A fast implementation of wavelet transform for m-band filter banks*

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ABSTRACT

An orthogonal m-band discrete wavelet transform has an $O(m^2)$ complexity. In this paper we present a fast implementation of such a discrete wavelet transform. In an orthonormal m-band wavelet system, the vanishing moments (which corresponds to the approximation order and smoothness) and orthogonality conditions are imposed on the scaling filter (or lowpass filter) only. Given a scaling filter, one can design the other $m-1$ wavelet filters (or highpass filters). It's well-known that there are infinitely many solutions in such designing procedure. Here we choose one specific type of solutions and implement the corresponding wavelet transform in a scheme which has complexity $O(m)$. Thus for any scaling filter, one can always construct a full orthogonal m-band wavelet matrix with an $O(m)$ discrete wavelet transform.

Keywords: discrete wavelet transform, fast wavelet transform, filter bank, polyphase decomposition, wavelet matrix, wavelet matrix factorization, wavelet matrix construction, characteristic Haar matrix, canonical Haar matrix

1. INTRODUCTION

The discrete wavelet transform (DWT) has become a very powerful tool in signal processing, such as noise removal[1–3] and image compression,[4–7] etc. Due to its localization property in both the spatial domain and spectral domain, the DWT gives a compact representation of data/signal and often leads to a better understanding of data/signal and possible improvement in post-processing. A wavelet system can also be thought as a filter bank with an additional functional space structure. With such a functional space structure (usually smooth bases), a wavelet representation is considered more preferable than other representation for smooth signals. For more details about wavelet analysis, we refer to,[8–15] etc.

An m-band (or rank m, scale factor of m) wavelet matrix consists of one scaling filter (or lowpass filter) and $m-1$ wavelet filters (or highpass filters). In the case $m=2$, the wavelet filter is uniquely determined (up to a translation) by the scaling filter. In the general m-band case, there is some degree of freedom in the choices of the wavelet filters. And several approaches[16–18] have been presented to construct the $m-1$ wavelet filters from the scaling filter.

In this paper we are interested to find the "optimal" wavelet filters from a given scaling filter. The optimality is measured in the sense of DWT implementation, that is, the fastest DWT implementation. From the wavelet matrix theory, we know that, the DWT of a wavelet matrix, in general, requires $O(m^2)$ operations. Thus a universal DWT implementation can not achieve better than $O(m^2)$. However, since the approximation order, the orthogonality, and the smoothness of a wavelet system are all imposed on the scaling filter only, one can fix the scaling filter and change the $m-1$ wavelet filters to some new $m-1$ wavelet filters, without changing the approximation order, the orthogonality and the smoothness. Then one derives a new wavelet matrix with the scaling filter unchanged. With some careful choice of the new wavelet filters, we show that there is a fast DWT implementation for the new wavelet matrix with $O(m)$ complexity. Thus from a given scaling filter (with all the approximation order, orthogonality, and smoothness conditions imposed on it), one can design a full wavelet matrix with an $O(m)$ DWT implementation. In fact, in most of wavelet applications, the design criteria of a wavelet system is on the scaling filter only, and one has the freedom to choose wavelet filters. So our fast DWT implementation is quite general.

The paper is organized as follows. In Section 2 we give a brief review of orthogonal wavelet matrix factorization and construction. We will focus on Vaidyanathan type factorization[19] and Heller's construction procedure.[17] In

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Section 3 we present the $O(m)$ DWT implementation. We choose a specific type of characteristic Haar matrices (more precisely, canonical Haar matrices) in the wavelet filters designing procedure and derive a fast DWT implementation with $O(m)$ complexity. The $O(m)$ DWT implementation is further illustrated by an example in Section 4. The paper is concluded in Section 5.

2. WAVELET MATRIX FACTORIZATION AND CONSTRUCTION

For a given sequence $a = \{a_k, k \in \mathbf{Z}\}$ which has only a finite number of nonzero elements, the *Laurent polynomial* $a(z)$ of a is defined by

$$a(z) \ := \sum_{k\in\mathbf{Z}} a_k z^{-k} \ = \ \sum_{k=k_1}^{k_2} a_k z^{-k}$$

where k_1 and k_2 are the smallest and largest indices that a_k is nonzero, respectively. The *Laurent degree* of $a(z)$ is defined by

$$\deg(a) \ := \ k_2 - k_1$$

In signal processing, a will be called a *finite impulse response* (FIR) *filter* and $a(z)$ is the *z-transform* of a.

