e-02	1.8730840e-03	-3.7024693e-04
	-1.8272254e-02	-3.4245084e-02	-3.6242294e-02
	1.1994086e-02	7.7548199e-03	8.7664593e-03
	1.3312700e-03	8.5202119e-03	9.9684916e-03
	-2.5858330e-03	-4.1161177e-03	-5.0563973e-03
	5.5309845e-04	-5.6915261e-04	-7.8883182e-04
	0	7.1114008e-04	1.0098456e-03
	0	-1.2552254e-04	-1.8974566e-04
$N=11$		$N=12$	
II_N	III_N	II_N	III_N
5.0990841e-02	5.8650144e-02	3.5835244e-02	4.1226583e-02
3.2185761e-01	3.5004362e-01	2.4934331e-01	2.7267127e-01
8.0438071e-01	8.3064877e-01	7.0088560e-01	7.3062223e-01
9.2953806e-01	9.0840298e-01	9.5451270e-01	9.4612801e-01
2.9935504e-01	2.4900389e-01	4.8265648e-01	4.3357844e-01
-3.6763176e-01	-3.8407619e-01	-2.7266496e-01	-3.0134381e-01
-2.5860794e-01	-2.3375959e-01	-3.6140910e-01	-3.4198738e-01
1.8288859e-01	1.9936301e-01	1.0219433e-01	1.2569725e-01
1.6604687e-01	1.5212248e-01	2.3033047e-01	2.2179903e-01
-1.1172649e-01	-1.2213405e-01	-5.7985257e-02	-7.3575952e-02
-9.1580463e-02	-8.3355371e-02	-1.3869797e-01	-1.3354473e-01
7.4548419e-02	7.9053392e-02	4.6014929e-02	5.4323638e-02
3.6769323e-02	3.3593364e-02	7.2904363e-02	7.0139540e-02
-4.3635829e-02	-4.5857290e-02	-3.6641104e-02	-4.0195014e-02
-6.1885919e-03	-5.6144422e-03	-2.8309929e-02	-2.7776905e-02
1.8036774e-02	1.9804872e-02	2.2346828e-02	2.4199209e-02
-2.5166700e-03	-2.9435298e-03	5.7158112e-03	5.8546834e-03
-4.2168789e-03	-5.0779239e-03	-8.9542939e-03	-1.0138251e-02
1.6229818e-03	2.0436984e-03	6.0611297e-04	7.3659492e-04
2.9839146e-04	4.1233219e-04	1.9925778e-03	2.4560769e-03
-2.7209272e-04	-3.8941618e-04	-6.2030083e-04	-7.9678361e-04
4.3106754e-05	6.5247055e-05	-1.4422580e-04	-1.9993439e-04
0	0	1.0322851e-04	1.4868815e-04
0	0	-1.4835845e-05	-2.2480933e-05

TABLE 5.2

$M=2$ scaling sequences from the families II_N and III_N, smoother than Daubechies' wavelets of the same lengths.

Similarly, when $M = 4$ we examined two PPZ wavelet families, the maximal vanishing moment family with filter length $4N$:

I_N : $(\pi,\ N;\ \pi/2,\ N;\ 3\pi/2,\ N)$ (generalized Daubechies),

and a new family with filter length $4N - 1$:

II_N : $(\pi,\ N-1;\ \pi/2,\ N-1;\ 3\pi/2,\ N-1;\ 4\pi/5,\ 1;\ 6\pi/5,\ 1)$.

The new PPZ family has one vanishing moment relaxed, with the resulting degrees of freedom used to assert a zero at the periodic point $4\pi/5$. The example scaling functions from this family proved to have Sobolev regularity that grew significantly faster than for the maximal vanishing moment family I_N for scaling sequence length up to 60, as shown in Figure 5.5.

