Reconstruction of self-similar functions from scale-space

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Abstract. We provide a solution to the problem of reconstructing a fractal interpolation function from its scale-space zeros. Every fractal interpolation function f has a graph that is the attractor of an iterated functions system defined by contractive maps $w_1, \ldots, w_N$. We construct approximations of these maps from fingerprints in scale-space.

Introduction

During the last years multiscale analysis of signals has attracted a lot of interest in the field of scale-space and wavelet theory. Scale-space theory has focused on using the Gaussian function $\theta(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)$ as a convolution kernel (see e.g. [4]), whereas in wavelet theory a larger class of kernels and properties of convolutions with these kernels are studied. In this paper, we combine methods from both fields.

The convolution of a function $f \in L^2(\mathbb{R})$ with the Gaussian kernel is defined by $L_\theta[f](x,t) = \frac{1}{t} \int \theta(\frac{z-x}{t}) f(z) dz$, where $t > 0$. The set $\mathcal{N}(L_{\theta^{(n)}}[f]) = \{(x,t) \in \mathbb{R} \times \mathbb{R}_+ \mid L_{\theta^{(n)}}[f](x,t) = 0\}$ of zeros of $L_{\theta^{(n)}}[f]$ is called the *fingerprint* of $L_{\theta^{(n)}}[f]$, where $\theta^{(n)}$ denotes the n-th derivative of θ.

One of the main problems in scale-space theory is the reconstruction of a function from its fingerprints. For references on this topic see, e. g., [4]. In [5],[6], Mallat et al. studied the reconstruction problem in terms of wavelet theory, which is strongly connected to scale-space theory. They have shown that zero-crossings of wavelet transforms form a tool to characterize singularities of the signal. Furthermore, they discuss how to obtain approximations of signals by reconstruction from selected zero-crossings.

In this paper, we study fingerprints of fractal interpolation functions. These functions are defined on a compact interval, and their graphs are self-affine curves. Since many signals seem to have self-affine properties it is interesting to ask which information can be extracted from fingerprints of self-affine functions. We apply results from wavelet theory to find a finite number of points in scale-space that contain all information necessary to compute the graph of a fractal interpolation function.

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Fingerprints for continuous wavelet transforms

The theory of wavelet transforms is based on convolutions and is connected to scale-space theory. In this section, we summarize how fingerprints in scale-space can be interpreted in terms of continuous wavelet theory. For references regarding wavelet theory we refer the reader, e.g., to [3] and [5].

The continuous wavelet transform of a function $f \in L^2(\mathbb{R})$ with wavelet ψ is typically denoted by $W_\psi[f](b,a) = \frac{1}{\sqrt{a}} \int \psi(\frac{x-b}{a}) f(x) dx$, where ψ has the wavelet property. We consider a function $\phi \in L^2(\mathbb{R})$ which is an n-times differentiable smoothing function, i.e., $\int \phi(x)dx \neq 0$ and $\phi(x) = O(1/(1+x^2))$. If $\phi^{(k)} \in L^2(\mathbb{R})$ and $\phi^{(k)} \not\equiv 0$ for $k \in \{1, \ldots n\}$, then $\phi^{(k)}$ is a wavelet with k vanishing moments.

Since the Gaussian function θ is a smoothing function we can translate the scale-space notations into "wavelet language". The n-th derivative of the Gaussian function is a wavelet with n vanishing moments. Moreover, the convolution $L_{\theta^{(n)}}[f]$ is, up to the normalizing factor $a^{-\frac{1}{2}}$, a wavelet transform, where the parameters (x,t) correspond to (b,a). Since we are interested in zeros of $W_\psi[f]$ rsp. $L_{\theta^{(n)}}[f]$, the normalizing factor is not important and we can also speak about fingerprints of wavelet transforms.

It has been observed in many experiments and studied in various papers that fingerprints of the Gaussian kernel characterize features of the original signal f (cf. [4] for more references). In the following section we show how zeros of wavelet transforms can be used to reconstruct self-affine functions.

Reconstruction of fractal interpolation functions

During the last years, wavelet analysis has been applied to various kinds of fractal objects. In this paper, we consider fractal interpolation functions. A graph of a fractal interpolation function is an attractor of an IFS with affine maps. First, we introduce some notation and state some results concerning fractal interpolation function theory (see e.g. [1]).

For a compact set $D \subset \mathbb{R}$ we consider affine transforms $w_1, \ldots, w_N, w_i : [0,1] \times D \to [0,1] \times D$ represented by

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha_i & 0 \\ \gamma_i & \delta_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \epsilon_i \\ \zeta_i \end{pmatrix},$$

where $\alpha_i, \gamma_i, \delta_i, \epsilon_i, \zeta_i \in \mathbb{R}$ such that $0 < \alpha_i < 1$ and $|\delta_i| < 1$ for each i. Let $p_1 = (0, y_1)$ be the fixed point of w_1 and $p_{N+1} = (1, y_{N+1})$ the fixed point of w_N. Choose the parameters so that $w_i(p_{N+1}) = w_{i+1}(p_1)$ for all $i = 1, \ldots, N-1$, and that $w_1(p_1), \ldots, w_N(p_1), p_{N+1}$ are not all collinear. The choice of α_i and δ_i ensures that the maps w_i are contractions. The attractor of the IFS $= \{w_1, \ldots, w_N\}$ is the graph of a continuous function $f : [0,1] \to \mathbb{R}$, which is nowhere differentiable. The function f is called *fractal interpolation function* with interpolation points $w_1(p_1), \ldots, w_N(p_1), p_{N+1}$.