Now consider a matrix $A = (a_{i,j})$ consisting of m rows of vectors of the form

$$A \ = \ \begin{pmatrix} \cdots & a_{0,-1} & a_{0,0} & a_{0,1} & a_{0,2} & \cdots \\ \cdots & a_{1,-1} & a_{1,0} & a_{1,1} & a_{1,2} & \cdots \\ & & \vdots & \vdots & & \\ \cdots & a_{m-1,-1} & a_{m-1,0} & a_{m-1,1} & a_{m-1,2} & \cdots \end{pmatrix}$$

where only a finite number of entries $a_{i,j}$ are nonzero, and $m \in \mathbf{N}, m \geq 2$. Define submatrices A_k of size $m \times m$ of A in the following manner

$$A_k \ = \ (a_{i,km+j}) \, , \quad i = 0, \cdots, m-1, \ j = 0, \cdots, m-1$$

for $k \in \mathbf{Z}$. In other words, A is expressed in terms of block matrices in the form

$$A \ = \ (\cdots, A_{-1}, A_0, A_1, \cdots)$$

where, for instance

$$A_0 \ = \ \begin{pmatrix} a_{0,0} & \cdots & a_{0,m-1} \\ \vdots & & \vdots \\ a_{m-1,0} & \cdots & a_{m-1,m-1} \end{pmatrix}$$

From the matrix A, we construct the formal power series

$$A(z) \ := \sum_{k\in\mathbf{Z}} A_k z^{-k} \ = \ \sum_{k=k_1}^{k_2} A_k z^{-k} \tag{2.1}$$

where k_1 and k_2 are the smallest and largest indices that $A_k \neq 0$, respectively. We call $A(z)$ the *Laurent series* of the matrix A. We can equally well write $A(z)$ as an $m \times m$ matrix

$$A(z) \ = \ \begin{pmatrix} \sum_k a_{0,km} z^{-k} & \cdots & \sum_k a_{0,km+m-1} z^{-k} \\ \vdots & & \vdots \\ \cdots & \sum_k a_{i,km+j} z^{-k} & \cdots \\ \vdots & & \vdots \\ \sum_k a_{m-1,km} z^{-k} & \cdots & \sum_k a_{m-1,km+m-1} z^{-k} \end{pmatrix}$$

which we will refer to as the *polyphase decomposition* of A. For the case of $m = 2$, we find

$$A(z) \ = \ \begin{pmatrix} \cdots a_{0,-2} z + a_{0,0} + a_{0,2} z^{-1} \cdots, & \cdots a_{0,-1} z + a_{0,1} + a_{0,3} z^{-1} \cdots \\ \cdots a_{1,-2} z + a_{1,0} + a_{1,2} z^{-1} \cdots, & \cdots a_{1,-1} z + a_{1,1} + a_{1,3} z^{-1} \cdots \end{pmatrix}$$

and we see that the even and odd coefficients along the rows are blocks in the left and right columns, respectively.

Let

$$g \;:=\; k_2 - k_1 + 1$$

be the number of terms in the summation (2.1) and call g the *genus* of the Laurent series $A(z)$ and the matrix A. We define the *adjoint* $\tilde{A}(z)$ of the Laurent series $A(z)$ by

$$\tilde{A}(z) \;:=\; A^*(z^{-1}) \;:=\; \sum_{k=k_1}^{k_2} A_k^* z^k \;=\; \sum_{k=-k_2}^{-k_1} A_{-k}^* z^{-k}$$

where $A_k^* := \bar{A}_k^t$ is the Hermitian adjoint of the $m \times m$ matrix A_k.

A matrix $A = (a_{i,j})$ is said to be an *orthogonal m-band wavelet matrix*[20] if

$$A(z)\tilde{A}(z) \;=\; mI_m \tag{2.2}$$

and

$$\sum_j a_{i,j} \;=\; \begin{cases} m & \text{if } i = 0 \\ 0 & \text{if } 1 \le i \le m-1 \end{cases} \tag{2.3}$$

where I_m is the $m \times m$ identity matrix. We will call the first row vector of an orthogonal wavelet matrix the *scaling filter* and the other $m-1$ rows of vectors the *wavelet filters*.

Note that in the theory of wavelet analysis, we will systematically employ the additional linear constraint (2.3) in addition to the orthogonality condition (2.2). This is one of the main difference between wavelet systems and perfect reconstruction FIR filter banks.

Comparison of coefficients of corresponding powers of z in (2.2) yields quadratic orthogonality relations for the rows of A

$$\sum_{j \in \mathbf{Z}} \bar{a}_{i_1,k_1 m+j} a_{i_2,k_2 m+j} \;=\; m\delta_{i_1,i_2}\delta_{k_1,k_2} \tag{2.4}$$

where δ is defined by

$$\delta_{i_1,i_2} \;:=\; \begin{cases} 1 & \text{if } i_1 = i_2 \\ 0 & \text{otherwise} \end{cases}$$

We will refer to Equations (2.2) and (2.3) or equivalently (2.4) and (2.3) as the *quadratic* and *linear* conditions defining an orthogonal wavelet matrix, respectively.

