L^2 Convergence of Biorthogonal Coifman Wavelet Systems

Jun Tian and Raymond O. Wells, Jr.

Abstract. Biorthogonal Coifman wavelet (BCW) systems are biorthogonal wavelet systems with the vanishing of moments equally distributed between the scaling functions and wavelet functions. It has been shown that these wavelet systems provide an optimal wavelet sampling approximation with an exponential convergence rate, where the optimality is measured over all possible vanishing moments distributions. The scaling filters and wavelet filters are all dyadic rational, which means we can implement a very fast multiplication-free discrete wavelet transform. Recent work showed that the scaling filters of BCW systems converge to the sinc scaling filter in $l^2(\mathbb{Z})$, while the $L^2(\mathbb{R})$ convergence of the scaling functions remained open. In this work we will prove the scaling functions of BCW systems converges to the sinc function in $L^2(\mathbb{R})$. Following the pioneering work of Chui and Wang, we establish a quantitative relationship between the orthonormal Daubechies wavelet systems and the BCW systems. Based on the convergence property of the Daubechies systems, we are able to show the $L^2(\mathbb{R})$ convergence of BCW systems.

§1. Introduction

Wavelet analysis (see [3,4,10,16,19,23,31,32], etc) has been proven to be a very powerful tool in harmonic analysis, neural networks, numerical analysis, and signal processing, especially in the area of image compression [20,27,29,33] and data denoising [8,13,14]. The theoretical work of orthogonal wavelets was done in the late eighties [9,15,16] and the framework of biorthogonal wavelets was established in the early nineties [5,7,30]. Wavelet theory is closely connected with subband coding, and it provides a functional space structure for subband coding, often leading to better understanding and tremendous improvement.

In this paper we will focus on biorthogonal Coifman wavelet (BCW) systems, a family of compactly supported biorthogonal wavelet systems with the vanishing of moments equally distributed between the scaling functions and wavelet functions. These wavelet systems provide a wavelet sampling approximation with an exponential convergence rate. Recent work ([19,26]) showed that the scaling filters of BCW systems converge to the sinc scaling filter in $l^2(\mathbb{Z})$, where the $L^2(\mathbb{R})$ convergence of the scaling functions remained open. We will show that the scaling functions of BCW systems converge to the sinc function in $L^2(\mathbb{R})$.

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§2. Biorthogonal Coifman Wavelet Systems

BCW systems are the generalization of orthogonal Coifman wavelet systems (or Coiflets, see [2,11]) to the biorthogonal setting. Here we give a brief review of BCW systems, for more details, we refer to [1,12,17,18,21–26,28].

Definition 1. *A biorthogonal wavelet system with compact support is called a* biorthogonal Coifman wavelet *(BCW) system of degree N if the following two conditions are satisfied,*

- *the vanishing moments of the synthesis scaling function $\tilde{\phi}(x)$ and wavelet function $\tilde{\psi}(x)$ are both of degree N,*

$$\begin{aligned} \mathrm{Mom}_p(\tilde{\phi}) &:= \int_{\mathbb{R}} x^p \tilde{\phi}(x)\, dx \;=\; 0, \qquad p = 1, \ldots, N, \\ \mathrm{Mom}_p(\tilde{\psi}) &:= \int_{\mathbb{R}} x^p \tilde{\psi}(x)\, dx \;=\; 0, \qquad p = 0, \ldots, N, \end{aligned}$$

- *the vanishing moment of the analysis wavelet function $\psi(x)$ is of degree N,*

$$\mathrm{Mom}_p(\psi) \;:=\; \int_{\mathbb{R}} x^p \psi(x)\, dx \;=\; 0, \qquad p = 0, \ldots, N.$$

Lemma 2. *For a biorthogonal Coifman wavelet system of degree N, the vanishing moment of the analysis scaling function $\phi(x)$ is also of degree N,*

$$\mathrm{Mom}_p(\phi) \;:=\; \int_{\mathbb{R}} x^p \phi(x)\, dx \;=\; 0, \qquad p = 1, \ldots, N.$$

Theorem 3. (Wavelet Sampling Approximation) *For $f(x) \in C^N(\mathbb{R})$ with compact support, and the N-th derivative $f^{(N)}(x)$ is Lipschitz, let $j \in \mathbb{N}$, define*

$$S^j(f)(x) \;:=\; \sum_{k \in \mathbb{Z}} f\left(\frac{k}{2^j}\right) \phi(2^j x - k),$$

where $\phi(x)$ is the analysis (or synthesis) scaling function of a biorthogonal Coifman wavelet system of degree N. Then

$$\left|\left|f(x) - S^j(f)(x)\right|\right|_{L^2} \;\leq\; C 2^{-j(N+1)},$$

where C is a constant, depending only on f and ϕ. If in addition, $\phi(x) \in C^n(\mathbb{R})$, then

$$\left|\left|f(x) - S^j(f)(x)\right|\right|_{H^n} \;\leq\; C 2^{-j(N+1-n)}.$$

It can be shown that (see, for example, [26]), for an optimal wavelet sampling approximation, vanishing moments should be equally distributed between the scaling function and the wavelet function. The closed form solution of the minimum length biorthogonal Coifman wavelet system of degree N is given by the scaling filters $\{a_k\}$ and $\{\tilde{a}_k\}$:

