An Algebraic Structure of Orthogonal Wavelet Space*
(revised version)

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Abstract

In this paper we study the algebraic structure of the space of compactly supported orthonormal wavelets over real numbers. Based on the parametrization of wavelet space, one can define a parameter mapping from the wavelet space of rank 2 (or 2-band, scale factor of 2) and genus g to the $(g-1)$ dimensional real torus (the products of unit circles). By the uniqueness and exactness of factorization, this mapping is well-defined and one-to-one. Thus we can equip the rank 2 orthogonal wavelet space with an algebraic structure of the torus. Because of the degenerate phenomenon of the paraunitary matrix, the parametrization map is not onto. However, there exists an onto mapping from the torus to the "closure" of the wavelet space. And with such mapping, a more complete parametrization is obtained. By utilizing the factorization theory, we present a fast implementation of discrete wavelet transform (DWT). In general, an orthogonal rank m DWT has an $O(m^2)$ complexity. In this paper we starts with a given scaling filter and construct an additional $(m-1)$ wavelet filters so that the DWT can be implemented in $O(m)$. With a fixed scaling filter, the approximation order, the orthogonality, and the smoothness remain unchanged, thus our fast DWT implementation is quite general.

List of Symbols

S^1	the unit circle in $\mathbf{R}^2$
S^{m-1}	the unit sphere in $\mathbf{R}^m$
A	a matrix with m rows
$A(z)$	the Laurent series or polyphase decomposition of A
m	the rank of a wavelet matrix, or the scale factor of a wavelet system
g	the genus of a wavelet matrix, which implies the matrix has size $m \times mg$, or the $m-$ band filter bank has length mg
$\tilde{A}(z)$	the adjoint of $A(z)$
A^*	the Hermitian adjoint of A
$\bar{A}$	the complex conjugate of A
A^t	the transpose of A
I_m	the $m \times m$ identity matrix
v	a unit column vector
$V(z)$	primitive paraunitary matrix
d	the number of factor in paraunitary factorization
k_1	the smallest index that $A_k \neq 0$
$\delta_{i,j}$	the delta function
$\hat{\phi}(\xi)$	the Fourier transform of $\phi(x)$
U	a unitary matrix
$\mathbf{H}$	the canonical Haar matrix
$\chi(A)$	the characteristic Haar matrix of A
$\det(A)$	the determinant of A
$\diamond_H$	the Pollen product
$WM(2,g)$	the collection of all real orthogonal wavelet matrices of rank 2 and genus g
$WM(2)$	the collection of all real orthogonal wavelet matrices of rank 2
T^{g-1}	$(g-1)$-dimensional real torus
$WM(m,g)$	the collection of all real orthogonal wavelet matrices of rank m and genus g
$WM(m)$	the collection of all real orthogonal wavelet matrices of rank m
$\mathcal{T}^{g-1}$	$(g-1)$-dimensional real torus

1 Introduction

Wavelet analysis (see [2, 3, 9, 22, 24, 28, 34, 35], etc) has been proven to be a very powerful tool in harmonic analysis, neural networks, numerical analysis, and signal processing, especially in the area of image compression [25, 29, 36] and data denoising [7, 11, 18]. The theoretical work of orthogonal wavelets was done in the late eighties [8, 21, 22] and the framework of biorthogonal wavelets was established in the early nineties [4, 6, 33]. Wavelet theory is closely connected with subband coding, and it provides a functional space structure for subband coding, often leading to better understanding and tremendous improvement.

The invention of the polyphase decomposition [1] is one of the reasons why multirate filter banks processing [30] became practically attractive. It enables the designer to perform all the computations at the "lowest rate permissible within the given context", and reduces the speed requirements on the processors. It is valuable not only in practical design and actual implementation of filter banks, but also in theoretical study. Actually with the polyphase decomposition, P. P. Vaidyanathan and his colleagues [31] derive factorizations of paraunitary matrices and apply such factorizations to design quadrature mirror filter (QMF) banks for digital signal processing problems. D. Pollen, P. N. Heller, H. L. Resnikoff, and R. O. Wells, Jr. [15, 23, 24] use the polyphase decomposition to develop a parametrization theory of compactly supported orthonormal wavelets.

In this paper, we will discuss the algebraic structure of the space of compactly supported orthonormal wavelets over real numbers. The factorization work mentioned above suggests a mapping from the orthogonal wavelet space to the space of products of unit spheres. But without the uniqueness of the factorization, the mapping cannot be well-defined. The uniqueness result for rank $m = 2$ (or 2-band, scale factor of 2) was first proved by Pollen [23] and one can define a mapping from the orthogonal wavelet space of rank 2 to products of unit circles. However, this mapping doesn't provide a quantitative relationship between the size of a wavelet system (or more precisely, the genus of a wavelet matrix) and the number of unit circles in the product. Here we present a new result of the wavelet factorization, the *exactness* of factorization, which says that the number of unit circles in the product will be exactly the genus minus 1. Then with the uniqueness and exactness of wavelet factorization, one can define a one-to-one mapping from the orthogonal wavelet space of rank 2 and genus g (which means the scaling filter has length $2g$) to the $(g-1)$ dimensional real torus

$$\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{g-1}$$

Because of the degenerate phenomenon of the paraunitary matrix, this mapping is not onto. Considering even genus and odd genus orthogonal wavelets as two different classes and taking the union in each class up to some genus, that is, adding wavelets with smaller genus but with the same parity to build up a larger space (which is called the "closure" in this paper), then there exists an onto mapping from the torus to the "closure" with which an inclusion diagram commutes. When $m > 2$, there is no well-defined mapping from the orthogonal rank m wavelet space to the products of unit spheres. However, one can still set up a parameter mapping from the real torus (products of unit spheres) to the orthogonal wavelet space.

This paper is organized as follows. In Section 2 we give a brief review of previous work on paraunitary matrices and orthonormal wavelets. The exactness and uniqueness of factorization of rank $m = 2$ is discussed in Section 3. In particular we present a new proof of Pollen's uniqueness result based on our exactness result. This leads to a particular algebraic topological structure for the space of compactly supported orthonormal wavelets over real numbers. More specifically, there exists a well-defined one-to-one mapping from the orthogonal wavelet space of rank 2 and genus g to the $(g-1)$ dimensional real torus. In Section 4 we study a fast wavelet transform based on factorization. An orthogonal rank m discrete wavelet transform has, in general, an $O(m^2)$ complexity. Here we starts with a fixed scaling filter and construct a full wavelet matrix of rank m with an $O(m)$ discrete wavelet transform. Since we don't change the scaling filter, the approximation order and the smoothness of the wavelet system remain unchanged, while the complexity of the discrete wavelet system has been improved from $O(m^2)$ to $O(m)$. We conclude the paper in Section 5.

2 A Review of Previous Work

Most of the material in this section is from [1, 15, 17, 23, 24, 26, 30, 31, 32]. We refer to these papers and books for more details.

2.1 Laurent Polynomial and Polyphase Decomposition

For a given sequence $a = \{a_k, k \in \mathbf{Z}\}$ which has only finite nonzero elements, the *Laurent polynomial* $a(z)$ of a is defined by

$$a(z) := \sum_{k \in \mathbf{Z}} a_k z^{-k} = \sum_{k=k_1}^{k_2} a_k z^{-k}$$

where k_1 and k_2 are the smallest and largest indices that a_k is nonzero, respectively. In signal processing, a will be called a *finite impulse response* (FIR) *filter* and $a(z)$ is the *z-transform* of a.