The set of orthogonal wavelet matrices with genus equal to 1 plays a special role in the theory of orthogonal wavelets. We shall call them *orthogonal Haar wavelet matrices*. The set of orthogonal m-band Haar wavelet matrices is a homogeneous space which is isomorphic to the Lie group U_{m-1} of unitary $(m-1) \times (m-1)$ matrices, and there is a distinguished orthogonal Haar wavelet matrix which corresponds to the identity element of the group U_{m-1}, which will be called the *canonical Haar matrix*.

LEMMA 2.1. *An $m \times m$ matrix H is an orthogonal Haar wavelet matrix if and only if*

$$H \;=\; \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \mathbf{H}$$

where $U \in U_{m-1}$ is a unitary matrix, that is, $U^*U = I_{m-1}$, and $\mathbf{H}$ is the canonical Haar matrix, *which is defined by*

$$\mathbf{H} \;:=\; \begin{pmatrix} 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ -\sqrt{m-1} & \sqrt{\frac{1}{m-1}} & \cdots & \cdots & \cdots & \cdots & \sqrt{\frac{1}{m-1}} \\ \vdots & \ddots & \ddots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & -\sqrt{\frac{im}{i+1}} & \sqrt{\frac{m}{i^2+i}} & \cdots & \sqrt{\frac{m}{i^2+i}} \\ \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & -\sqrt{\frac{m}{2}} & \sqrt{\frac{m}{2}} \end{pmatrix}$$

where $i = m-1, \cdots, 2, 1$.

The proof of the above lemma can be found in.[20,12] Let A be an orthogonal wavelet matrix and let $A(z)$ be its Laurent series. Define the *characteristic Haar matrix* $\chi(A)$ of the wavelet matrix A by

$$\chi(A) \;:=\; A(1) \;=\; \sum_{k=k_1}^{k_2} A_k$$

It can be easily checked that χ is a well-defined mapping from orthogonal m-band wavelet matrices to orthogonal m-band Haar wavelet matrices. When $m = 2$, we have

$$\chi(A) \;=\; \begin{pmatrix} \cdots + a_{0,-2} + a_{0,0} + a_{0,2} + \cdots, & \cdots + a_{0,-1} + a_{0,1} + \cdots \\ \cdots + a_{1,-2} + a_{1,0} + a_{1,2} + \cdots, & \cdots + a_{1,-1} + a_{1,1} + \cdots \end{pmatrix}$$

A fundamental result of the wavelet matrix theory is that an orthogonal wavelet matrix can be factored into the product of small size "prime factors". The following theorem tells us the basic structure of an orthogonal wavelet matrix and its proof can be found in.[19,20,18]

THEOREM 2.2 (VAIDYANATHAN FACTORIZATION THEOREM). *If A is an orthogonal m-band wavelet matrix of genus g, then there exist unit column vectors $v_1, v_2, \cdots, v_d$ such that*

$$A(z) \;=\; z^{-k_1} V_1(z) V_2(z) \cdots V_d(z) H \tag{2.5}$$

where $d \in \mathbf{N}$, $d \geq g-1$, $H = \chi(A)$ is the characteristic Haar matrix of A, k_1 is the smallest index such that $A_k \neq 0$ in the Laurent series (2.1), and

$$V_i(z) \;:=\; I_m - v_i v_i^* + v_i v_i^* z^{-1}, \quad i = 1, 2, \cdots, d$$

Remark: We will call a matrix V of the form

$$V(z) \;:=\; I_m - vv^* + vv^* z^{-1}$$

a *primitive paraunitary matrix*, here v is a unit column vector, $v^* v = 1$. Any primitive paraunitary matrix has determinant z^{-1}. The proof, can be found, for example, in.[19]

LEMMA 2.3. *If V is a primitive paraunitary matrix, then* $\det(V(z)) = z^{-1}$, *and*

$$V(z)\tilde{V}(z) \;=\; I_m$$

Now we will repeat Heller's procedure to construct a full wavelet matrix from a given scaling filter and a characteristic Haar matrix. A more general construction method is described in.[18] The next theorem is Theorem 4.1 of[17] (see also[12]).