- if N is even, $N = 2n$,

$$a_{2k+1} \;=\; \tilde{a}_{2k+1} \;=\; \frac{(-1)^k(2n+1)}{2^{4n-1}(2k+1)}\binom{2n-1}{n-1}\binom{2n}{n+k}, \tag{1}$$

- if N is odd, $N = 2n-1$,

$$a_{2k+1} \;=\; \tilde{a}_{2k+1} \;=\; \frac{(-1)^k(2n-1)}{2^{4n-3}(2k+1)}\binom{2n-2}{n-1}\binom{2n-1}{n+k}, \tag{2}$$

and

$$\tilde{a}_{2k} = \delta_{0,k}, \tag{3}$$

$$a_{2k} = 2\delta_{0,k} - \sum_{l\in\mathbb{Z}} \tilde{a}_{2l+1}\tilde{a}_{2l+1-2k}. \tag{4}$$

In the remainder of this paper, a biorthogonal Coifman wavelet system will always be refered to the one with minimum length, which is given by the above formula, unless it is otherwise stated. An attractive feature of the formula is that all a_k's, $\tilde{a}_k$'s are either zero or a dyadic rational number (see [26] for proof), that is, an integer divided by a power of 2. Thus one can implement a very fast multiplication-free discrete wavelet transform, which consists of only addition and shift operations, on digital computers. Also from the above formula, when the degree N is odd, the BCW system is symmetric, i.e., $a_k = a_{-k}$, and $\tilde{a}_k = \tilde{a}_{-k}$. An interesting observation is that BCW systems and orthonormal Daubechies wavelet systems ([9,10]) are closely related by their scaling filters, more precisely,

$$\tilde{m}_{2n+1}(\xi) \;=\; \left|m_n^D(\xi)\right|^2, \tag{5}$$

where m is defined by the scaling filter $\{a_k\}$,

$$m(\xi) \;:=\; \frac{1}{2}\sum_{k\in\mathbb{Z}} a_k e^{-ik\xi},$$

and $\tilde{m}_{2n+1}$ corresponds to the synthesis part of the BCW system of degree $2n+1$, m_n^D corresponds to the orthonormal Daubechies wavelet system of degree n, whose wavelet function ψ_n^D has vanishing moments up to degree n,

$$\int_{\mathbb{R}} x^p \psi_n^D(x)\,dx \;=\; 0, \qquad p = 0, \ldots, n. \tag{6}$$

And one can prove (see [19,26]) that the scaling filters of BCW systems converge to the sinc scaling filter, which is defined by

$$a_{2k}^{\mathrm{sinc}} \;:=\; \delta_{0,k}, \qquad a_{2k+1}^{\mathrm{sinc}} \;:=\; \frac{(-1)^k 2}{(2k+1)\pi}.$$

The sinc scaling filter is the scaling filter of the sinc wavelet system, whose scaling function is exactly $\mathrm{sinc}(\pi x) = \frac{\sin \pi x}{\pi x}$.

Lemma 4. *Suppose $\{\tilde{a}_k^N, k \in \mathbb{Z}\}$ to be the synthesis scaling filter of the BCW system of degree N, then*

$$\lim_{N\to\infty} \left|\left|\tilde{a}^N - a^{\mathrm{sinc}}\right|\right|_{l^2} \;=\; \lim_{N\to\infty}\left(\sum_{k\in\mathbb{Z}}\left(\tilde{a}_k^N - a_k^{\mathrm{sinc}}\right)^2\right)^{1/2} \;=\; 0.$$

§3. L^2 Convergence Property

It had been conjectured (see [28]) for some time that the scaling functions of BCW systems converge to $\operatorname{sinc}(\pi x)$ in $L^2(\mathbb{R})$. Such convergence could also explain the good sampling approximation property of BCW systems (see Theorem 3).

3.1. Chui and Wang's Convergence Theorem

In [6], Chui and Wang proved the following theorem for *orthonormal* wavelet systems.