The following theorem (proof see [2]) explains the self-similarities in fingerprints of wavelet transforms of fractal interpolation functions.

Theorem 1. *Let f be a fractal interpolation function and $\mathbf{w} = \{w_1, \ldots, w_N\}$ an associated IFS with parameters $\alpha_i, \epsilon_i, \gamma_i, \delta_i, \zeta_i$. Furthermore, let ψ be a wavelet with compact support $[-K, K]$ and k vanishing moments, $k \geq 2$. Then for $(b, a) \in \mathbb{R} \times \mathbb{R}_+$ with $[-aK + b, aK + b] \subset [0, 1]$ the following equation holds:*

$$W_\psi[f](b, a) = \frac{1}{\delta_i \alpha_i} W_\psi[f](\alpha_i b + \epsilon_i, \alpha_i a) \tag{1}$$

for each $i = 1, \ldots, N$.

In the following, we assume that the wavelet ψ is the k-th ($k \geq 2$) derivative of a $(k+1)$-times differentiable smoothing function with compact support $[-K, K]$ and f a fractal interpolation function.

It can be shown that the set

$$S(\psi, f) = \{(b, a) \in \mathbb{R} \times \mathbb{R}_+ \mid W_\psi[f](b, a) = 0 \text{ and } \frac{\partial W_\psi[f]}{\partial b}(b, a) = 0\}$$

is discrete in scale-space.

If f is a fractal interpolation function one can show with the results from [5] and Theorem 1 that the set $S(\psi, f)$ is non-empty (proof see [2]). Since for given parameters α_i and ϵ_i the parameters γ_i, δ_i and ζ_i can be computed from three non-collinear points of the graph of f, the following lemma presents a solution to the reconstruction problem of fractal interpolation functions.

Lemma 2. *Let $\mathbf{w} = \{w_1, \ldots, w_N\}$ be an IFS with parameters $\alpha_i, \epsilon_i, \gamma_i, \delta_i, \zeta_i$ and f the corresponding fractal interpolation function. Furthermore, let $(b, a) \in S(\psi, f)$. Then there are points $(b_i, a_i) \in S(\psi, f)$, $i = 1, \ldots, N$, such that*

$$a_i = \alpha_i a, \; b_i = \alpha_i b + \epsilon_i, \; \sum_{i=1}^{N} \alpha_i = 1 \;\; \textit{and} \;\; \alpha_i + \epsilon_i = \epsilon_{i+1}. \tag{2}$$

For determing the set $S(\psi, f)$ we have to compute the wavelet transforms at all possible scales. Therefore, we can not use the algorithm presented in [5] which is based on dyadic wavelet transforms. At this point we encounter the link between scale-space and wavelet theory. We interpret a truncated derivative of the Gaussian function, $\theta^{(n)}$, as a good approximation of a wavelet with compact support which is the n-th derivative of a smoothing function. Furthermore, scaling theorems for Gaussian kernels allow an efficient computation of the set $S(\theta^{(n)}, f) \subset L_{\theta^{(n)}}$. The $S(\theta^{(n)}, f)$ is used to find the maps w_i by following the points of $S(\theta^{(n)}, f)$ from coarsest scale to finer scales. In contrast to the method presented in [6], only a finite set of zeros of wavelet transforms at selected scales is necessary to reconstruct the graph of the fractal interpolation function.

In Table 1, the result of an algorithm is presented which is based on a coarse-to-fine tracking of $S(\psi, f)$. We use the discretized second derivative of the Gaussian function as an approximation of the wavelet ψ. The analyzed fractal interpolation function has a corresponding IFS with equal scaling factors α_i. Examples with different scaling factors are analyzed in [2]. The errors in the computation are caused by the discretization of the scale parameter a.

Table 1. Theoretical parameters (*left*), numerical computations (*right*).

i	α_i	ϵ_i	γ_i	δ_i	ζ_i
1	1/3	0	42	0.5	0
2	1/3	1/3	-5	-0.5	50
3	1/3	2/3	-32.3	0.23	40

i	α_i	ϵ_i	γ_i	δ_i	ζ_i
1	0.334	0.000	42.615	0.538	0.000
2	0.334	0.332	-4.029	-0.518	50.089
3	0.334	0.666	-32.202	0.232	39.874

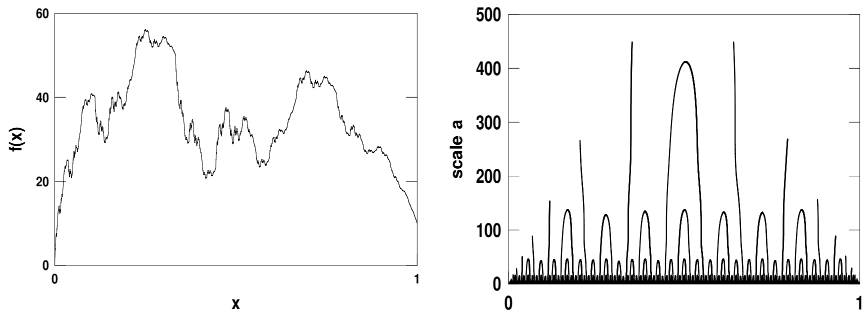

Fig. 1. *Left*: Graph of the fractal interpolation function from Table 1. *Right*: Corresponding fingerprints of $L_{\theta^{(2)}}[f]$.

Remarks. The presented results based on scale-space zeros yield a theoretical basis for applications in time series and image analysis. In the literature, fractal interpolation functions are used to model signals with almost self-affine properties. Therefore it is obvious to ask whether fingerprints of a given signal can be used to construct a self-affine function whose graph is close to the given signal. Further work is going on to answer this question.

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