THEOREM 2.4. *Given a scaling filter a_0 of genus g and a characteristic Haar matrix H, there exists an orthogonal wavelet matrix A of genus g whose first row is a_0 and with characteristic Haar matrix H.*

The proof in[17] is constructive. It suffices to obtain vectors v_i such that the relationship (2.5) holds with $d = g-1$. Without loss of generality, we can assume $k_1 = 0$ (otherwise one can always shift a_0 to get $k_1 = 0$). The factorization (2.5) now looks

$$A_0 \;+\; A_1 z^{-1} \;+\; \cdots \;+\; A_{g-1} z^{-g+1} \;=\; \left(\prod_{i=1}^{g-1} \left(I_m - v_i v_i^* + v_i v_i^* z^{-1} \right) \right) H$$

where the first row of each A_k is known. Right-multiplying by H^{-1}, it becomes

$$B_0 \;+\; B_1 z^{-1} \;+\; \cdots \;+\; B_{g-1} z^{-g+1} \;=\; \prod_{i=1}^{g-1} \left(I_m - v_i v_i^* + v_i v_i^* z^{-1} \right) \tag{2.6}$$

Again the first row of each B_k is known. By comparing the coefficient of z^{-g+1} on both side of Equation (2.6), one gets

$$B_{g-1} \;=\; v_1 {v_1}^* v_2 {v_2}^* \cdots v_{g-1} {v_{g-1}}^*$$

The right hand side is a rank-1 matrix, and each of whose rows is proportional to ${v_{g-1}}^*$. Since the first row of B_{g-1} is known, and let it be β, it follows

$$v_{g-1} \;=\; \frac{\beta^*}{||\beta||}$$

Right-multiply Equation (2.6) by

$$I_m - v_{g-1}{v_{g-1}}^* + v_{g-1}{v_{g-1}}^* z \,,$$

the inverse of the newly determined prime factor, to obtain

$$C_0 \,+\, C_1 z^{-1} \,+\, \cdots \,+\, C_{g-2} z^{-g+2} \;=\; \prod_{i=1}^{g-2} \left(I_m - v_i {v_i}^* + v_i {v_i}^* z^{-1} \right)$$

Again the first row of each C_k is known, and we can repeat the pattern to get $v_{g-2}, v_{g-3}, \cdots, v_1$. It has been shown in[17] that the resulting matrix

$$\left(\prod_{i=1}^{g-1} (I_m - v_i {v_i}^* + v_i {v_i}^* z^{-1}) \right) H$$

is an orthogonal wavelet matrix of genus g whose first row is a_0 and with characteristic Haar matrix H.

3. FAST WAVELET TRANSFORM

The discrete wavelet transform is a filtering by the scaling filter and wavelet filters, followed by a downsampling. For a given data/signal $x = \{x_l, l \in \mathbf{Z}\}$ which has only a finite number of nonzero elements, consider the polyphase decomposition of x

$$x(z) \;:=\; \left(\sum_l x_{lm} z^{-l}, \sum_l x_{lm+1} z^{-l}, \cdots, \sum_l x_{lm+m-1} z^{-l} \right)^* \,,$$

with an orthogonal m-band wavelet matrix A of genus g, the DWT of x is the multiplication of $A(z^{-1})$ and $x(z)$, that is,

$$A(z^{-1})\, x(z) \;=\; \begin{pmatrix} \sum_k a_{0,km} z^k & \cdots & \sum_k a_{0,km+m-1} z^k \\ \vdots & & \vdots \\ \cdots & \sum_k a_{i,km+j} z^k & \cdots \\ \vdots & & \vdots \\ \sum_k a_{m-1,km} z^k & \cdots & \sum_k a_{m-1,km+m-1} z^k \end{pmatrix} \begin{pmatrix} \sum_l x_{lm} z^{-l} \\ \vdots \\ \sum_l x_{lm+i} z^{-l} \\ \vdots \\ \sum_l x_{lm+m-1} z^{-l} \end{pmatrix}$$

A direct DWT implementation based on the matrix multiplication will have $m^2 g$ multiplications and $(m^2 - m)g$ additions. Note that we do not consider the length of the data/signal x in the DWT complexity for comparison purpose. One can equally add the length as an additional parameter and carry out the same analysis.

Replacing z with z^{-1} in Theorem 2.2, one can get a factorization of the matrix $A(z^{-1})$ of the same type,

$$A(z^{-1}) \;=\; z^{k_1} V_1(z^{-1}) V_2(z^{-1}) \cdots V_d(z^{-1}) H$$

From Lemma 2.1, the set of characteristic Haar matrices H is isomorphic to the unitary matrices. Since the space of unitary matrices U_{m-1} has dimension $(m-1)(m-2)/2$, the space of orthogonal m-band wavelet matrices will be at least $(m-1)(m-2)/2$ dimensional (actually much larger). Thus one derives

LEMMA 3.1. *A universal DWT implementation can not achieve better than $O(m^2)$.*

An interesting observation on the DWT complexity is that it can be reduced to the one of its characteristic Haar matrix.