Theorem 5. (Chui-Wang) *Let ϕ_n be orthonormal scaling functions defined by*

$$\begin{aligned}
\phi_n(x) &= \sum_k a_{n,k}\phi_n(2x-k), \\
m_n(\xi) &= \frac{1}{2}\sum_k a_{n,k}e^{-ik\xi}, \\
m_n(\xi) &= \left(\frac{1+e^{-ik\xi}}{2}\right)^n \mathcal{L}_n(\xi), \\
\mathcal{L}_n(\pi) \neq 0 \quad &\text{and} \quad \mathcal{L}_n(0) = 1, \\
|\mathcal{L}_n(\xi)| \leq C2^n &\left|\sin^n\left(\frac{\xi}{2}\right)\right|, \qquad \frac{\pi}{2} \leq |\xi| \leq \pi,
\end{aligned}$$

for some absolute constant C (independent of n). Then

$$\lim_{n\to\infty}\left|\left|\hat{\phi_n}| - \chi_{(-\pi,\pi)}(\xi)\right|\right|_{L^2} = 0.$$

As an immediate consequence, let ϕ_n^D be an (arbitrarily chosen) orthonormal Daubechies scaling function of degree n (see Eqn. (6)), then

$$\lim_{n\to\infty}\left|\left|\hat{\phi_n^D}| - \chi_{(-\pi,\pi)}(\xi)\right|\right|_{L^2} = 0. \tag{7}$$

3.2. L^2 Convergence to $\operatorname{sinc}(\pi x)$

Since $\hat{\phi}(\xi) = \prod_{j=1}^{\infty} m(2^{-j}\xi)$, Eqn. (5) implies a relationship between the synthesis scaling functions of BCW systems and orthonormal Daubechies scaling functions,

$$\hat{\tilde{\phi}}_{2n+1}(\xi) = \left|\hat{\phi_n^D}(\xi)\right|^2.$$

Thus

$$\begin{aligned}
\left|\hat{\tilde{\phi}}_{2n+1}(\xi) - \chi_{(-\pi,\pi)}(\xi)\right| &= \left|\left|\hat{\phi_n^D}(\xi)\right|^2 - \chi_{(-\pi,\pi)}(\xi)\right| \\
&= \left|\left(|\hat{\phi_n^D}| + \chi_{(-\pi,\pi)}(\xi)\right)\cdot\left(|\hat{\phi_n^D}| - \chi_{(-\pi,\pi)}(\xi)\right)\right| \\
&\leq 2\left||\hat{\phi_n^D}| - \chi_{(-\pi,\pi)}(\xi)\right|,
\end{aligned}$$

since $|\hat{\phi_n^D}| \leq 1$. By Eqn. (7) and Parseval's identity, it follows

$$\begin{aligned}\lim_{n\to\infty}\left|\left|\tilde{\phi}_{2n+1}(x) - \frac{\sin\pi x}{\pi x}\right|\right|_{L^2} &= (2\pi)^{-1/2}\lim_{n\to\infty}\left|\left|\hat{\tilde{\phi}}_{2n+1}(\xi) - \chi_{(-\pi,\pi)}(\xi)\right|\right|_{L^2} \\ &= 0.\end{aligned}$$

For the even degree part, from Eqns (1) and (2), we have

$$\tilde{a}_{2n,2k+1} = \tilde{a}_{2n+1,2k+1} + \frac{(-1)^k}{2^{4n+1}}\binom{2n}{n}\binom{2n+1}{n+k+1},$$

where $\{\tilde{a}_{2n,k}|k \in \mathbb{Z}\}$ and $\{\tilde{a}_{2n+1,k}|k \in \mathbb{Z}\}$ are the synthesis scaling filters of BCW systems of degree $2n$ and $2n+1$, respectively. By summing up, one gets

$$\tilde{m}_{2n}(\xi) = \tilde{m}_{2n+1}(\xi) - \frac{(2n-1)!!}{(2n)!!} i \sin^{2n+1}\xi,$$

where

$$(2n-1)!! := \prod_{k=1}^{n}(2k-1), \qquad (2n)!! := \prod_{k=1}^{n}(2k) = 2^n n!.$$

Notice that

$$1 - C\sin^{2n}(\xi) \leq \left|m_n^D(\xi)\right|^2 \leq 1, \qquad \text{for } |\xi| \leq \frac{\pi}{2},$$

for some absolute constant C, which is independent of n. Choose a positive integer k such that $|\xi/2^k| < \frac{\pi}{2}$, then

$$1 - C\sin^{2n}(\xi/2^k) \leq \tilde{m}_{2n+1}(\xi/2^k) \leq 1.$$

So

$$\begin{aligned}\left|\tilde{m}_{2n}(\xi/2^k)\right| &= \left|\tilde{m}_{2n+1}(\xi/2^k) - \frac{(2n-1)!!}{(2n)!!} i \sin^{2n+1}(\xi/2^k)\right| \\ &= \left|\tilde{m}_{2n+1}(\xi/2^k)\right| \cdot \left|1 - \frac{(2n-1)!!}{(2n)!!} i \frac{\sin^{2n+1}(\xi/2^k)}{\tilde{m}_{2n+1}(\xi/2^k)}\right|.\end{aligned}$$

