Coifman Wavelet Systems: Approximation, Smoothness, and Computational Algorithms[1]

J. Tian, R. O. Wells, Jr., J. E. Odegard, and C. S. Burrus

Computational Mathematics Laboratory
Rice University
Houston, Texas 77005-1892, USA
http://cml.rice.edu/

1. INTRODUCTION

The theory of wavelet analysis has grown explosively in the last decade. The terminology "wavelet" was first introduced, in the context of a mathematical transform, in 1984 by A. Grossmann and J. Morlet [GM84]. In 1988, I. Daubechies, in her celebrated paper [Dau88], introduced a class of compactly supported orthonormal wavelet systems in general, as well as a family with growing smoothness for increasing support, the *Daubechies wavelet systems*. In 1989, S. G. Mallat [Mal89] and Y. Meyer [Mey92] presented the theory of multiresolution analysis. The spline family was studied by G. Battle [Bat87], P. G. Lemarié [Lem88], and C. K. Chui [Chu92b]. The necessary and sufficient conditions for an orthonormal wavelet system were given by A. Cohen [Coh90] and W. M. Lawton [Law91]. Except for the *Haar wavelet system*, compactly supported orthogonal wavelet systems can't be symmetric, though symmetry is highly desired, for example, in the applications in signal processing, where symmetry corresponds to linear phase. To obtain symmetry and keep the property of perfect reconstruction, A. Cohen, I. Daubechies, and J.-C. Feauveau [CDF92] replaced the orthogonality condition with biorthogonality and thus established the theory of biorthogonal wavelet systems. At the same time, pioneer work has been done by many

[1] in *Computational Science for the 21st Century*, M-O. Bristeau, G. Etgen, W. Fitzgibbon, J. L. Lions, J. Périaux, M. F. Wheeler, Editors, 831-840, Tours, France, 1997. This work was supported in part by ARPA and Texas ATP.

scientists from mathematics, physics and engineering. For more details of wavelet theory, we refer to [BGG97, Chu92a, Dau92, Mey92, RW97, VK95].

In this paper, we will present an overview of *Coifman wavelet systems*, a family of wavelet systems with very nice properties both in the theoretical sense and application sense. The original idea goes back to R. Coifman of Yale University. In the spring of 1989, he suggested that it might be worthwhile to construct orthogonal wavelet systems with vanishing moments not only for the wavelet functions (which is the hypothesis posed on the Daubechies wavelet systems), but also for the scaling functions. This turned out to be an interesting success. One key property of these orthogonal wavelet systems with vanishing moments equally distributed between scaling functions and wavelet functions (which we call *orthogonal Coifman wavelet systems*) is that they have very good approximation properties with exponential decay. *Biorthogonal Coifman wavelet systems*, the biorthogonal counterparts of orthogonal Coifman wavelet systems, also have good approximation properties with exponential decay. An attractive feature of biorthogonal Coifman wavelet systems is that all the scaling filters have dyadic rational coefficients. Thus we have obtained a family of biorthogonal wavelet systems with a multiplication-free discrete wavelet transform. Moreover they are compactly supported, and converge to the *Sinc wavelet system*. Half of the members (more precisely, those with odd degrees) of the family are symmetric, which indicates the potential promising of these biorthogonal Coifman wavelet systems in the applications of signal processing. Actually, based on a complete coding performance evaluation [WTWB97], biorthogonal Coifman wavelet systems are very useful for image transform coding and seem to be quite comparable to the wavelet systems used in the state-of-the-art compression systems.

2. A WAVELET SAMPLING APPROXIMATION THEOREM

As usual, in orthonormal wavelet systems, we define the *wavelet orthogonal projection* of an $L^2(\mathbf{R})$ function $f(x)$ at the j-th level by

$$P^j(f) \;:=\; \sum_{k\in\mathbf{Z}} \left(\int_{-\infty}^{\infty} f(t)\phi_{j,k}(t)dt \right) \cdot \phi_{j,k}(x) \,,$$

where $\phi(x)$ is the scaling function of the orthonormal wavelet system, and $\phi_{j,k}(x) = 2^{j/2}\phi(2^jx-k)$, $j,k\in\mathbf{Z}$. The wavelet orthogonal projection is exactly the orthogonal projection of $f(x)$ onto the j-th subspace V_j in the multiresolution analysis of the orthonormal wavelet system [Mal89].