LEMMA 3.2. *The DWT complexity of an orthogonal m-band wavelet matrix A constructed in* Theorem 2.4 *is the DWT complexity of the given characteristic Haar matrix H plus* $O(m)$.

Proof: From Theorem 2.4, a_0 is a given scaling filter of genus g, H is a given characteristic Haar matrix, and A is the orthogonal wavelet matrix of genus g with first row vector a_0 and characteristic Haar matrix H. Also from Theorem 2.4, $A(z)$ is constructed by

$$A(z) \;=\; \left(\prod_{i=1}^{g-1}(I_m - v_i v_i^* + v_i v_i^* z^{-1})\right) H \tag{3.1}$$

The multiplication by a primitive paraunitary matrix

$$I_m - v_i v_i^* + v_i v_i^* z^{-1}$$

can be implemented with $O(m)$ complexity because of its special structure as the product of vectors. Since there are exactly $g-1$ prime factors in the factorization, totally they will contribute to $O(m(g-1))$ operations. By Equation (3.1), the DWT complexity of A is equal to the sum of the DWT complexity of H and $O(m(g-1))$. The lemma follows. □

Remark: The $O(m)$ complexity of the multiplication by a primitive paraunitary matrix can be also illustrated by the Gaussian elimination method.

Set the unit vector $v_i = (w_1, w_2, \cdots, w_m)^*$, then

$$V_i(z) = \begin{pmatrix} 1+\bar{w}_1 w_1(z^{-1}-1) & \bar{w}_1 w_2(z^{-1}-1) & \cdots & \bar{w}_1 w_m(z^{-1}-1) \\ \bar{w}_2 w_1(z^{-1}-1) & 1+\bar{w}_2 w_2(z^{-1}-1) & \cdots & \bar{w}_2 w_m(z^{-1}-1) \\ \vdots & \vdots & \vdots & \vdots \\ \bar{w}_m w_1(z^{-1}-1) & \bar{w}_m w_2(z^{-1}-1) & \cdots & 1+\bar{w}_m w_m(z^{-1}-1) \end{pmatrix}$$

Now using the Gaussian elimination method, we get

$$V_i(z) \;=\; X(z)X_1X_2\cdots X_{m-1} \tag{3.2}$$

where $X_j = (x_{k,l})_{m\times m}$, $j = 1, 2, \cdots, m-1$, is defined by

$$X_j \;:=\; \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & w_j/w_m & 0 & \cdots & 0 & 1 \end{pmatrix}$$

that is

$$x_{k,l} \;=\; \begin{cases} 1 & \text{if } k = l \\ w_j/w_m & \text{if } k = m \text{ and } l = j \\ 0 & \text{otherwise} \end{cases}$$

and $X(z)$ is of the form

$$X(z) \;=\; \begin{pmatrix} 1 & 0 & \cdots & 0 & \bar{w}_1 w_m(z^{-1}-1) \\ 0 & 1 & \cdots & 0 & \bar{w}_2 w_m(z^{-1}-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \bar{w}_{m-1} w_m(z^{-1}-1) \\ -w_1/w_m & -w_2/w_m & \cdots & -w_{m-1}/w_m & 1+\bar{w}_m w_m(z^{-1}-1) \end{pmatrix}$$

Note that if $w_m = 0$, then one can move to w_{m-1} and apply the same procedure.

Continuing the process and using the identity

$$\bar{w}_1 w_1 + \bar{w}_2 w_2 + \cdots + \bar{w}_m w_m \;=\; 1$$

we have

$$X(z) \;=\; Y_1 Y_2 \cdots Y_{m-1} Y(z) \tag{3.3}$$

where Y_j, $j = 1, 2, \cdots, m-1$, is defined by

$$Y_j \;:=\; \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -w_j/w_m & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and

$$Y(z) \;=\; \begin{pmatrix} 1 & 0 & \cdots & 0 & \bar{w}_1 w_m (z^{-1} - 1) \\ 0 & 1 & \cdots & 0 & \bar{w}_2 w_m (z^{-1} - 1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \bar{w}_{m-1} w_m (z^{-1} - 1) \\ 0 & 0 & \cdots & 0 & z^{-1} \end{pmatrix}$$

Finally

$$Y(z) \;=\; P_1 P_2 \cdots P_{m-1} P(z) \tag{3.4}$$

where P_j, $j = 1, 2, \cdots, m-1$, is defined by

$$P_j \;:=\; \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \bar{w}_j w_m \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and

$$P(z) \;=\; \begin{pmatrix} 1 & 0 & \cdots & 0 & -\bar{w}_1 w_m \\ 0 & 1 & \cdots & 0 & -\bar{w}_2 w_m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\bar{w}_{m-1} w_m \\ 0 & 0 & \cdots & 0 & z^{-1} \end{pmatrix}$$