Since

$$\begin{aligned}\left|\frac{(2n-1)!!}{2n!!} i \frac{\sin^{2n+1}(\xi/2^k)}{\tilde{m}_{2n+1}(\xi/2^k)}\right| &\leq \left|\frac{\sin^{2n+1}(\xi/2^k)}{\tilde{m}_{2n+1}(\xi/2^k)}\right| \leq \left|\frac{\sin^{2n+1}(\xi/2^k)}{1 - C\sin^{2n}(\xi/2^k)}\right| \\ &\leq \left|C\sin^{2n+1}(\xi/2^k)\right| \leq \left|C(\xi/2^k)^{2n+1}\right|,\end{aligned}$$

for n large enough, where C is an absolute constant independent of n. Thus

$$\lim_{n\to\infty}\left(\prod_{j=k+1}^{\infty}\left|1 - \frac{(2n-1)!!}{(2n)!!} i \frac{\sin^{2n+1}(\xi/2^j)}{\tilde{m}_{2n+1}(\xi/2^j)}\right|\right) = 1.$$

This implies

$$\begin{aligned}
\lim_{n\to\infty}\left|\hat{\tilde{\phi}}_{2n}(\xi)\right| &= \lim_{n\to\infty}\left(\prod_{j=1}^{\infty}\left|\tilde{m}_{2n}(\xi/2^j)\right|\right) \\
&= \lim_{n\to\infty}\left(\prod_{j=1}^{k}\left|\tilde{m}_{2n}(\xi/2^j)\right| \cdot \prod_{j=k+1}^{\infty}\left|\tilde{m}_{2n}(\xi/2^j)\right|\right) \\
&= \lim_{n\to\infty}\left(\prod_{j=1}^{k}\left|\tilde{m}_{2n+1}(\xi/2^j)\right| \cdot \prod_{j=k+1}^{\infty}\left|\tilde{m}_{2n+1}(\xi/2^j)\right|\right) \\
&= \lim_{n\to\infty}\left|\hat{\tilde{\phi}}_{2n+1}(\xi)\right| \;=\; \chi_{(-\pi,\pi)}(\xi).
\end{aligned}$$

Now by taking $\{2\hat{\tilde{\phi}}_{2n+1}, n \in \mathbb{Z}\}$ as a dominating sequence, the sequence $\{\hat{\tilde{\phi}}_{2n}, n \in \mathbb{Z}\}$ converge to $\chi_{(-\pi,\pi)}$ in L^2. By using the concept of *asymptotically linear phase* (see [6]), the sequence $\{\tilde{\phi}_{2n}, n \in \mathbb{Z}\}$ converge to $\frac{\sin \pi x}{\pi x}$ in $L^2(\mathbb{R})$.

For the analysis scaling function $\phi_n(x)$, again, based on Eqns (1), (2), and (4), we have

$$\begin{aligned}
m_n(\xi) = &\ \frac{1}{2}\sum_{k\in\mathbb{Z}}\left((2\delta_{0,k} - \sum_{l\in\mathbb{Z}}\tilde{a}_{n,2l+1}\tilde{a}_{n,2l+1-2k})e^{-2ik\xi} + \tilde{a}_{n,2k+1}e^{-i(2k+1)\xi}\right) \\
= &\ 2\tilde{m}_n(\xi) + \tilde{m}_n(-\xi) - 2\tilde{m}_n(\xi)\tilde{m}_n(-\xi).
\end{aligned}$$

Now one can apply the previous approach to prove the convergence.

Thus we derive

Theorem 6. *Let $\tilde{\phi}_n(x)$ be the synthesis (or analysis) scaling function of biorthogonal Coifman wavelet system of degree n, then*

$$\lim_{n\to\infty}\left|\left|\tilde{\phi}_n(x) - \frac{\sin \pi x}{\pi x}\right|\right|_{L^2} \;=\; 0.$$

§4. Conclusions

We proved the scaling functions of BCW systems converge to $\text{sinc}(\pi x)$ in $L^2(\mathbb{R})$. This provides a family of smooth wavelet systems to approximate the sinc wavelet system. It can also be shown that the wavelet functions of BCW systems converge to the difference of two sinc functions. An open question concerning Coiflets is the existence of large degree Coiflets, and there is no explicit formula for Coiflets. If Coiflets do exist for large degrees, one would expect that they possess the same convergence property as of BCW systems.

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