Theorem 2..1 *In an orthonormal wavelet system with scaling function $\phi(x)$ and wavelet function $\psi(x)$, if $\psi(x)$ has vanishing moments up to degree N, i.e.,*

$$\int_{-\infty}^{\infty} x^n\psi(x)\,dx \;=\; 0\,, \quad \text{for}\, n = 0,1,\cdots,N\,,$$

then for $f(x)\in C_0^{N,1}(\mathbf{R})$,

$$\left|\left|f(x) \,-\, P^j(f)\right|\right|_{L^2} \;\le C2^{-j(N+1)}\,,$$

where C is a constant, independent of j and N.

The proof of Theorem 2..1 can be found in [GLRT90].

We define the *wavelet sampling approximation* of an $L^2(\mathbf{R})$ function $f(x)$ at the j-th level by

$$S^j(f) \;:=\; 2^{-j/2} \sum_{k\in\mathbf{Z}} f\left(\frac{k}{2^j}\right) \phi_{j,k}(x)\,.$$

A similar result to Theorem 2..1 is that imposing vanishing moments on the scaling function $\phi(x)$ will reduce the difference between the wavelet orthogonal projection $P^j(f)$ and the wavelet sampling approximation $S^j(f)$. The proof can be found in [RW97, Tia96].

Theorem 2..2 *In an orthonormal wavelet system with scaling function $\phi(x)$, if $\phi(x)$ has vanishing moments up to degree N, i.e.,*

$$\int_{-\infty}^{\infty} \phi(x)\,dx \;=\; 1\,,$$

$$\int_{-\infty}^{\infty} x^n \phi(x)\,dx \;=\; 0\,, \quad \text{for}\, n = 1, \cdots, N\,,$$

then for $f(x) \in C_0^{N,1}(\mathbf{R})$,

$$\left|\left|P^j(f) \,-\, S^j(f)\right|\right|_{L^2} \;\leq C 2^{-j(N+1)}\,,$$

where C is a constant, independent of j and N.

The wavelet sampling approximation is what is used in most applications of wavelets, as it is the easiest approximation to compute. (Simply let the sampling values of the given function be the corresponding wavelet expansion coefficients.) In an orthonormal wavelet system, by definition,

$$\left|\left|f(x) - S^j(f)\right|\right|_{L^2}^2 \;=\; \left|\left|f(x) - P^j(f)\right|\right|_{L^2}^2 \,+\, \left|\left|P^j(f) - S^j(f)\right|\right|_{L^2}^2\,.$$

Thus, to minimize the wavelet sampling approximation error (the left hand side), we need to minimize the two terms of the right hand side to the same degree. So based on Theorem 2..1 and Theorem 2..2, the optimal distribution of vanishing moments between the scaling function and the wavelet function will be the equal distribution, the scaling function and the wavelet function will have the same degrees of vanishing moments. Also it follows that vanishing moments of the wavelet function will reduce the error in the wavelet orthogonal projection, or the distance from the original L^2 function to the projection subspace, while vanishing moments of the scaling function will reduce the difference between the wavelet orthogonal projection and the wavelet sampling approximation, where both of these two are in the projection subspace. The wavelet sampling approximation is illustrated in Figure 1. The illustration shows the nature and relationships of the two types of approximations, the wavelet orthogonal projection P^j and the wavelet sampling approximation S^j.

Surprisingly, the orthogonality is not a necessary condition for the wavelet sampling approximation estimation, though it is used in the above argument. All we need is the

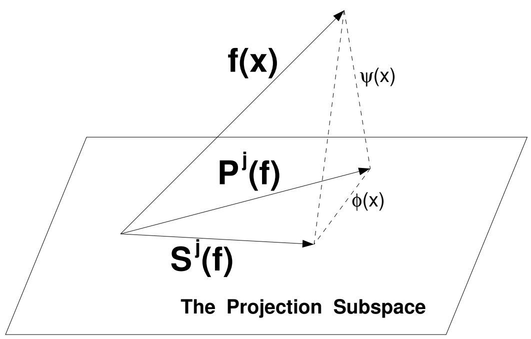

Figure 1 Wavelet Sampling Approximation

vanishing moments conditions on the scaling function and the wavelet function. Recall that the vanishing moments conditions of the scaling function and the wavelet function are equivalent to the linear conditions on the scaling filter [SHGB93, Hel95]. Thus we can present the wavelet approximation theorem with conditions on $\{a_k\}$ only.