Now consider a matrix $A = (a_{i,j})$ consisting of m rows of vectors of the form

$$A = \begin{pmatrix} \cdots & a_{0,-1} & a_{0,0} & a_{0,1} & a_{0,2} & \cdots \\ \cdots & a_{1,-1} & a_{1,0} & a_{1,1} & a_{1,2} & \cdots \\ & & \vdots & \vdots & & \\ \cdots & a_{m-1,-1} & a_{m-1,0} & a_{m-1,1} & a_{m-1,2} & \cdots \end{pmatrix}$$

where only a finite number of entries $a_{i,j}$ are nonzero, and $m \in \mathbf{N}, m \geq 2$. Define submatrices A_k of size $m \times m$ of A in the following manner

$$A_k = (a_{i,km+j}), \quad i = 0, \cdots, m-1, \ j = 0, \cdots, m-1$$

for $k \in \mathbf{Z}$. In other words, A is expressed in terms of block matrices in the form

$$A = (\cdots, A_{-1}, A_0, A_1, \cdots)$$

where, for instance

$$A_0 = \begin{pmatrix} a_{0,0} & \cdots & a_{0,m-1} \\ \vdots & & \vdots \\ a_{m-1,0} & \cdots & a_{m-1,m-1} \end{pmatrix}$$

From the matrix A, we construct the formal power series

$$A(z) := \sum_{k \in \mathbf{Z}} A_k z^{-k} = \sum_{k=k_1}^{k_2} A_k z^{-k} \tag{2.1}$$

where k_1 and k_2 are the smallest and largest indices that $A_k \neq 0$, respectively. We call $A(z)$ the *Laurent series* of the matrix A. We can equally well write $A(z)$ as an $m \times m$ matrix

$$A(z) \;=\; \begin{pmatrix} \sum_k a_{0,km} z^{-k} & \cdots & \sum_k a_{0,km+m-1} z^{-k} \\ \vdots & & \vdots \\ \cdots & \sum_k a_{i,km+j} z^{-k} & \cdots \\ \vdots & & \vdots \\ \sum_k a_{m-1,km} z^{-k} & \cdots & \sum_k a_{m-1,km+m-1} z^{-k} \end{pmatrix}$$

which we will refer to as the *polyphase decomposition* of A. For the case of $m = 2$, we find

$$A(z) \;=\; \begin{pmatrix} \cdots a_{0,-2} z + a_{0,0} + a_{0,2} z^{-1} \cdots, & \cdots a_{0,-1} z + a_{0,1} + a_{0,3} z^{-1} \cdots \\ \cdots a_{1,-2} z + a_{1,0} + a_{1,2} z^{-1} \cdots, & \cdots a_{1,-1} z + a_{1,1} + a_{1,3} z^{-1} \cdots \end{pmatrix}$$

and we see that the even and odd coefficients along the rows are blocks in the left and right columns, respectively.

Let

$$g \;:=\; k_2 - k_1 + 1$$

be the number of terms in the summation (2.1) and call g the *genus* of the Laurent series $A(z)$ and the matrix A.

Finally we define the *adjoint* $\tilde{A}(z)$ of the Laurent series $A(z)$ by

$$\tilde{A}(z) \;:=\; A^*(z^{-1}) \;:=\; \sum_{k=k_1}^{k_2} A_k^* z^k \;=\; \sum_{k=-k_2}^{-k_1} A_{-k}^* z^{-k}$$

where $A_k^* := \bar{A}_k^t$ is the Hermitian adjoint of the $m \times m$ matrix A_k. When A_k is a real matrix, $A_k^* = A_k^t$.

2.2 Paraunitary Matrices

For an $m \times mg$ matrix A, we say that A is *paraunitary* if its Laurent series $A(z)$ satisfies

$$A(z)\tilde{A}(z) \;=\; cI_m \tag{2.2}$$

where $c \neq 0$ is a constant and I_m is the $m \times m$ identity matrix. This generalizes the classical notion of unitary matrix (the case where $g = 1$ and

$c = 1$). In signal processing, one can design perfect reconstruction (PR) FIR filter banks with particular properties from a paraunitary matrix A.

Let v be a unit column vector, that is, $v^*v = 1$. Define the Laurent matrix

$$V(z) \; := \; I_m - vv^* + vv^*z^{-1} \tag{2.3}$$

and let V be the corresponding $m \times 2m$ matrix. It's easy to verify that V is a paraunitary matrix. We will say that a paraunitary matrix V of the form (2.3) is *primitive*.

Note that a primitive paraunitary matrix has the constant $c = 1$ in (2.2). And any primitive paraunitary matrix has determinant z^{-1}. The proof, can be found, for example, in [31].

Lemma 2.1 *If V is a primitive paraunitary matrix, then* $\det(V(z)) = z^{-1}$.

A fundamental result is that an arbitrary paraunitary matrix can be factored into products of primitive paraunitary matrices. We define the *paraunitary product* of two paraunitary matrices A and B of the same size $m \times mg$ to be given by

$$AB \; = \; C$$

if

$$A(z)B(z) \; = \; C(z) \tag{2.4}$$

where $A(z)$, $B(z)$, and $C(z)$ are the Laurent series of A, B, and C, and the product in (2.4) is simply the matrix product.

Theorem 2.1 (Paraunitary Factorization Theorem) *Let A be an $m \times mg$ paraunitary matrix, then there must exist primitive paraunitary matrices $V_1, \cdots, V_d$ and a square $m \times m$ paraunitary matrix U such that*

$$A \; = \; V_1 V_2 \cdots V_d U$$

where $d \in \mathbf{N}, d \geq g - 1$, and $U = z^{-k_1}A(1)$, where $A(z)$ is the Laurent series of A, and k_1 is the smallest index that $A_k \neq 0$ in (2.1).

See [31] for a complete proof.

2.3 Orthogonal Wavelet Matrices

An $m \times mg$ matrix $A = (a_{i,j})$ is said to be an *orthogonal wavelet matrix* of *rank* m if

$$A(z)\tilde{A}(z) \;=\; mI_m \tag{2.5}$$

and

$$\sum_j a_{i,j} \;=\; \begin{cases} m & \text{if } i = 0 \\ 0 & \text{if } 1 \leq i \leq m-1 \end{cases} \tag{2.6}$$

It is easy to verify that an orthogonal wavelet matrix with m rows has rank m in the classical sense.

Note that in the theory of wavelet analysis, we will systematically employ the additional linear constraint (2.6) in addition to the paraunitary condition (2.5). This is one of the main difference between wavelet systems and PR FIR filter banks.

Comparison of coefficients of corresponding powers of z in (2.5) yields quadratic orthogonality relations for the rows of A

$$\sum_{j \in \mathbf{Z}} \bar{a}_{i_1,k_1 m+j} a_{i_2,k_2 m+j} \;=\; m\delta_{i_1,i_2}\delta_{k_1,k_2} \tag{2.7}$$

where δ is defined by

$$\delta_{i_1,i_2} \;:=\; \begin{cases} 1 & \text{if } i_1 = i_2 \\ 0 & \text{otherwise} \end{cases}$$

We will refer to (2.5) and (2.6) or equivalently (2.7) and (2.6) as the *quadratic* and *linear* conditions defining an orthogonal wavelet matrix, respectively.

From the theory of orthogonal wavelets [8, 9, 24, 27], one can define a *scaling function* $\phi(x)$ and $m-1$ *wavelet functions* $\psi_1(x), \cdots, \psi_{m-1}(x)$, all are compactly supported L^2 functions, by

$$\hat{\phi}(\xi) \;=\; (2\pi)^{-1/2} \prod_{j=1}^{\infty} M_0(m^{-j}\xi)$$

where $\hat{\phi}(\xi)$ is the Fourier transform of $\phi(x)$

$$\hat{\phi}(\xi) \;:=\; \frac{1}{\sqrt{2\pi}} \int \phi(x) e^{-i\xi x}\, dx$$

and $M_0(\xi)$ is the discrete Fourier transform of $\{a_{0,k}, k \in \mathbf{Z}\}$

$$M_0(\xi) \ := \ \frac{1}{m} \sum_k a_{0,k} e^{-ik\xi}$$

and

$$\psi_i(x) \ := \ \sum_k a_{i,k} \phi(mx-k)\,, \quad i = 1, \cdots, m-1$$

For $j, k \in \mathbf{Z}$ we define the *rescaled* and *translated* scaling and wavelet functions by

$$\phi_{j,k}(x) \ := \ m^{j/2} \phi(m^j x - k)$$
$$(\psi_i)_{j,k}(x) \ := \ m^{j/2} \psi_i(m^j x - k)\,, \quad i = 1, \cdots, m-1$$

With the quadratic condition (2.5) and linear condition (2.6), the wavelet functions $\psi_1(x), \cdots, \psi_{m-1}(x)$ will generate a tight frame [19].