Combining Equations (3.2), (3.3), and (3.4), we obtain

$$V_i(z) \;=\; Y_1 \cdots Y_{m-1} P_1 \cdots P_{m-1} P(z) X_1 \cdots X_{m-1}\,. \tag{3.5}$$

It follows then the multiplication by $V_i(z)$ can be implemented with $4(m-1)$ multiplications and $4(m-1)$ additions. Note that $Y_i, P_i, X_i, i = 1, \cdots, m-1$ are all constant matrices with at most one nonzero entry off the diagonal, and $P(z)$ is an upper triangular matrix whose nonzero entries (except the diagonal) have Laurent degree zero and located on the last column. Conversely, given a unit vector $(w_1, w_2, \cdots, w_m)^*$, one can design a primitive paraunitary matrix from $Y_i, P_i, X_i, i = 1, \cdots, m-1$ and $P(z)$. Thus the above analysis also give an approach to construct orthogonal wavelet matrices. Similarly, the characteristic Haar matrix H can also be factored as the product of upper and lower triangular matrices. Thus for any orthogonal m-band wavelet matrix, it can always be

written as the product of upper and lower triangular matrices. Recall that in[21] Daubechies and Sweldens proved that any biorthogonal 2-band DWT (which includes orthogonal 2-band DWT) can be obtained with a finite number of lifting steps from the Lazy wavelet, that is, any biorthogonal 2-band wavelet matrix is the product of upper and lower triangular matrices. The above argument generalizes Daubechies and Sweldens' result in the sense that it provides a factorization into upper and lower triangular matrices for any orthogonal m-band filter banks and scale factor m wavelets. On the other hand, the factorization in[21] is valid for biorthogonal DWT. Thus there is a trade-off between these two approaches.

Now given a scaling filter a_0 and a characteristic Haar matrix H, the orthogonal wavelet matrix A constructed from Theorem 2.4 will have the same DWT complexity as of H, since the DWT complexity of H will be at least $O(m)$. (Notice that the first row of a characteristic Haar matrix will always be $(1, 1, \cdots, 1)$ from Lemma 2.1) If the multiplication by the characteristic Haar matrix H requires $O(m^2)$ operations, then the orthogonal wavelet matrix A will have an $O(m^2)$ DWT complexity. If the multiplication by H can be implemented in $O(m \log m)$ (for example, choose the discrete cosine transform matrix), then the DWT of A can also be implemented in $O(m \log m)$. Further, if the multiplication of the characteristic Haar matrix has an $O(m)$ complexity, then we will derive an $O(m)$ DWT implementation of an orthogonal m-band wavelet matrix. The next lemma tells us that the DWT complexity of a characteristic Haar matrix is fully determined by its unitary matrix.

LEMMA 3.3. *For a characteristic Haar matrix H, the DWT complexity is equal to the sum of $O(m)$ and the complexity of a multiplication by a matrix U, where U is the unitary matrix in* Lemma 2.1.

Proof: From Lemma 2.1, it suffices to show that the multiplication by the canonical Haar matrix $\mathbf{H}$ has an $O(m)$ complexity. By definition, $\mathbf{H}$ is equal to the product of two matrices

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\frac{1}{m-1}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & \sqrt{\frac{m}{i^2+i}} & \cdots & 0 \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \sqrt{\frac{m}{2}} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ 1-m & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ \vdots & \ddots & \ddots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & -j & 1 & \cdots & 1 \\ \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 1 \end{pmatrix}$$

The first matrix is diagonal, and a multiplication by it will be $m-1$ multiplications and no additions. For the second matrix, let's denote it by G, and assume

$$G \cdot (x_0, x_1, \cdots, x_{m-1})^* \;=\; (y_0, y_1, \cdots, y_{m-1})^*$$

Since

$$\begin{aligned} y_0 &= x_0 + x_1 + \cdots + x_{m-1} \\ y_1 &= y_0 - m \cdot x_0 \\ y_2 &= y_1 - (m-1) \cdot (x_1 - x_0) \\ \vdots &\quad \vdots \\ y_{m-1} &= y_{m-2} - 2 \cdot (x_{m-2} - x_{m-3}) \end{aligned}$$

So the multiplication of G can be implemented within $m-1$ multiplications and $3m-4$ additions. Totally the multiplication by $\mathbf{H}$ has $2m-2$ multiplications and $3m-4$ additions. The lemma is proved. □

Combining Theorem 2.4, Lemma 3.2, and 3.3, we obtain

THEOREM 3.4. *Given a scaling filter a_0 of genus g, there exists an orthogonal wavelet matrix A of genus g whose first row is a_0, and the DWT complexity of A is $O(m)$.*