Theorem 2..3 (Wavelet Sampling Approximation Theorem) *Suppose $\phi(x)$ is an $L^2(\mathbf{R})$ solution of the two-scale difference equation*

$$\phi(x) \;=\; \sum_{k\in\mathbf{Z}} a_k \phi(2x-k)\,,$$

and it is normalized

$$\int_{-\infty}^{\infty} \phi(x)\,dx \;=\; 1\,,$$

where $\{a_k\}$ is a sequence with finite nonzero elements, satisfying

$$\sum_{k\in\mathbf{Z}} (2k)^n a_{2k} \;=\; \sum_{k\in\mathbf{Z}} (2k+1)^n a_{2k+1} \;=\; 0\,,$$

for $n = 1, \cdots, N$, *and*

$$\sum_{k\in\mathbf{Z}} a_{2k} \;=\; \sum_{k\in\mathbf{Z}} a_{2k+1} \;=\; 1\,.$$

Then for $f(x) \in C_0^{N,1}(\mathbf{R})$,

$$||f\,(x) - S^j(f)\,(x)\,||_{L^2} \;\le\; C2^{-j(N+1)}\,,$$

where C is a constant, depending only on $f(x)$ and $\{a_k\}$.
If in addition, $\phi(x) \in C^m(\mathbf{R})$, where $m \in \mathbf{Z}, 0 \le m \le N$, then

$$||f\,(x) - S^j(f)\,(x)\,||_{H^m} \;\le\; C2^{-j(N+1-m)}\,,$$

where C is a constant, depending only on $f(x)$ and $\{a_k\}$.

The wavelet sampling approximation theorem doesn't require the orthogonality condition. It is true not only for general wavelet systems (semiorthogonal and biorthogonal), but also for an L^2 function which satisfies a two-scale difference equation with the coefficients $\{a_k\}$ having vanishing moments. Space limits preclude our reproducing the proof of Theorem 2..3; however it can be found in [RW97, Tia96].

3. BIORTHOGONAL COIFMAN WAVELET SYSTEMS

A significant advantage of compactly supported biorthogonal wavelet systems [CDF92, VH92] over compactly supported orthonormal wavelet systems is that biorthogonal systems can be symmetric, while orthonormal systems can't, except the *Haar wavelet system.*

3.1. DEFINITION

Definition 3..1 *A biorthogonal wavelet system with compact support is called a* biorthogonal Coifman wavelet system *of degree N if the following two conditions are satisfied,*

- *the vanishing moments of the synthesis scaling function $\tilde{\phi}(x)$ and wavelet function $\tilde{\psi}(x)$ are of degree N, i.e.,*

$$\int_{-\infty}^{\infty} x^n \tilde{\phi}(x)\,dx \;=\; 0\,, \quad \text{for}\, n = 1, \cdots, N\,,$$

$$\int_{-\infty}^{\infty} x^n \tilde{\psi}(x)\,dx \;=\; 0\,, \quad \text{for}\, n = 0, \cdots, N\,.$$

- *the vanishing moment of the analysis wavelet function $\psi(x)$ is of degree N,*

$$\int_{\mathbf{R}} x^n \psi(x)\,dx \;=\; 0\,, \quad \text{for}\, n = 0, \cdots, N\,.$$

Note that in the definition of the biorthogonal Coifman wavelet system, although there is no vanishing moment requirement on the analysis scaling function $\phi(x)$, it turns out that $\phi(x)$ also has vanishing moments up to degree N, because of the perfect reconstruction condition

$$\sum_{k\in\mathbf{Z}} a_k \tilde{a}_{k+2l} \;=\; 2\delta_{0,l}\,, \quad \forall l \in \mathbf{Z}\,.$$

Lemma 3..1 *For a biorthogonal Coifman wavelet system of degree N, the vanishing moments' degree of the analysis scaling function $\phi(x)$ is also N,*

$$\int_{-\infty}^{\infty} x^n \phi(x)\,dx \;=\; 0\,, \quad \text{for}\, n = 1, \cdots, N\,.$$

The proof of Lemma 3..1 (and all other results in this section) can be found in [Tia96]. So based on Theorem 2..3, biorthogonal Coifman wavelet systems will provide very good wavelet sampling approximation with fast convergence.