Theorem 2.2 (Lawton) *Let A be an orthogonal wavelet matrix of rank m and genus g, and let $\psi_1(x), \cdots, \psi_{m-1}(x)$ and their rescaled and translated version be defined as above. Then, for all $f \in L^2(\mathbf{R})$, there exists an L^2 convergent expansion*

$$f(x) \ = \ \sum_{i=1}^{m-1} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (d_i)_{j,k} (\psi_i)_{j,k}(x)$$

where the coefficients are given by

$$(d_i)_{j,k} \ = \ \int f(x)\, (\psi_i)_{j,k}(x)\, dx$$

The necessary and sufficient conditions that $(\psi_i)_{j,k}(x)$ constitute an orthonormal basis for $L^2(\mathbf{R})$ are given by A. Cohen [5] and W. M. Lawton [20]. One needs to either find a "good" compact set congruent to $[-\pi, \pi]$ modulo 2π or check whether 1 is a non-degenerate eigenvalue of a multiresolution operator to ensure orthogonality. We refer to [5, 20] for more details.

Now we return to orthogonal wavelet matrices. The set of orthogonal wavelet matrices with genus equal to 1 play a special role in the theory of orthogonal wavelets. We shall call them *orthogonal Haar wavelet matrices.* The set of orthogonal Haar wavelet matrices of rank m is a homogeneous space which is isomorphic to the Lie group U_{m-1} of unitary $(m-1) \times (m-1)$

matrices, and there is a distinguished Haar matrix which corresponds to the identity element of the group U_{m-1}, which will be called the *canonical Haar matrix.*

Lemma 2.2 *An $m \times m$ matrix H is an orthogonal Haar wavelet matrix if and only if*

$$H = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \mathbf{H}$$

where $U \in U_{m-1}$ is a unitary matrix, that is, $U^*U = I_{m-1}$, and $\mathbf{H}$ is the canonical Haar matrix *of rank m, which is defined by*

$$\mathbf{H} := \begin{pmatrix} 1 & 1 & & & & & 1 \\ -\sqrt{m-1} & \sqrt{\frac{1}{m-1}} & & & & & \sqrt{\frac{1}{m-1}} \\ \vdots & \ddots & \ddots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & -\sqrt{\frac{im}{i+1}} & \sqrt{\frac{m}{i^2+i}} & \cdots & \sqrt{\frac{m}{i^2+i}} \\ \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & -\sqrt{\frac{m}{2}} & \sqrt{\frac{m}{2}} \end{pmatrix}$$

where $i = m-1, \cdots, 2, 1$.

The proof of the above lemma, can be found, for example, in [15, 24]. Let A be an orthogonal wavelet matrix and let $A(z)$ be its Laurent series. Define the *characteristic Haar matrix* $\chi(A)$ of the wavelet matrix A by

$$\chi(A) := A(1)$$

It can be easily checked that χ is a well-defined mapping from orthogonal wavelet matrices of rank m to orthogonal Haar wavelet matrices of rank m. When $m = 2$, we have

$$\chi(A) = \begin{pmatrix} \cdots + a_{0,-2} + a_{0,0} + a_{0,2} + \cdots, & \cdots + a_{0,-1} + a_{0,1} + a_{0,3} + \cdots \\ \cdots + a_{1,-2} + a_{1,0} + a_{1,2} + \cdots, & \cdots + a_{1,-1} + a_{1,1} + a_{1,3} + \cdots \end{pmatrix}$$

With Lemma 2.2, the following theorem is a simple consequence of Theorem 2.1.

Theorem 2.3 (First Orthogonal Factorization Theorem) *If A is an orthogonal wavelet matrix of rank m and genus g, then there exist primitive paraunitary matrices $V_1, \cdots, V_d$ such that*

$$A \;=\; z^{-k_1} V_1 V_2 \cdots V_d H \tag{2.8}$$

where $d \in \mathbf{N}$, $d \geq g-1$, $H = \chi(A)$ is the characteristic Haar matrix of A, and k_1 is the smallest index that $A_k \neq 0$ in the Laurent series (2.1).

The number d in (2.8) can by determined by the following way. From (2.5) it follows that $\det(A(z)) = cz^{-b}$ for some constant $c \neq 0$ and integer b. Now taking the determinant on both sides of (2.8) and using the result of Lemma 2.1, we get

$$cz^{-b} \;=\; z^{-mk_1} z^{-d} \det(H)$$

Note that

$$\det(H) \;=\; \det(A(1)) \;=\; c$$

so

$$b \;=\; mk_1 + d$$

Thus

$$d = b - mk_1$$

In particular, if $k_1 = 0$, then $d = b$.

For two orthogonal wavelet matrices A_1 and A_2, whose characteristic Haar matrix are the same, we can define the *Pollen product* by the formula

$$A_1 \diamond_H A_2 \;:=\; A$$

if their Laurent series satisfy

$$A(z) \;=\; A_1(z) H^{-1} A_2(z)$$

where

$$H \;=\; \chi(A_1) \;=\; \chi(A_2)$$

It's easy to check that the characteristic Haar matrix of A is also H. Thus the set of orthogonal wavelet matrices with the same characteristic Haar matrix is a group with the (noncommutative) Pollen product. Thus we may restate Theorem 2.3 in the language of Pollen products.

Theorem 2.4 (Second Orthogonal Factorization Theorem) *If A is an orthogonal wavelet matrix A of rank m and genus g, then there exist orthogonal wavelet matrices $A_1, \cdots, A_d$ such that*

$$A \;=\; z^{-k_1} A_1 \diamond_H A_2 \diamond_H \cdots \diamond_H A_d$$

where $H = \chi(A)$ is the characteristic Haar matrix of A, and

$$A_j(z) \;=\; V_j(z)H \;=\; (I_m - v_j {v_j}^* + v_j {v_j}^* z^{-1})H\,, \quad j = 1, \cdots, d$$

where for each v_j, ${v_j}^* v_j = 1$.

3 The Case of Rank $m = 2$

From the factorization theorems in Section 2, we know that an orthogonal wavelet matrix can be written as a product of some primitive paraunitary matrices and a Haar matrix. Thus it suggests a mapping from the space of orthogonal wavelet matrices to the space of products of primitive paraunitary matrices (which is, actually, unit spheres, by definition) and Haar matrices. However such mapping can not be well-defined without knowing

1. the factorization form is unique. From an orthogonal wavelet matrix, there is only one set of primitive paraunitary matrices and a Haar matrix which correspond to the factorization form.

2. the number of primitive paraunitary matrices, d, can be quantitatively determined by the genus g. Then one can set up a mapping from the subspace of orthogonal wavelet matrices of genus g to a product of a torus and Haar matrices.

We will refer to these two requirements as the uniqueness and exactness of factorization, respectively. In this section, we will present the exactness of factorization for rank $m = 2$ and also give a new proof of Pollen's uniqueness result for rank $m = 2$. It turns out only for $m = 2$, the uniqueness and exactness of factorization are valid. In this section, we will mainly study the case of $m = 2$. We will also discuss the case of $m > 2$ as well.