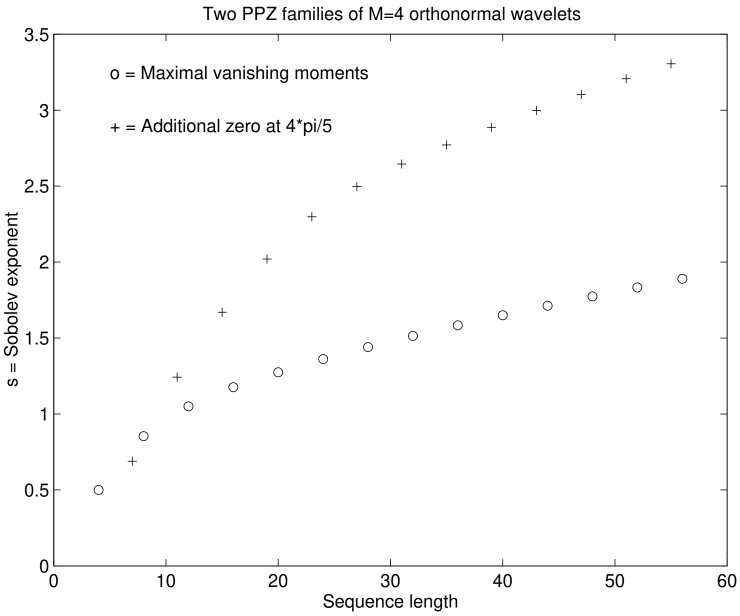

FIG. 5.5. *Sobolev exponents for the families I_N and II_N of $M = 4$ wavelets.*

For $M = 3$ we define the families:

I_N : $(2\pi/3,\ N;\ 4\pi/3,\ N)$ (sequence length $3N$)

II_N : $(2\pi/3,\ N-1;\ 4\pi/3,\ N-1;\ \pi,\ 1)$ (sequence length $3N - 1$).

The family I_N comprises the generalized Daubechies wavelets, and family II_N has a additional zero at $\omega_c = \pi$ in return for relaxing the approximation order by 1. Again, we applied the formula (3.29) to determine the critical Sobolev exponents of the members of these two families with scaling sequence length $L \leq 60$, and found that the smoothness of family I_N grew logarithmically with L (as predicted by Theorem 4.7), and the smoothness of family II_N appeared to grow algebraically with sequence length L. For the II_N examples studied, smoothness grew at a rate greater than $L^{.55}$. Figure 5.6 shows the Sobolev exponents for scaling functions from the new family and the minimal-length family, for scaling sequence lengths up to 60. Scaling functions from the two $M = 3$ families with similar support lengths (17 and 17.5) are compared in Figure 5.7.

These examples begin to illustrate the relation between smoothness and assertion of zeros of the Fourier transform of the filters at specified preperiodic points, which has also been observed by other authors [6], [10]. The patterns described indicate that a family of PPZ wavelets with vanishing specified at preperiodic points other than just the first preimages of 2π under τ may, in general, lead to greater smoothness than the maximal vanishing moment family. The methods of [46] may be useful for further theoretical studies in this direction.

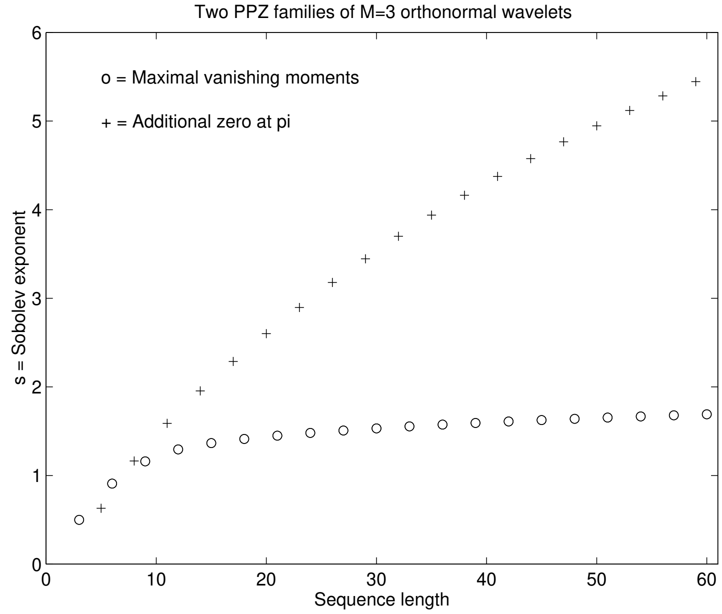

FIG. 5.6. *Sobolev exponents for the two families I_N and II_N of $M = 3$ scaling functions, $N = 1, 2, \ldots, 15$.*