3.2. CONSTRUCTION

Using a time domain design method, the closed form solutions of the minimum length biorthogonal Coifman wavelet systems of all degrees are obtained.

Theorem 3..1 *The minimum length biorthogonal Coifman wavelet systems of degree N have the analysis scaling filter $\{a_k\}$ and the synthesis scaling filter $\{\tilde{a}_k\}$ of the form*

$$\tilde{a}_0 = 1, \quad \tilde{a}_{2k} = 0 \text{ when } k \neq 0,$$

- *if N is even, $N = 2n$,*

$$a_{2k+1} = \tilde{a}_{2k+1} = \frac{(-1)^k (2n+1)}{2^{4n-1}(2k+1)} \binom{2n-1}{n-1} \binom{2n}{n+k}$$

- *if N is odd, $N = 2n-1$,*

$$a_{2k+1} = \tilde{a}_{2k+1} = \frac{(-1)^k (2n-1)}{2^{4n-3}(2k+1)} \binom{2n-2}{n-1} \binom{2n-1}{n+k}$$

and

$$a_{2k} = 2\delta_{0,k} - \sum_{l \in \mathbf{Z}} \tilde{a}_{2l+1} \tilde{a}_{2l+1-2k}\,.$$

The scaling filters of the minimum length biorthogonal Coifman wavelet systems with degrees $N = 0, 1, 2$ and 3 are listed in Table 1.

3.3. MULTIPLICATION-FREE DISCRETE WAVELET TRANSFORM

As it can be seen in Table 1, the scaling filters $\{a_k\}$ and $\{\tilde{a}_k\}$ are all dyadic rational, i.e., all the nonzero elements in the scaling filters are of the form $(2p+1)/2^q$, for some integers p and q. Actually this assertion is true for scaling filters of the minimum length biorthgonal Coifman wavelet systems of all degrees. Thus we have obtained a family of biorthogonal wavelet systems on which we can implement a very fast multiplication-free discrete wavelet transform [Mal89], which consists of only addition and shift operations, on digital computers. This is one of the main advantage of biorthogonal Coifman wavelet systems over other widely used irrational wavelet systems, like the Cohen-Daubechies-Feauveau 9-7 biorthogonal filters (CDF-9-7) [CDF92].

Theorem 3..2 *In a biorthogonal Coifman wavelet system of degree N, the scaling filters are dyadic rationals, i.e., $\forall k \in \mathbf{Z}$, there must exist four integers p_1, p_2, q_1, q_2, such that*

$$a_k = \frac{2p_1+1}{2^{q_1}}, \quad \tilde{a}_k = \frac{2p_2+1}{2^{q_2}},$$

whenever a_k or $\tilde{a}_k$ are nonzero.

Table 1 Biorthogonal Coifman Wavelet Systems

N	a_k	$\tilde{a}_k$	N	a_k	$\tilde{a}_k$
$N=0$	$a_0 = 1$	$\tilde{a}_0 = 1$	$N=3$	$a_{-6} =$ -1/256	
	$a_1 = 1$	$\tilde{a}_1 = 1$		$a_{-5} = 0$	
$N=1$	$a_{-2} =$ -1/4			$a_{-4} = 9/128$	
	$a_{-1} = 1/2$	$\tilde{a}_{-1} = 1/2$		$a_{-3} =$ -1/16	$\tilde{a}_{-3} =$ -1/16
	$a_0 = 3/2$	$\tilde{a}_0 = 1$		$a_{-2} =$ -63/256	$\tilde{a}_{-2} = 0$
	$a_1 = 1/2$	$\tilde{a}_1 = 1/2$		$a_{-1} = 9/16$	$\tilde{a}_{-1} = 9/16$
	$a_2 =$ -1/4			$a_0 = 87/64$	$\tilde{a}_0 = 1$
$N=2$	$a_{-4} = 3/64$			$a_1 = 9/16$	$\tilde{a}_1 = 9/16$
	$a_{-3} = 0$			$a_2 =$ -63/256	$\tilde{a}_2 = 0$
	$a_{-2} =$ -3/16			$a_3 =$ -1/16	$\tilde{a}_3 =$ -1/16
	$a_{-1} = 3/8$	$\tilde{a}_{-1} = 3/8$		$a_4 = 9/128$	
	$a_0 = 41/32$	$\tilde{a}_0 = 1$		$a_5 = 0$	
	$a_1 = 3/4$	$\tilde{a}_1 = 3/4$		$a_6 =$ -1/256	
	$a_2 =$ -3/16	$\tilde{a}_2 = 0$			
	$a_3 =$ -1/8	$\tilde{a}_3 =$ -1/8			
	$a_4 = 3/64$				