3.1 Uniqueness and Exactness of Factorization

In the literature, a two-band (or two-channel) filter bank has been refered to as the *quadrature mirror filter* (QMF) bank [12, 13, 16]. In wavelet analysis, the most common setting for the rank m is $m = 2$, in which we will have one scaling function $\phi(x)$ and one wavelet function $\psi(x)$. Thus when we decompose a function, there will be exactly two parts in the wavelet representation, the average part of $\phi(x)$ and the difference part of $\psi(x)$.

In $\mathbf{R}^2$ a unit vector can be represented by a point $(\cos\theta, \sin\theta)'$ on the unit circle S^1, and its primitive paraunitary matrix

$$
\begin{aligned}
V_\theta(z) &= I_2 - \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} (\cos\theta, \sin\theta) + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} (\cos\theta, \sin\theta) z^{-1} \\
&= I_2 - \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} + \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} z^{-1} \\
&= \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix} + \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} z^{-1} \\
&= S(\theta) + C(\theta) z^{-1}
\end{aligned}
$$

where

$$
\begin{aligned}
S(\theta) &:= \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix} \\
C(\theta) &:= \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}
\end{aligned}
$$

Note that $S(\theta)$ and $C(\theta)$ are related by

$$S(\theta) = C(\theta + \pi/2)$$

and

$$C(\theta) = S(\theta + \pi/2)$$

With this special form of primitive paraunitary matrix over real numbers, the orthogonal wavelet matrix over real numbers will have exactly $g-1$ primitive paraunitary matrix factors in the factorization.

Theorem 3.1 (Exactness of Factorization) *If A is a real orthogonal wavelet matrix of rank 2 and genus g, then there exist exactly $g-1$ primitive paraunitary matrices $V_1, \cdots, V_{g-1}$ such that*

$$A = z^{-k_1} V_1 V_2 \cdots V_{g-1} H \tag{3.1}$$

where k_1 is the smallest index that $A_k \neq 0$ in the Laurent series (2.1).

Proof: Let

$$A(z) \;=\; \sum_{k\in\mathbf{Z}} A_k z^{-k} \;=\; \sum_{k=k_1}^{k_2} A_k z^{-k}$$

where k_1 and k_2 are the smallest and largest indices that $A_k \neq 0$, respectively, and $g = k_2 - k_1 + 1$. From Theorem 2.3, there exist d primitive paraunitary matrices $V_1, \cdots, V_d$, such that

$$\begin{aligned} A(z) &= z^{-k_1} V_1(z)\cdots V_d(z) A(1) \\ &= z^{-k_1}\left(S(\theta_1) + C(\theta_1)z^{-1}\right)\cdots\left(S(\theta_d) + C(\theta_d)z^{-1}\right) A(1) \\ &= z^{-k_1}\left(S(\theta_1)\cdots S(\theta_d) + \cdots + C(\theta_1)\cdots C(\theta_d)z^{-d}\right) A(1) \end{aligned}$$

To prove $d = g-1$, it suffices to show that

$$C(\theta_1)C(\theta_2)\cdots C(\theta_d) \;\neq\; 0$$

If not, that is

$$C(\theta_1)C(\theta_2)\cdots C(\theta_d) \;=\; 0$$

since

$$\begin{aligned} & C(\theta_1)\cdots C(\theta_d) \\ &= \begin{pmatrix} \cos^2\theta_1 & \cos\theta_1\sin\theta_1 \\ \cos\theta_1\sin\theta_1 & \sin^2\theta_1 \end{pmatrix} \cdots \begin{pmatrix} \cos^2\theta_d & \cos\theta_d\sin\theta_d \\ \cos\theta_d\sin\theta_d & \sin^2\theta_d \end{pmatrix} \\ &= \left(\prod_{j=1}^{d-1}\cos(\theta_j - \theta_{j+1})\right)\begin{pmatrix} \cos\theta_1\cos\theta_d & \cos\theta_1\sin\theta_d \\ \sin\theta_1\cos\theta_d & \sin\theta_1\sin\theta_d \end{pmatrix} \end{aligned}$$

it follows that for some $j_0 \in \{1, \cdots, d-1\}$, $\cos(\theta_{j_0} - \theta_{j_0+1}) = 0$. Now

$$\begin{aligned} & S(\theta_1)\cdots S(\theta_d) \\ &= C(\theta_1 + \pi/2)\cdots C(\theta_d + \pi/2) \\ &= \left(\prod_{j=1}^{d-1}\cos(\theta_j - \theta_{j+1})\right)\begin{pmatrix} \sin\theta_1\sin\theta_d & -\sin\theta_1\cos\theta_d \\ -\cos\theta_1\sin\theta_d & \cos\theta_1\cos\theta_d \end{pmatrix} \\ &= 0 \end{aligned}$$

which is impossible, since

$$S(\theta_1)\cdots S(\theta_d) \ = \ A_{k_1} \ \neq \ 0$$

Thus d should be equal to $g-1$.

$\square$

The following uniqueness theorem was first derived by Pollen [23].

Theorem 3.2 (Uniqueness of Factorization) *For an orthogonal wavelet matrix A of rank 2 and genus g, the factorization is unique. If*

$$A \ = \ z^{-k_1}V_1V_2\cdots V_dH \ = \ z^{-k_1}W_1W_2\cdots W_dG$$

where V_j,W_j are primitive paraunitary matrices, $j=1,\cdots,d$, H and G are two square constant matrices, then $V_j = W_j$, $j=1,\cdots,g-1$, and $H=G$.

Note that from Section 2, we know that if $\det(A(z)) = cz^{-b}$, then

$$d \ = \ b - mk_1$$

thus the number of factors, d, will be invariant if $A(z)$ has two or more different factorizations, which actually can not happen when $m=2$.

The uniqueness of factorization actually can be obtained from the exactness result. The reason is that if

$$A \ = \ z^{-k_1}V_1V_2\cdots V_{g-1}H \ = \ z^{-k_1}W_1W_2\cdots W_{g-1}G$$

then evaluating at $z=1$ will imply $H=G$, and

$$V_1V_2\cdots V_{g-1}(W_{g-1})^{-1}\cdots(W_2)^{-1}(W_1)^{-1} \ = \ I_2$$

Note that W_j is a primitive paraunitary matrix

$$W_j \ = \ S(\eta_j) + C(\eta_j)z^{-1}$$

so

$$\begin{aligned}
(W_j)^{-1} &= (W_j)^* \\
&= S(\eta_j) + C(\eta_j)z \\
&= z\left(C(\eta_j) + S(\eta_j)z^{-1}\right) \\
&= z\left(S(\eta_j+\pi/2) + C(\eta_j+\pi/2)z^{-1}\right) \\
&= zU_j
\end{aligned}$$

where

$$U_j \;=\; S(\eta_j + \pi/2) + C(\eta_j + \pi/2)z^{-1}$$

Thus we have

$$V_1 V_2 \cdots V_{g-1} U_{g-1} \cdots U_2 U_1 \;=\; z^{-g+1}$$

Since z^{-g+1} has genus 1, and $d = g - 1 = 0$, we must have

$$V_j U_j \;=\; z^{-1} I_2 \,, \quad j = 1, \cdots, g-1$$

that is

$$V_j \;=\; W_j$$

which implies the uniqueness of factorization.