Proof: By choosing the canonical Haar matrix $\mathbf{H}$ as the characteristic Haar matrix, we can construct an orthogonal wavelet matrix A of genus g whose first row is a_0 by Theorem 2.4. Now for the canonical Haar matrix $\mathbf{H}$, the unitary matrix $U = I_{m-1}$. Applying Lemma 3.2 and 3.3, it follows that the DWT complexity of A is $O(m)$. □

When designing an orthogonal wavelet system, the approximation order, the orthogonality, and the smoothness are three of the most frequent criteria. The orthogonality condition (2.2) is already in the definition of the orthogonal wavelet matrix. (From an orthogonal wavelet matrix, one can derive a tight frame of $L^2(\mathbf{R})$. For orthonormal bases, Cohen/Lawton conditions[22,23] are required. For more details, see[20,12] etc.) The approximation order and smoothness can be translated into the vanishing moments of the scaling filter. (For the smoothness, the vanishing moments are necessary conditions.) Once the scaling filter is fixed, the approximation order and the smoothness of the wavelet system is fixed. From a scaling filter, there is some degree of freedom in the choices of the wavelet filters. (In the case $m = 2$, the wavelet filter is uniquely determined up to a translation.) All these different wavelet filters will give the same approximation order and smoothness. For the fast wavelet transform, we will construct $m - 1$ wavelet filters to have an $O(m)$ DWT implementation, based on Theorem 3.4.

In the design problem for orthogonal m-band wavelet matrices, oftenly the scaling filter is constructed first, and the only requirement on the wavelet filters is the quadratic condition (2.4) and the linear condition (2.3). In these situations, one can simply choose the canonical Haar matrix (or some other simple structured orthogonal Haar wavelet matrix whose unitary matrix has an $O(m)$ multiplication complexity) as the characteristic Haar matrix and construct additional $m-1$ wavelet filters (see Theorem 2.4 and 3.4). Then the full wavelet matrix will have an $O(m)$ DWT. In the case that an orthogonal m-band wavelet matrix is already given, because changing the wavelet filters doesn't change the approximation order and the smoothness, one can fix the scaling filter and modify the wavelet filters to achieve $O(m)$ DWT. One can ignore the wavelet filters and construct a new wavelet matrix from the given scaling filter by Theorem 2.4 and 3.4. Or one can employ the eigenfilter approach[19] to find the unit column vectors $v_1, v_2, \cdots, v_d$ in the factorization (2.5) and force the characteristic Haar matrix to be the canonical Haar matrix (or some other orthogonal Haar wavelet matrix with $O(m)$ DWT). There is a slight difference between these two approaches. For the eigenfilter approach, the number of prime factors, d, is greater than or equal to $g - 1$. When $d > g - 1$ (which actually seldom happens), the multiplication complexity of the primitive paraunitary matrices may be larger than $O(m)$ (more precisely, larger than $O(mg)$), and the DWT complexity of the constructed wavelet matrix will be larger than the DWT complexity of its characteristic Haar matrix. So with a given full wavelet matrix, the construction method from its scaling filter, as illustrated in Theorem 2.4 and 3.4, would seem preferable.

4. EXAMPLES

In this section we will work on a specific example on how to construct a full orthogonal wavelet matrix from a given scaling filter to have an $O(m)$ DWT. The fast wavelet transform implementation will also be illustrated.

This example is from,[16] a 3-band scaling filter with approximation order two,

$$a_0 = \begin{pmatrix} a_{0,0} \\ a_{0,1} \\ a_{0,2} \\ a_{0,3} \\ a_{0,4} \\ a_{0,5} \end{pmatrix}^* = \begin{pmatrix} 0.58610191307059 \\ 0.91943524640393 \\ 1.25276857973726 \\ 0.41389808692940 \\ 0.08056475359608 \\ -0.25276857973727 \end{pmatrix}^*$$

Recall that an m-band scaling filter is said to have approximation order K if it has a polynomial factor of the form $(P(z))^K$, with $P(z) = (1 + z^{-1} + \cdots + z^{-(m-1)})/m$ for maximal possible K. Also note that we use a different normalization (2.3). In,[16] they have

$$\sum_j a_{0,j} = \sqrt{m}$$

Now a_0 is a scaling filter of genus 2, and the canonical Haar matrix $\mathbf{H}$ is

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 \\ -\sqrt{2} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ 0 & -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} \end{pmatrix}$$