3.4. CONVERGENCE TO SINC WAVELET SYSTEM

The *Sinc wavelet system* is a basic wavelet system whose scaling filter $\{a_k^{sinc}, k \in \mathbf{Z}\}$ is defined by

$$a_{2k}^{sinc} = \delta_{0,k}\,, \quad a_{2k+1}^{sinc} = \frac{(-1)^k 2}{(2k+1)\pi}\,.$$

It had been a problem for some time to find a sequence of scaling functions with compact supports which approximates the function $\mathrm{sinc}(\pi x) = \frac{\sin \pi x}{\pi x}$, the scaling function of the Sinc wavelet system. This problem is important because of the special relation of sinc function to signal processing applications. As observed by H. L. Resnikoff, the family of biorthogonal Coifman wavelet systems just provides a very suitable candidate.

Theorem 3..3 *For $N \in \mathbf{N}$, let $\{\tilde{a}_k^N, k \in \mathbf{Z}\}$ be the synthesis scaling filter of the biorthogonal Coifman wavelet system of degree N. Then*

$$\lim_{N\to\infty} \left|\left|\left(\tilde{a}^N\right) - \left(a^{sinc}\right)\right|\right|_{l^2} = \lim_{N\to\infty} \left(\sum_{k\in\mathbf{Z}} \left(\tilde{a}_k^N - a_k^{sinc}\right)^2\right)^{1/2} = 0\,.$$

4. A VARIATION OF ORTHOGONAL COIFMAN SYSTEMS

Several methods were proposed (for example, see [Dau93, TW95]) to construct *orthogonal Coifman wavelet systems*, a family of compactly supported orthogonal wavelet systems with vanishing moments equally distributed between scaling functions and wavelet functions. In examining these orthogonal Coifman wavelet systems, it was discovered that the odd degree systems always achieved (at least) one more vanishing moment for scaling functions than was specified and, therefore, did not have equal numbers of vanishing moments. Actually this observation can be proved based on the perfect reconstruction condition

$$\sum_{k\in\mathbf{Z}} a_k a_{k+2l} \;=\; 2\delta_{0,l}\,, \quad \forall l \in \mathbf{Z}\,.$$

Those with scaling filter lengths of $L = 6n + 2$ (the degree $N = 2n$ is even) have equal number of vanishing moments for scaling function and wavelet function, but always have even-order "extra" vanishing moment for scaling function located after the first non-zero one. Lengths $L = 6n$ (the degree $N = 2n - 1$ is odd) always have an "extra" vanishing moment for scaling function. Indeed, both lengths $L = 6n + 2$ and $L = 6n$ will have several even-order "extra" vanishing moments for longer L as a result of the perfect reconstruction condition. Lengths $L = 6n - 2$ do not occur for the original definition of an orthogonal Coifman wavelet system. If we generalize the definition to specify systems with approximately equal degrees of vanishing moments for scaling function and wavelet function, all lengths become possible and a larger class of orthogonal Coifman wavelet systems are available [BGG97, BO96].