When $m > 2$, the uniqueness result or the exactness does not hold in general. Here is a counterexample

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} z^{-1} \\
&= \left(I_3 - e_2 {e_2}^* + e_2 {e_2}^* z^{-1}\right)\left(I_3 - e_3 {e_3}^* + e_3 {e_3}^* z^{-1}\right) \\
&= \left(I_3 - v_2 {v_2}^* + v_2 {v_2}^* z^{-1}\right)\left(I_3 - v_3 {v_3}^* + v_3 {v_3}^* z^{-1}\right)
\end{aligned}$$

where

$$e_2 = (0, 1, 0)' \,, \quad e_3 = (0, 0, 1)'$$

and

$$v_2 = \left(0, \sqrt{2}/2, \sqrt{2}/2\right)' \,, \quad v_3 = \left(0, \sqrt{2}/2, -\sqrt{2}/2\right)'$$

In this example, we have two different factorizations and $d = 2 > g - 1 = 1$. Even when $m = 2$, if A is a complex matrix, the number of factors d can be greater than $g - 1$, even though $A(z)$ has a unique factorization. One can choose two complex unit vectors $v_1, v_2 \in \mathbf{C}^2$ such that they are orthogonal to each other but

$$(I_2 - v_1 v_1^*)\,(I_2 - v_2 v_2^*) \neq 0$$

then the product of their primitive paraunitary matrices will have genus $g = 2$, but $d = 2 > g - 1$.

Another remark is that in the factorization, the exponent k_1 is set to be the smallest index that $A_k \neq 0$ in the Laurent series $A(z)$. If we drop this requirement on k_1, then $A(z)$ will have different factorization, because of the degenerate phenomenon

$$V_{\theta_1}(z)V_{\theta_2}(z) \;=\; z^{-1}I_2$$

if $\theta_1 - \theta_2 = k\pi + \pi/2$ for some integer k. However the factorization given by Theorem 2.3 is minimal in the sense that the number of factors is minimal. And in the minimal case, the factorization is also unique and exact.

3.2 Algebraic Structure of Orthogonal Wavelet Space

Combining Theorem 3.1 and Theorem 3.2, if A is a real orthogonal wavelet matrix of rank 2 and genus g, there exist exactly and uniquely $g-1$ primitive paraunitary matrices $V_1, \cdots, V_{g-1}$ such that

$$A(z) \;=\; z^{-k_1}V_1 \cdots V_{g-1}H \tag{3.2}$$

where $H = A(1)$ is the characteristic Haar matrix, and since $m = 2$, we have

$$H \;=\; \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \text{or } H \;=\; \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Thus for any real orthogonal wavelet matrix of rank 2 and genus g, there exists a unique set $\{k_1, V_1, V_2, \cdots, V_{g-1}, H\}$ such that (3.2) holds. By definition,

$$V_j \;=\; I_m - v_j v_j^* + v_j v_j^* z^{-1}$$

where v_j is a unit vector, $v_j^* v_j = 1$, $j = 1, 2, \cdots, g-1$. Then $V_1 V_2 \cdots V_{g-1}$ can be identified as a point on the $(g-1)$ dimensional real torus

$$\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{g-1}$$

So the uniqueness of factorization sets up a well-defined mapping from the space of orthogonal wavelets to the space of real tori (with additional parameter k_1 and Haar matrix H, which will be discussed next). And the exactness of factorization guarantees the image set of a subspace of genus g orthogonal wavelets will be inside the $(g-1)$ dimensional real torus.

Since the multiplication by z^{-k} is just a shift of index j in $a_{i,j}$, we will discard the difference between two orthogonal wavelet matrices if one is the other multiplied by z^{-k} for some integer k. Note that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

if we assume

$$A(z) = \begin{pmatrix} h_e(z) & h_o(z) \\ g_e(z) & g_o(z) \end{pmatrix}$$

then

$$A(z)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} h_e(z) & h_o(z) \\ g_e(z) & g_o(z) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} h_o(z) & h_e(z) \\ g_o(z) & g_e(z) \end{pmatrix}$$

which means that a multiplication by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is a flip on the components (even and odd coefficients of A at each row) and doesn't change the structure of $A(z)$. Without loss of generality, we will always assume that

$$H = A(1) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

or equivalently

$$\sum_{j \in \mathbf{Z}} a_{1,2j} = -1$$

$$\sum_{j \in \mathbf{Z}} a_{1,2j+1} = 1$$

Then a real orthogonal wavelet matrix A of rank 2 and genus g can by uniquely characterized by $g-1$ primitive paraunitary matrices, or $g-1$ unit vectors, which is an element of the $(g-1)$ dimensional real torus. So the uniqueness and exactness of factorization builds up a bridge between the space of compactly supported orthonormal wavelets over real numbers and a torus, which is the product of unit circles.

Let $WM(2,g)$ be the collection of all real orthogonal wavelet matrices of rank 2 and genus g, and $WM(2)$ be the collection of all real orthogonal wavelet matrices of rank 2, then

$$WM(2) = \bigcup_{1 \le g < \infty} WM(2,g)$$

And let T^n denote the n-dimensional real torus

$$T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n}$$

As we know, the image set of $WM(2,g)$ under the factorization map is a subset of T^{g-1}. Thus the factorization map will induce a mapping from $WM(2,g)$ to T^{g-1}

$$\begin{array}{rccc} \rho: & WM(2,g) & \longrightarrow & T^{g-1} \\ & A(z) & \longmapsto & (\theta_1, \theta_2, \cdots, \theta_{g-1}) \end{array}$$

By the uniqueness and exactness of factorization, this mapping ρ is well-defined. And it is also clear that ρ is one-to-one. Because of the degenerate phenomenon of paraunitary matrix

$$V_{\theta_1}(z) V_{\theta_2}(z) = z^{-1} I_2 \tag{3.3}$$

if $\theta_1 - \theta_2 = k\pi + \pi/2$ for some integer k, this mapping is not onto. The image set will exclude the degenerate points $(\theta_1, \cdots, \theta_{g-1})$ where $\theta_{j_0} - \theta_{j_0+1} = k\pi + \pi/2$ for some $j_0 \in \{1, \cdots, g-2\}$ and $k \in \mathbf{Z}$. Thus one derives

Theorem 3.3 *The mapping from $WM(2,g)$, the space of real orthogonal wavelet matrices of rank 2 and genus g, to the real torus T^{g-1}*

$$\begin{array}{rccc} \rho: & WM(2,g) & \longrightarrow & T^{g-1} \\ & A(z) & \longmapsto & (\theta_1, \theta_2, \cdots, \theta_{g-1}) \end{array}$$

is well-defined and one-to-one.

Conversely, there is a natural mapping τ from the torus T^{g-1} to $WM(2)$

$$\begin{array}{rccc} \tau: & T^{g-1} & \longrightarrow & WM(2) \\ & (\theta_1, \theta_2, \cdots, \theta_{g-1}) & \longmapsto & V_{\theta_1}(z) V_{\theta_2}(z) \cdots V_{\theta_{g-1}}(z) \mathbf{H} \end{array}$$

where $\mathbf{H}$ is the canonical Haar matrix of rank 2, which is exactly

$$\mathbf{H} \;=\; \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Again, because of (3.3), the image set $\tau(T^{g-1}) \neq WM(2,g)$. The degenerate property (3.3) cancels out two primitive paraunitary matrices at each time, and the image set $\tau(T^{g-1})$ will be

$$\tau(T^{g-1}) \;=\; WM(2,g) \cup WM(2,g-2) \cup WM(2,g-4) \cup \cdots WM(2,\mathrm{rem})$$

where rem $= 1$ is g is odd, otherwise rem $= 2$. In the sense of the mapping τ, we define the "closure" $\overline{WM(2,g)}$ of $WM(2,g)$ by

- if g is odd

$$\overline{WM(2,g)} := \bigcup_{j=1}^{(g+1)/2} WM(2,2j-1) = WM(2,1) \cup \cdots \cup WM(2,g)$$

- if g is even

$$\overline{WM(2,g)} := \bigcup_{j=1}^{g/2} WM(2,2j) = WM(2,2) \cup \cdots \cup WM(2,g)$$