Using the notation in the proof of Theorem 2.4, the first row of A_1 is

$$(0.41389808692940, \quad 0.08056475359608, \quad -0.25276857973727)$$

Right-multiplying by $\mathbf{H}^{-1}$ (which is equal to $\mathbf{H}^*/m$), we get the first row of B_1, which is

$$\begin{aligned}\beta &= (0.41389808692940, \quad 0.08056475359608, \quad -0.25276857973727) \cdot \frac{1}{3}\begin{pmatrix} 1 & -\sqrt{2} & 0 \\ 1 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{3}{2}} \\ 1 & \sqrt{\frac{1}{2}} & \sqrt{\frac{3}{2}} \end{pmatrix} \\ &= (0.08056475359607, \quad -0.23570226039551, \quad -0.13608276348796)\end{aligned}$$

Now normalize β to obtain

$$\begin{aligned}v_1 &= \frac{\beta^*}{||\beta||} \\ &= (0.28383930946236, \quad -0.83040739086483, \quad -0.47943593065288)^*\end{aligned}$$

Since $g - 1 = 1$, from Theorem 2.4 and 3.4, the full wavelet matrix is equal to

$$\begin{aligned}A(z) &= (I_3 - v_1 v_1^* + v_1 v_1^* z^{-1})\mathbf{H} \\ &= \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.08056475359607 & -0.23570226039551 & -0.13608276348796 \\ -0.23570226039551 & 0.68957643480293 & 0.39812714026032 \\ -0.13608276348796 & 0.39812714026032 & 0.22985881160100 \end{pmatrix} + \right. \\ &\quad \left.\begin{pmatrix} 0.08056475359607 & -0.23570226039551 & -0.13608276348796 \\ -0.23570226039551 & 0.68957643480293 & 0.39812714026032 \\ -0.13608276348796 & 0.39812714026032 & 0.22985881160100 \end{pmatrix} z^{-1}\right) \cdot \begin{pmatrix} 1 & 1 & 1 \\ -\sqrt{2} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ 0 & -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} \end{pmatrix} \\ &= \begin{pmatrix} 0.58610191307059 & 0.91943524640393 & 1.25276857973726 \\ -0.20330295558638 & 0.94280904158208 & -0.03239930480915 \\ 0.69911956479291 & -1.08866210790362 & 0.79779083357459 \end{pmatrix} + \\ &\quad \begin{pmatrix} 0.41389808692940 & 0.08056475359608 & -0.25276857973727 \\ -1.21091060678671 & -0.23570226039553 & 0.73950608599570 \\ -0.69911956479291 & -0.13608276348797 & 0.42695403781700 \end{pmatrix} z^{-1}\end{aligned}$$

For the fast DWT implementation, we apply the decomposition (3.5) to the primitive paraunitary matrix for

$$v_1 = (0.28383930946236, \quad -0.83040739086483, \quad -0.47943593065288)^*$$

to get

$$\begin{aligned}&I_3 - v_1 v_1^* + v_1 v_1^* z^{-1} = \\ &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.59202761269027 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1.73205080756882 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -0.13608276348796 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \\ &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.39812714026031 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0.13608276348796 \\ 0 & 1 & -0.39812714026031 \\ 0 & 0 & z^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.59202761269027 & 0 & 1 \end{pmatrix} \cdot \\ &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1.73205080756882 & 1 \end{pmatrix}\end{aligned}$$

And for $\mathbf{H}$, it can be factored as

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1}2 & 0 \\ 0 & 0 & \sqrt{3}2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

Combining the factorization of the primitive paraunitary matrix and the canonical Haar matrix, we derive a fast implementation of the DWT

$$
\begin{aligned}
A(z) = &\\
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.59202761269027 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1.73205080756882 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & -0.13608276348796 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \\
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.39812714026031 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0.13608276348796 \\ 0 & 1 & -0.39812714026031 \\ 0 & 0 & z^{-1} \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.59202761269027 & 0 & 1 \end{pmatrix} \cdot \\
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1.73205080756882 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{12} & 0 \\ 0 & 0 & \sqrt{32} \end{pmatrix}
\begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}
\end{aligned}
$$

5. CONCLUSIONS

In this paper we provide an $O(m)$ implementation of DWT. From a given scaling filter, we construct $m-1$ wavelet filters such that the full orthogonal wavelet matrix will have a fast DWT. Starting from a given orthogonal wavelet matrix, we fix the scaling filter and modify the $m-1$ wavelet filters to obtain a new orthogonal wavelet matrix. The new one will have the same approximation order and smoothness as the old one, while the DWT complexity of the new one is just $O(m)$. Thus without changing the scaling filter, one can always achieve an $O(m)$ DWT. Since in most wavelet applications, the design criteria is on the scaling filter only, and one has the freedom to choose wavelet filters, our fast DWT implementation is quite general and will be quite useful in large size data processing.

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