We define *generalized orthogonal Coifman wavelet systems* to allow the degree of vanishing moments for scaling function and wavelet function to differ by at most one. That will include all the existing orthogonal Coifman wavelet systems plus $L = 6n-2$. The length-10 was designed by setting the scaling function has 3 vanishing moments and the wavelet function has 2 vanishing moments rather than 2 and 2 for the length-8 or 3 and 3 for the length-12 orthogonal Coifman wavelet systems. We obtained a length-10 generalized orthogonal Coifman wavelet system with the scaling filter

$$\begin{aligned}
\{&a_{-2} = 0.045436534779, && a_{-1} = -0.106828662892, && a_0 = -0.137086882888,\\
&a_1 = 0.695155395345, && a_2 = 1.138641439990, && a_3 = 0.430505791324,\\
&a_4 = -0.047768370432, && a_5 = -0.019163117110, && a_6 = 0.000777278551,\\
&a_7 = 0.000330593334\}.
\end{aligned}$$

It was found that the length-10 design again gives one extra vanishing moments for the scaling function which is two more than the wavelet function. A similar approach was used to design length-16, 22, and 28. Adding these new lengths to our traditional orthogonal Coifman wavelet systems gives Table 2.

The fourth and sixth columns in Table 2 contain the number of vanishing moments of the wavelet function, excluding the 0th moment because of orthogonality in all of these systems. The extra vanishing moment of the scaling function that occurs just after a non zero moment for $L = 6n + 2$ is also excluded from the count. This table shows generalized orthogonal Coifman wavelet systems for all even lengths. It shows the extra vanishing moments of the scaling function moments that are sometime achieved and how the total number of vanishing moments monotonically increases and how the "smoothness" as measured by the Hölder exponent [Rio92, HW96] increases with L and N.

When the moments of both scaling function and wavelet function are set to zero, a larger number can be obtained than is expected from considering the degrees

Table 2 Vanishing Moments for Various Length and Degree Generalized Orthogonal Coifman Systems

L Length	N Degree	ϕ set	ψ set	ϕ actual	ψ actual	Hölder exponent
4	0	1	0	1	0	0.2075
6	1	1	1	2	1	1.0137
8	2	2	2	2	2	1.3887
10	2	3	2	4	2	1.0909
12	3	3	3	4	3	1.9294
14	4	4	4	4	4	1.7353
16	4	5	4	6	4	1.5558
18	5	5	5	6	5	2.1859
20	6	6	6	6	6	2.8531
22	6	7	6	8	6	2.5190
24	7	7	7	8	7	2.8300
26	8	8	8	8	8	3.4404
28	8	9	8	10	8	2.9734
30	9	9	9	10	9	3.4083

of freedom available. As noted earlier, the generalized orthogonal Coifman wavelet systems fall into three classes. Those with scaling filter lengths of $L = 6n+2$ have equal degree of vanishing moments for scaling function and wavelet function, but always has "extra" vanishing moment for scaling function located after the first non-zero one. Lengths $L = 6n$ always have one more vanishing moment for scaling function than wavelet function and lengths $L = 6n-2$ always have two more vanishing moments for scaling function than wavelet function. There are additional even-order zero moments for longer lengths. We have observed that within each of these classes, the Hölder continuity exponent increases monotonically.

The lengths $L = 6n-2$ were not found by earlier investigators because they have the same degree of vanishing moments as the system just two shorter. However, they achieve two more vanishing moments for scaling function than the shorter length with the same degree.

Table 2 is just the beginning of a large collection of vanishing moment wavelet system designs with a wide variety of trade-offs that would be tailored to a particular application. In addition to the variety illustrated here, many (perhaps all) of these sets of specified vanishing moments have multiple solutions. The variety of solutions for each length can have different shifts, different Hölder exponents, and different degrees of being approximately symmetric.

REFERENCES

[Bat87] Battle G. (1987) A block spin construction of ondelettes. Part I: Lemarié functions. *Commun. Math. Phys.* 110: 610–615.

[BGG97] Burrus C. S., Gopinath R. A., and Guo H. (1997) *Introduction to Wavelets and*

Wavelet Transform. Prentice-Hall, Englewood Cliffs, NJ. to appear.

[BO96] Burrus C. S. and Odegard J. E. (submitted, November 1 1996) Wavelet systems and zero moments. *IEEE Trans. Signal Proc.* .

[CDF92] Cohen A., Daubechies I., and Feauveau J.-C. (1992) Biorthogonal bases of compactly supported wavelets. *Comm. Pure Appl. Math.* XLV: 485–560.