Then the mapping τ with the image set restricted to $\overline{WM(2,g)}$ will be onto

$$\tau :\; T^{g-1} \;\longrightarrow\; \overline{WM(2,g)}$$

However this mapping is not one-to-one, because

$$\left(I_2 - e_1{e_1}^* + e_1{e_1}^* z^{-1}\right)\left(I_2 - e_2{e_2}^* + e_2{e_2}^* z^{-1}\right) =$$
$$\left(I_2 - v_1{v_1}^* + v_1{v_1}^* z^{-1}\right)\left(I_2 - v_2{v_2}^* + v_2{v_2}^* z^{-1}\right)$$

where

$$e_1 = (1,0)', \quad e_2 = (0,1)'$$

and

$$v_1 = \left(\sqrt{2}/2, \sqrt{2}/2\right)', \quad v_2 = \left(\sqrt{2}/2, -\sqrt{2}/2\right)'$$

The structure of the image set of τ suggests that we may divide orthogonal wavelet matrices into two classes, $WM(2,\text{odd})$, those with odd genus, and $WM(2,\text{even})$, those with even genus. Inside each class, the following diagram commutes

$$\begin{array}{ccc} T^{g-1} & \xrightarrow{\tau} & \overline{WM(2,g)} \\ \downarrow & & \downarrow \\ T^{g+1} & \xrightarrow{\tau} & \overline{WM(2,g+2)} \end{array}$$

where the inclusion mapping i_1 from T^{g-1} to T^{g+1} is adding a pair of degenerate point (θ_g, θ_{g+1})

$$\begin{array}{rccc} i_1 : & T^{g-1} & \longrightarrow & T^{g+1} \\ & (\theta_1, \cdots, \theta_{g-1}) & \longmapsto & (\theta_1, \cdots, \theta_{g-1}, \theta_g, \theta_{g+1}) \end{array}$$

where $\theta_g - \theta_{g+1} = k\pi + \pi/2$ for some integer k. For example, one can choose $\theta_g = \pi/2$ and $\theta_{g+1} = 0$. And the inclusion mapping i_2 from $\overline{WM(2,g)}$ to $\overline{WM(2,g+2)}$ is a multiplication by z^{-1}

$$\begin{array}{rccc} i_2 : & \overline{WM(2,g)} & \longrightarrow & \overline{WM(2,g+2)} \\ & A(z) & \longmapsto & z^{-1}A(z) \end{array}$$

From (3.3), it is easy to verify that the above diagram is truly commutative. We formulate it as the following theorem.

Theorem 3.4 *Define a mapping τ from the real torus T^{g-1} to $\overline{WM(2,g)}$, the "closure" of $WM(2,g)$ by*

$$\begin{array}{rccc} \tau : & T^{g-1} & \longrightarrow & \overline{WM(2,g)} \\ & (\theta_1, \theta_2, \cdots, \theta_{g-1}) & \longmapsto & V_{\theta_1}(z)V_{\theta_2}(z)\cdots V_{\theta_{g-1}}(z)\mathbf{H} \end{array}$$

where

$$\mathbf{H} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Then τ is well-defined and onto, and the following diagram commutes

$$\begin{array}{ccc} T^{g-1} & \xrightarrow{\tau} & \overline{WM(2,g)} \\ \downarrow & & \downarrow \\ T^{g+1} & \xrightarrow{\tau} & \overline{WM(2,g+2)} \end{array}$$

For the rank $m > 2$, let $WM(m,g)$ be the collection of all real (or complex) orthogonal wavelet matrices of rank m and genus g, and $WM(m)$ be the collection of all real (or complex) orthogonal wavelet matrices of rank m,

$$WM(m) \;=\; \bigcup_{1 \le g < \infty} WM(m,g)$$

And let $\mathcal{T}^n$ denote the n-dimensional real (or complex) torus

$$\mathcal{T}^n \;=\; \underbrace{S^{m-1} \times S^{m-1} \times \cdots \times S^{m-1}}_{n}$$

Because of the lack of the uniqueness and exactness of factorization, the factorization map from $WM(m,g)$ to $\mathcal{T}^{g-1}$ is not well-defined. However one can always define the parameter mapping from $\mathcal{T}^n$ to $WM(m)$ by

$$\begin{array}{cccc} \tau : & \mathcal{T}^n & \longrightarrow & WM(m) \\ & (v_1, v_2, \cdots, v_n) & \longmapsto & V_1(z)V_2(z)\cdots V_n(z)\mathbf{H} \end{array}$$

where $\mathbf{H}$ is the canonical Haar matrix of rank m. Similar to the case of $m = 2$, we can define the "closure" of $WM(m,g)$ by

$$\overline{WM(m,g)} \;:=\; WM(m,g) \cup WM(m,g-m) \cup \cdots \cup WM(m,\mathrm{rem})$$

where rem is the reminder of g under module m. And one can also generalize Theorem 3.4 to $m > 2$, which will be omitted here.

4 A Fast Wavelet Transform based on Factorization

The discrete wavelet transform (DWT) [21] can be implemented via the multiplication by the polyphase decomposition of the wavelet matrix. Thus the complexity of the DWT is the same as the multiplication complexity of the wavelet matrix.

Assume we have an orthogonal wavelet matrix of rank m and genus g, then a direct DWT implementation based on matrix multiplication will have $m^2 g$ multiplications and $(m^2 - m)g$ additions. Note that in the DWT complexity, we do not consider the length of the data/signal with which the DWT will be applied to. One can equally add the length as an additional parameter and carry out a similar analysis.

From Lemma 2.2, the set of characteristic Haar matrices H is isomorphic to a specific unitary group. Since the unitary group U_{m-1} has dimension $(m-1)(m-2)/2$, the space of orthogonal m-band wavelet matrices will be at least $(m-1)(m-2)/2$ dimensional (actually much larger). Thus one derives

Lemma 4.1 *A universal DWT implementation can not achieve better than* $O(m^2)$.

Thus in general, for a given wavelet matrix, there is no $O(m)$ DWT implementation. However, when one designs a wavelet matrix, in most cases, all preimposed conditions (for examples, the approximation order, the orthogonality, the smoothness, etc.) are on the scaling filter only, and the wavelet filters are then constructed from the scaling filter. When $m = 2$, the wavelet filter is uniquely determined (up to a translation) by the scaling filter. When $m > 2$, there is some degree of freedom in the choices of the wavelet filters. It suggests the possibility that from a given scaling filter, one may construct a full wavelet matrix with less DWT complexity. Several approaches [27, 14, 17] have been presented to construct the $m-1$ wavelet filters from the scaling filter.

Here we briefly repeat Heller's procedure to construct a full wavelet matrix from a given scaling filter and a characteristic Haar matrix. A more general construction method is described in [17]. The next theorem is Theorem 4.1 of [14] (see also [24]).

Theorem 4.1 *Given a scaling filter* a_0 *of genus* g *and a characteristic Haar matrix* H*, there exists an orthogonal wavelet matrix* A *of genus* g *whose first row is* a_0 *and whose characteristic Haar matrix is* H.