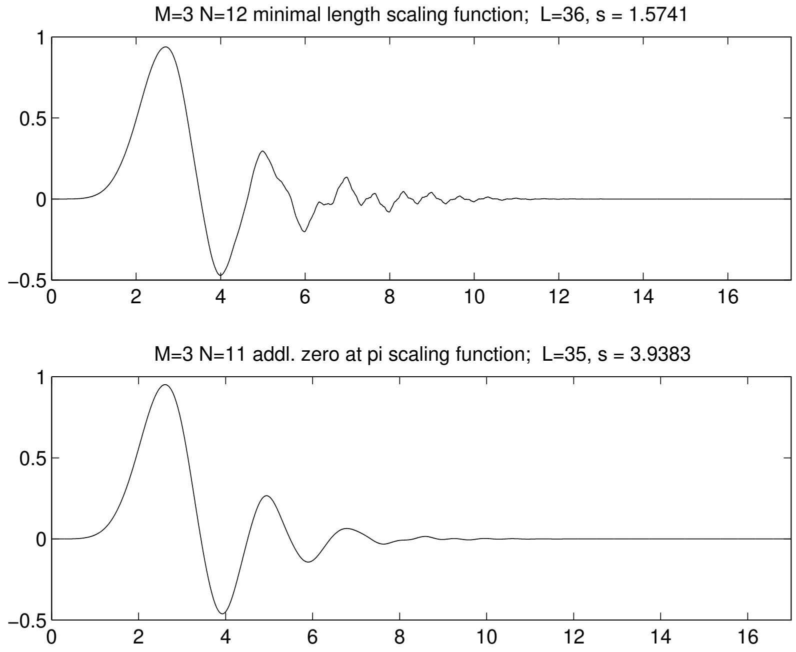

FIG. 5.7. *Two $M = 3$ scaling functions, one with support 18 and maximal approximation degree 12 (top), and the other with support $17\frac{1}{2}$, approximation degree 11, and an additional zero at π for the symbol (bottom).*

5.2. Maximally regular families. One may define a new family of wavelets as those having the maximal regularity possible for a given scaling sequence length, as the solution to an extremal problem. A priori, there is no reason that this family should have an algebraic definition similar to the general PPZ family described above. We present here some preliminary work for the $M = 2$ case, in which we solved the extremal problem numerically (see also [35]). The Sobolev exponent (as measured by the eigenvalues of a finite matrix) was used as the objective function for an optimization procedure. In fact, rather than work with the formula we have used so far in terms of T_R, we worked with the transfer operator T_P based on the full symbol $P(\omega)$. P must be a halfband filter (i.e. every other coefficient is zero, except the middle coefficient which is 1) or dyadic interpolant, in order to guarantee orthogonality:

$$P(\omega) = \sum_{n=-N}^{N} p_n e^{in\omega} \quad \text{and} \quad p_{2n} = 2\delta_{n,0} \ .$$

But then the remaining coefficients are free, except for the constraint

$$P(\omega) \geq 0 \ , \tag{5.56}$$

which is required so that we can obtain $A(\omega)$ as a spectral factor of P. Instead of the formula (3.29) one uses the variant

$$s(\phi) = -\frac{\log \mu(T_P)|_{\mathcal{F}_{L,N}}}{2 \log 2} \ ,$$

i.e. one examines the eigenvalues of T_P on the subspace of trigonometric polynomials in the space E_{2L-1} that have the form

$$F(\omega) = (1 - \cos\omega)^N G(\omega) \ .$$

In this way one is able to use a constrained minimization [13] of $-s(\phi)$ on the free coefficients p_{2n-1} of the halfband filter $P(\omega)$, subject to (5.56). All possible constellations of the roots of $P(z)$, on and off the unit circle were allowed. For sequence lengths 2, 4, and 6, the optimization procedure found no wavelets smoother than the Daubechies wavelets; however, smoother wavelets were found for sequence lengths 8 and larger. Families of wavelet filters with greater asymptotic growth than the Daubechies wavelets have been introduced by Volkmer [46]; the filters discussed here fill out the picture for relatively short sequence lengths. In fact, the smoother wavelets described here verify the results of [46]: for sequence lengths $L \leq 30$, we found the smoothness of the maximally regular filters to grow approximately as $.2L$, compared with $.12L$ for the Daubechies orthogonal wavelets. Volkmer [46] demonstrated a family with asymptotic growth at least 0.175, and the true asymptotic growth is usually less than the growth displayed by the first members of a family.

A graph of the smoothnesses of the maximally smooth wavelets is shown in Figure 5.8, and Table 5.3 gives the scaling sequences (lowpass filter coefficients) for the first five maximally smooth wavelets to be smoother than the Daubechies wavelets. Notice that the wavelets given here are maximally Sobolev-regular; it is also possible to define and obtain maximally Hölder-regular wavelets, as was done for short sequence lengths in [10], [32]. Further details are given in [19].