[Chu92a] Chui C. K. (1992) *An Introduction to Wavelets*. Wavelet Analysis and Its Applications. Academic Press, Boston.

[Chu92b] Chui C. K. (1992) On cardinal spline wavelets. In et al. M. B. R. (ed) *Wavelets and Their Applications*, pages 419–438. Jones and Bartlett Publishers.

[Coh90] Cohen A. (1990) Ondelettes, analyses multirésolutions et filtres miroir en quadrature. *Annales de l'institut Henri Poincaré* 7(5): 439–459.

[Dau88] Daubechies I. (1988) Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.* XLI: 906–966.

[Dau92] Daubechies I. (1992) *Ten Lectures on Wavelets*. CBMS-NSF regional conference series in applied mathematics. Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania.

[Dau93] Daubechies I. (1993) Orthonormal bases of compactly supported wavelets II. Variations on a theme. *SIAM J. Math. Anal.* 24(2): 499–519.

[GLRT90] Glowinski R., Lawton W., Ravachol M., and Tenenbaum E. (1990) Wavelet solution of linear and nonlinear elliptic, parabolic and hyperbolic problems in one dimension. In Glowinski R. and Lichnewski A. (eds) *Proceedings of the Ninth International Conference on Computing Methods in Applied Sciences and Engineering*. SIAM, Philadelphia.

[GM84] Grossmann A. and Morlet J. (July 1984) Decomposotion of Hardy functions into square integrable wavelets of constant shape. *SIAM J. Math. Anal.* 15(4): 723–736.

[Hel95] Heller P. N. (1995) Rank m wavelet matrices with n vanishing moments. *SIAM J. Matr. Anal.* 16: 502–518.

[HW96] Heller P. N. and Wells Jr. R. O. (1996) Sobolev regularity for rank M wavelets. *submitted to SIAM J. Math. Anal.* .

[Law91] Lawton W. M. (January 1991) Necessary and sufficient conditions for constructing orthogonal wavelet bases. *J. Math. Phys.* 32: 57–61.

[Lem88] Lemarié P. G. (1988) Une nouvelle bade d'ondelettes de $L^2(\mathbf{R}^n)$. *J. de Math. Pure et Appl.* 67(3): 227–236.

[Mal89] Mallat S. G. (September 1989) Multiresolution approximation and wavelet orthonormal bases of $L^2(\mathbf{R})$. *Trans. AMS* 315(1): 69–87.

[Mey92] Meyer Y. (1992) *Wavelets and Operators*. Cambridge University Press.

[Rio92] Rioul O. (November 1992) Simple regularity criteria for subdivision schemes. *SIAM J. Math. Anal.* 23(6): 1544–1576.

[RW97] Resnikoff H. L. and Wells Jr. R. O. (1997) *Wavelet Analysis and the Scalable Structure of Information*. Springer-Verlag, New York. To appear.

[SHGB93] Steffen P., Heller P. N., Gopinath R. A., and Burrus C. S. (December 1993) Theory of regular m-band wavelet bases. *IEEE Trans. on Signal Processing* 41(12).

[Tia96] Tian J. (February 1996) *The Mathematical Theory and Applications of Biorthogonal Coifman Wavelet Systems*. PhD thesis, Rice University, Houston, Texas.

[TW95] Tian J. and Wells Jr. R. O. (January 1995) Vanishing moments and wavelet approximation. Technical Report CML TR 95-01, Computational Mathematics Laboratory, Rice University. (ftp://cml.rice.edu/pub/reports/CML9501.ps.Z).

[VH92] Vetterli M. and Herley C. (1992) Wavelets and filter banks: Theory and design. *IEEE Trans. Acoust. Speech Signal Process.* 40: 2207–2232.

[VK95] Vetterli M. and Kovačević J. (1995) *Wavelets and Subband Coding*. Prentice Hall Signal Processing Series. Prentice Hall PTR, Englewood Cliffs, New Jersey.

[WTWB97] Wei D., Tian J., Wells Jr. R. O., and Burrus C. S. (1997) A new class of biorthogonal wavelet systems for image transform coding. *to appear in IEEE Trans. Image Processing* .