The proof in [14] is constructive. It suffices to obtain vectors v_i such that the relationship (2.8) holds with $d = g-1$. Without loss of generality, we can assume $k_1 = 0$ (otherwise one can always shift a_0 to get $k_1 = 0$). The factorization (2.8) now has the form

$$A_0 + A_1 z^{-1} + \cdots + A_{g-1} z^{-g+1} = \left(\prod_{i=1}^{g-1} \left(I_m - v_i {v_i}^* + v_i {v_i}^* z^{-1} \right) \right) H$$

where the first row of each A_k is known. Right-multiplying by H^{-1}, it becomes

$$B_0 \,+\, B_1 z^{-1} \,+\, \cdots \,+\, B_{g-1} z^{-g+1} \;=\; \prod_{i=1}^{g-1} \left(I_m - v_i {v_i}^* + v_i {v_i}^* z^{-1}\right) \tag{4.4}$$

Again the first row of each B_k is known. By comparing the coefficient of z^{-g+1} on both side of (4.4), one gets

$$B_{g-1} \;=\; v_1 {v_1}^* v_2 {v_2}^* \cdots v_{g-1} {v_{g-1}}^*$$

The right hand side is a rank-1 matrix, and each of whose rows is proportional to ${v_{g-1}}^*$. Since the first row of B_{g-1} is known, and let it be β, it follows

$$v_{g-1} \;=\; \frac{\beta^*}{||\beta||}$$

Right-multiply (4.4) by

$$I_m - v_{g-1} {v_{g-1}}^* + v_{g-1} {v_{g-1}}^* z$$

the inverse of the newly determined prime factor, to obtain

$$C_0 \,+\, C_1 z^{-1} \,+\, \cdots \,+\, C_{g-2} z^{-g+2} \;=\; \prod_{i=1}^{g-2} \left(I_m - v_i {v_i}^* + v_i {v_i}^* z^{-1}\right)$$

Again the first row of each C_k is known, and we can repeat the pattern to get $v_{g-2}, v_{g-3}, \cdots, v_1$. It has been shown in [14] that the resulting matrix

$$\left(\prod_{i=1}^{g-1} (I_m - v_i {v_i}^* + v_i {v_i}^* z^{-1})\right) H$$

is an orthogonal wavelet matrix of genus g whose first row is a_0 and with characteristic Haar H.

With the above construction procedure, let's examine very carefully the multiplication complexity of each factor in the wavelet matrix factorization and shows that, for any scaling filter, one can always construct a full orthogonal wavelet matrix with an $O(m)$ discrete wavelet transform. We first show that the factorization into primitive paraunitary factors can be further factored into almost diagonal matrices which is similar to the decomposition of Daubechies and Sweldens' [10] for biorthogonal wavelets of rank 2, using the lifting steps.

First, for the primitive paraunitary matrix

$$V_i(z) \;=\; I_m - v_i v_i^* + v_i v_i^* z^{-1}\,,$$

a multiplication by $V_i(z)$ has an $O(m)$ complexity because of its special structure as the product of vectors. This $O(m)$ complexity can be also illustrated by the Gaussian elimination method.

Set the unit vector $v_i = (w_1, w_2, \cdots, w_m)^*$, then

$$V_i(z) = \begin{pmatrix} 1+\bar{w}_1 w_1(z^{-1}-1) & \bar{w}_1 w_2(z^{-1}-1) & \cdots & \bar{w}_1 w_m(z^{-1}-1) \\ \bar{w}_2 w_1(z^{-1}-1) & 1+\bar{w}_2 w_2(z^{-1}-1) & \cdots & \bar{w}_2 w_m(z^{-1}-1) \\ \vdots & \vdots & \vdots & \vdots \\ \bar{w}_m w_1(z^{-1}-1) & \bar{w}_m w_2(z^{-1}-1) & \cdots & 1+\bar{w}_m w_m(z^{-1}-1) \end{pmatrix}$$

Now using the Gaussian elimination method, we get

$$V_i(z) \;=\; X(z) X_1 X_2 \cdots X_{m-1} \tag{4.5}$$

where $X_j = (x_{k,l})_{m\times m}$, $j = 1, 2, \cdots, m-1$, is defined by

$$X_j \;:=\; \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & w_j/w_m & 0 & \cdots & 0 & 1 \end{pmatrix}$$

that is

$$x_{k,l} \;=\; \begin{cases} 1 & \text{if } k = l \\ w_j/w_m & \text{if } k = m \text{ and } l = j \\ 0 & \text{otherwise} \end{cases}$$

and $X(z)$ is of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \bar{w}_1 w_m(z^{-1}-1) \\ 0 & 1 & \cdots & 0 & \bar{w}_2 w_m(z^{-1}-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \bar{w}_{m-1} w_m(z^{-1}-1) \\ -\frac{w_1}{w_m} & -\frac{w_2}{w_m} & \cdots & -\frac{w_{m-1}}{w_m} & 1+\bar{w}_m w_m(z^{-1}-1) \end{pmatrix}$$

Note that if $w_m = 0$, then one can move to w_{m-1} and apply the same procedure.

Continuing the process and using the identity

$$\bar{w}_1 w_1 + \bar{w}_2 w_2 + \cdots + \bar{w}_m w_m \; = \; 1$$

we have

$$X(z) \; = \; Y_1 Y_2 \cdots Y_{m-1} Y(z) \tag{4.6}$$

where Y_j, $j = 1, 2, \cdots, m-1$, is defined by

$$Y_j \; := \; \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -w_j/w_m & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and

$$Y(z) \; = \; \begin{pmatrix} 1 & 0 & \cdots & 0 & \bar{w}_1 w_m (z^{-1} - 1) \\ 0 & 1 & \cdots & 0 & \bar{w}_2 w_m (z^{-1} - 1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \bar{w}_{m-1} w_m (z^{-1} - 1) \\ 0 & 0 & \cdots & 0 & z^{-1} \end{pmatrix}$$

Since

$$Y(z) \; = \; P_1 P_2 \cdots P_{m-1} P(z) \tag{4.7}$$

where P_j, $j = 1, 2, \cdots, m-1$, is defined by

$$P_j \; := \; \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \bar{w}_j w_m \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and

$$P(z) = \begin{pmatrix} 1 & 0 & \cdots & 0 & -\bar{w}_1 w_m \\ 0 & 1 & \cdots & 0 & -\bar{w}_2 w_m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\bar{w}_{m-1} w_m \\ 0 & 0 & \cdots & 0 & z^{-1} \end{pmatrix}$$

Combining (4.5), (4.6), and (4.7), we obtain

$$V_i(z) = Y_1 \cdots Y_{m-1} P_1 \cdots P_{m-1} P(z) X_1 \cdots X_{m-1}.$$

Thus the multiplication by $V_i(z)$ can be implemented with $4(m-1)$ multiplications and $4(m-1)$ additions. The next lemma tells us that the DWT complexity of a characteristic Haar matrix is fully determined by its unitary matrix.

Lemma 4.2 *For a characteristic Haar matrix H, the DWT complexity is equal to the sum of $O(m)$ and the complexity of a multiplication by a matrix U, where U is the unitary matrix in* Lemma 2.2.

Proof: From Lemma 2.2, it suffices to show that the multiplication by the canonical Haar matrix $\mathbf{H}$ has an $O(m)$ complexity. By definition, $\mathbf{H}$ is equal to the product of two matrices

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\frac{1}{m-1}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & \sqrt{\frac{m}{i^2+i}} & \cdots & 0 \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \sqrt{\frac{m}{2}} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ 1-m & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ \vdots & \ddots & \ddots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & -j & 1 & \cdots & 1 \\ \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 1 \end{pmatrix}$$

The first matrix is diagonal, and a multiplication by it will be $m-1$ multiplications and no additions. For the second matrix, let's denote it by G, and assume

$$G \cdot (x_0, x_1, \cdots, x_{m-1})^* = (y_0, y_1, \cdots, y_{m-1})^*$$

Since

$$
\begin{aligned}
y_0 &= x_0 + x_1 + \cdots + x_{m-1} \\
y_1 &= y_0 - m \cdot x_0 \\
y_2 &= y_1 - (m-1) \cdot (x_1 - x_0) \\
&\vdots \\
y_{m-1} &= y_{m-2} - 2 \cdot (x_{m-2} - x_{m-3})
\end{aligned}
$$

So the multiplication of G can be implemented within $m-1$ multiplications and $3m-4$ additions. Totally the multiplication by $\mathbf{H}$ has $2m-2$ multiplications and $3m-4$ additions. The lemma is proved. $\square$

Combining Theorem 4.1 and Lemma 4.2, we obtain

Theorem 4.2 *Given a scaling filter a_0 of genus g, there exists an orthogonal wavelet matrix A of genus g whose first row is a_0, and the DWT complexity of A is $O(m)$.*