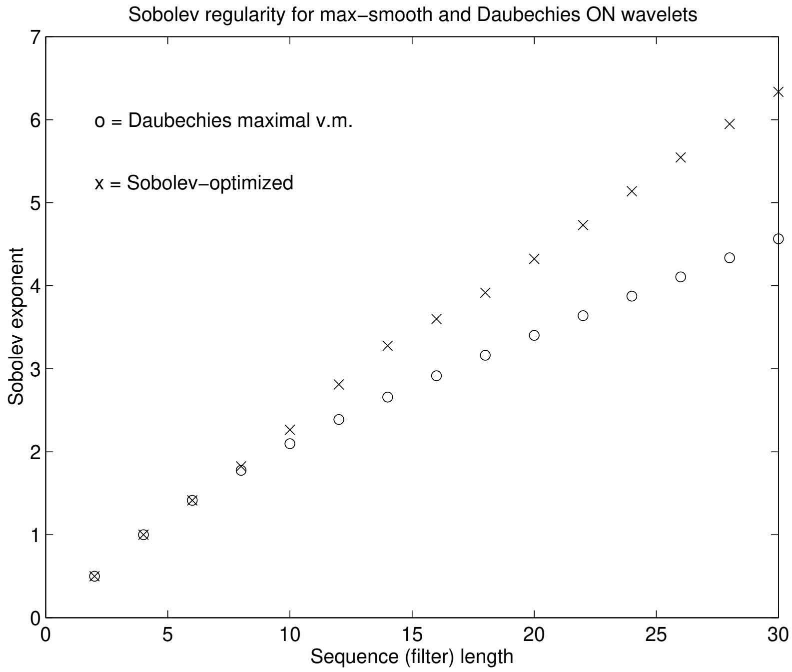

FIG. 5.8. *Sobolev regularity for maximally regular and Daubechies orthogonal wavelets (M=2).*

$L = 8$ $s = 1.8243$	$L = 10$ $s = 2.2649$	$L = 12$ $s = 2.8102$	$L = 14$ $s = 3.2767$	$L = 16$ $s = 3.5996$
3.3685703e-01	2.5557243e-01	1.9928930e-01	1.4417064e-01	1.1232325e-01
1.0165625e+00	8.8714362e-01	7.7189637e-01	6.4237732e-01	5.4610880e-01
8.8044013e-01	9.9293073e-01	1.0333604e+00	1.0352376e+00	9.9797890e-01
-4.4932326e-02	1.5839422e-01	3.5310140e-01	5.5649518e-01	6.9932470e-01
-2.6684133e-01	-3.4598373e-01	-3.3953222e-01	-2.5686447e-01	-1.3791439e-01
4.4787241e-02	-4.2007324e-02	-1.6585624e-01	-2.8954572e-01	-3.7329911e-01
4.9544171e-02	1.2047358e-01	1.5410368e-01	1.2325755e-01	5.0237485e-02
-1.6417390e-02	-1.0154456e-02	4.2668283e-02	1.1835784e-01	1.8540439e-01
0	-2.2993011e-02	-5.8691899e-02	-6.7385273e-02	-4.2381774e-02
0	6.6239345e-03	1.1517140e-03	-2.9688878e-02	-7.4940880e-02
0	0	1.1470713e-02	2.6416524e-02	3.0218639e-02
0	0	-2.9615249e-03	9.1966772e-04	1.9023799e-02
0	0	0	-4.8325378e-03	-1.2874778e-02
0	0	0	1.0845807e-03	-1.1254647e-03
0	0	0	0	2.4126693e-03
0	0	0	0	-4.9623600e-04

TABLE 5.3

$M = 2$ maximally Sobolev-regular scaling sequences, smoother than Daubechies' sequences of the same length.

Acknowledgements

The authors would like to thank Howard Resnikoff and Aware, Inc. for providing the environment for carrying out this mathematical work. This research was supported in part by the Advanced Research Projects Agency of the Department of Defense and was monitored by the Air Force Office of Scientific Research under contract no. F49620-92-C-0054. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation hereon.

Note: After finishing this work (the bulk of which was originally reported at the AMS meeting in Cincinnati, January 1994), we learned of results related to those of sections 3 and 4, obtained independently by Bi, Dai, and Sun [1].

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