Proof: By choosing the canonical Haar matrix $\mathbf{H}$ as the characteristic Haar, we can construct an orthogonal wavelet matrix A of genus g whose first row is a_0 by Theorem 4.1,

$$
A(z) = \left(\prod_{i=1}^{g-1} (I_m - v_i v_i^* + v_i v_i^* z^{-1}) \right) \mathbf{H}\,.
$$

Note that each prime factor

$$
I_m - v_i v_i^* + v_i v_i^* z^{-1}
$$

can be implemented with $4(m-1)$ multiplications and $4(m-1)$ additions. Since there are exactly $g-1$ prime factors in the factorization, totally they will contribute to $4(m-1)(g-1)$ multiplications and $4(m-1)(g-1)$ additions. Now for the canonical Haar matrix $\mathbf{H}$, the unitary matrix $U = I_{m-1}$. Applying Lemma 4.2, it follows that the DWT complexity of A is $O(m)$. $\square$

In the design problem for orthogonal wavelet matrices, oftenly the scaling filter is constructed first. In these situations, one can simply choose the canonical Haar matrix (or some other simple structured Haar matrix whose unitary matrix has an $O(m)$ multiplication complexity) as the characteristic Haar matrix and construct the additional $m-1$ wavelet filters (see Theorem 4.1 and 4.2). Then the full wavelet matrix will have an $O(m)$ DWT. In the case that an orthogonal m-band wavelet matrix is already given, because changing the wavelet filters doesn't change the approximation order and the smoothness, one can fix the scaling filter and modify the wavelet filters to achieve $O(m)$ DWT. One can ignore the wavelet filters and construct a new wavelet matrix from the given scaling filter by Theorem 4.1 and 4.2. Or one can employ the eigenfilter approach [31] to find the unit column vectors $v_1, v_2, \cdots, v_d$ in the factorization (2.8) and force the characteristic Haar matrix to be the canonical Haar matrix (or some other Haar matrix with $O(m)$ DWT). There is a slight difference between these two approaches. For the eigenfilter approach, the number of prime factors, d, is greater than or equal to $g-1$. When $d > g-1$ (which is the case when the exactness of factorization fails), the multiplication complexity of the prime factors may be larger than $O(m)$, and the DWT complexity of the constructed wavelet matrix will be larger than the DWT complexity of its characteristic Haar matrix. So with a given full wavelet matrix, the construction method from its scaling filter would seem preferable.

Finally we will discuss a specific example on how to construct a full wavelet matrix from a given scaling filter to have an $O(m)$ DWT. The fast wavelet transform implementation will also be illustrated.

This example is from [27], a rank 3 scaling filter with approximation order two [1],

$$a_0 = \begin{pmatrix} a_{0,0} \\ a_{0,1} \\ a_{0,2} \\ a_{0,3} \\ a_{0,4} \\ a_{0,5} \end{pmatrix}^* = \begin{pmatrix} 0.58610191307059 \\ 0.91943524640393 \\ 1.25276857973726 \\ 0.41389808692940 \\ 0.08056475359608 \\ -0.25276857973727 \end{pmatrix}^*$$

Note that we use a different normalization (2.6). In [27], they have

$$\sum_j a_{0,j} = \sqrt{m}$$

[1]An rank m scaling filter is said to have approximation order K if it has a polynomial factor of the form $P^K(z)$, with $P(z) = (1 + z^{-1} + \cdots + z^{-(m-1)})/m$ for maximal possible K.

Now a_0 is a scaling filter of genus 2, and the canonical Haar matrix $\mathbf{H}$ is

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 \\ -\sqrt{2} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ 0 & -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} \end{pmatrix}$$

Using the notation in the proof of Theorem 4.1, the first row of A_1 is

$$(0.41389808692940, \quad 0.08056475359608, \quad -0.25276857973727)$$

Right-multiplying by $\mathbf{H}^{-1}$ (which is equal to $\mathbf{H}^*/m$), we get the first row of B_1, which is

$$\beta = (0.08056475359607, \quad -0.23570226039551, \quad -0.13608276348796)$$

Now normalize this row vector to obtain

$$\begin{aligned} v_1 &= \frac{\beta^*}{||\beta||} \\ &= (0.28383930946236, \quad -0.83040739086483, \quad -0.47943593065288)^* \end{aligned}$$

So the full wavelet matrix is equal to

$$A(z) = (I_3 - v_1 v_1^* + v_1 v_1^* z^{-1})\mathbf{H} =$$

$$\begin{pmatrix} 0.58610191307059 & 0.91943524640393 & 1.25276857973726 \\ -0.20330295558638 & 0.94280904158208 & -0.03239930480915 \\ 0.69911956479291 & -1.08866210790362 & 0.79779083357459 \end{pmatrix} +$$

$$\begin{pmatrix} 0.41389808692940 & 0.08056475359608 & -0.25276857973727 \\ -1.21091060678671 & -0.23570226039553 & 0.73950608599570 \\ -0.69911956479291 & -0.13608276348797 & 0.42695403781700 \end{pmatrix} z^{-1}$$

For the fast DWT implementation, based on the factorization in the Appendix,

$$I_3 - v_1 v_1^* + v_1 v_1^* z^{-1} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.59202761269027 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1.73205080756882 & 1 \end{pmatrix} \cdot$$

$$
\begin{pmatrix} 1 & 0 & -0.13608276348796 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.39812714026031 \\ 0 & 0 & 1 \end{pmatrix} \cdot
$$

$$
\begin{pmatrix} 1 & 0 & 0.13608276348796 \\ 0 & 1 & -0.39812714026031 \\ 0 & 0 & z^{-1} \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.59202761269027 & 0 & 1 \end{pmatrix} \cdot
$$

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1.73205080756882 & 1 \end{pmatrix}
$$

And for $\mathbf{H}$, it can be factored as

$$
\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1}2 & 0 \\ 0 & 0 & \sqrt{3}2 \end{pmatrix}
\begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}
$$

Combining the factorization of the prime factor and the canonical Haar matrix, we derive a fast implementation of the DWT.

5 Conclusions

In this paper we define a one-to-one mapping ρ from the space of real orthogonal wavelet matrices of rank 2 and genus g to the space

$$
\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{g-1}
$$

Thus any real orthogonal wavelet matrix can be uniquely identified as a point on a real torus. Meanwhile one can define a mapping τ from the torus to the wavelet space. The image set of τ suggests a partition of all orthogonal wavelet matrices of rank 2. With the same parity on the genus, the mapping τ provides a commutative diagram.

Based on the factorization, we propose an $O(m)$ DWT implementation for an orthogonal wavelet matrix constructed from a given scaling filter. The scaling filter completely determined the approximation order, the orthogonality, and the smoothness of a full wavelet matrix. Thus from the scaling filter, one can always construct a full wavelet matrix with the additional $(m-1)$ wavelet filters whose DWT has an $O(m)$ complexity.

After finishing this work, we learned that some similar results to those in Section 3 were obtained by J. Kautsky and R. Turcajová [17]. In [17], they proved that

For any orthogonal wavelet matrix $A = (A_0, A_1, \cdots, A_{g-1})$, which is rank m and genus g, there exists a decomposition into linear factors

$$A \;=\; K_1 K_2 \cdots K_{g-1} H$$

where the linear factor

$$K_i \;=\; I_m - U_i U_i^* + U_i U_i^* z^{-1}\,, \quad i = 1, 2, \cdots, g-1\,,$$

where U_i is an $m \times r_i$ matrix, $1 \leq r_i < m$, and its r_i column vectors are unit vectors and orthogonal to each other. And H is the characteristic Haar matrix of A. The factorization is unique if and only if

$$\mathrm{rank}(A_0) + \mathrm{rank}(A_{g-1}) \;=\; m\,.$$

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