Translation Invariant Multiwavelet Transforms*

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June 17, 1998

Abstract

A translation invariant multiwavelet transform is introduced. It is shown that the associated correlation matrices satisfy a matrix-valued refinement equation implying a hidden basis wavelet representation. Properties of the correlation matrices such as regularity, approximation order, and vanishing moments are investigated. The results are illustrated for the DGHM multiwavelet.

Keywords and Phrases: Refinable function vectors, multiwavelet transform, correlation functions.

*Technical Report CML TR 98-06, Computational Mathematics Laboratory, Rice University.

†Partially supported by DARPA/AFOSR F49620-97-1-0513, and Alexander von Humboldt-Stiftung.

‡Partially supported by NSF grant DMS-9503282.

1 Introduction

In recent years wavelet analysis has become a powerful analyzing and synthesizing tool in pure and applied mathematics. The ability of wavelets to resolve different scales and to transfer information back and forth between these scales has been successfully applied to signal processing, data and image compression [4, 25]. The behavior of the continuous or discrete wavelet transform at different levels of resolution is one of the key features of the theory. The continuous wavelet transform gives a highly redundant two-dimensional representation of a function whereas the discrete (orthogonal) transform yields a more efficient representation in an appropriate sequence space.

In the theory of denoising a hybrid between the redundant continuous wavelet transform and the discrete wavelet transform was introduced which computes all shifts of an orthogonal wavelet transform and, therefore, keeps the advantage of fast computations in applications. In addition, a better approximation order is obtained. (See, for instance, [3, 6, 16].) This hybrid transform is based on translation invariance, a property not shared by the continuous or discrete wavelet transform.

More recently, multiwavelets have improved the performance of wavelets for several applications by providing added flexibility [26]. Multiwavelets are bases of $L^2(\mathbb{R})$ consisting of more than one base function or generator [25]. One of the advantages of multiwavelets is that unlike in the case of a single wavelet, the regularity and approximation order can be improved by increasing the number of generators instead of lengthening the support. These additional generators then provide more flexibility in approximating a given function.

In this article some of the results obtained in [3] are generalized to multiwavelets. The emphasis will be on regularity, approximation order, and vanishing moments. The existence of more than one generator adds a new feature to the representation of a function in terms of so-called redundant projectors. Whereas in the case of a single wavelet this redundant representation depends explicitly only on the autocorrelation functions, the cross-correlation functions enter implicitly into the representation if more than one wavelet is used. It will be seen that this is a direct consequence of the matrix-valued refinement equation satisfied by the correlation functions associated with a multiwavelet.

The structure of this article is as follows. In Section 2 a brief review of multiwavelet theory is provided, the relevant terminology and notation is introduced. Section 3 contains the main results on the refinement equation, the regularity, approximation order, and the number of vanishing moments for a translation invariant multiwavelet system. These results are then applied to a particular example, namely the DGHM multiwavelet system, in Section 4. Concluding remarks are made in the last section.

2 Preliminaries on multiwavelets

In this section we give a brief review of the theory of multiwavelets. For a more detailed presentation of multiwavelets, the reader is referred to the references given in the bibliography [11, 13, 14, 19, 25].

A finite collection of L^2-functions $\Psi := (\psi_1, \ldots, \psi_r)^T$ is called a *multiwavelet* if the two-parameter family $\{\Psi_{jk} := 2^{j/2}\Psi(2^j \cdot - k) : \ j, k \in \mathbb{Z}\}$ forms an orthonormal, or more generally, a Riesz basis of $L^2(\mathbb{R})$.

One way to construct a multiwavelet is through multiresolution analysis, which consists of a nested sequence $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$, of closed subspaces of $L^2(\mathbb{R})$ with the property that the closure of their union is $L^2(\mathbb{R})$ and their intersection is the trivial subspace $\{0\}$. Furthermore, each subspace V_j is spanned by the dyadic dilates and integer translates of a finite set of *scaling functions* $\{\phi_i : \ i = 1, \ldots, r\}$, sometimes also called the *generators* of the multiresolution analysis. Typically, the *scaling vector* $\Phi := (\phi_1, \ldots, \phi_r)^T$ has compact support or decays rapidly enough at infinity. (Here, the support of a scaling vector Φ is defined as the union of the supports of its individual components.) For $r = 1$ we obtain the classical wavelet systems as defined and discussed in, for instance, [5, 9, 21].

The condition that the spaces V_j be nested implies that the scaling vector Φ satisfies the following *two-scale matrix dilation equation* or *matrix refinement equation*

$$\Phi(x) = \sum_{k \in \mathbb{Z}} P(k)\, \Phi(2x - k), \tag{2.1}$$

where the *filter coefficient matrices* $\{P(k)\}_{k\in\mathbb{Z}}$ are $r \times r$ matrices satisfying $\sum_{k\in\mathbb{Z}} \|P(k)\|_{\ell^2(\mathbb{R}^{r\times r})} < \infty$. Denoting the L^2-orthogonal complement of V_j in V_{j+1} by W_j, it can be shown that there exists a multiwavelet Ψ such that W_j is spanned by the dyadic dilates and integer translates of Ψ. Moreover, a multiwavelet satisfies a two-scale matrix dilation equation of the form

$$\Psi(x) = \sum_{k\in\mathbb{Z}} Q(k)\Phi(2x-k), \tag{2.2}$$

where the $r \times r$ matrices $\{Q(k)\}_{k\in\mathbb{Z}}$ are in $\ell^2(\mathbb{R}^{r\times r})$.

The pair (Φ, Ψ) will be called a *multiwavelet system.* As our emphasis is entirely on compactly supported and orthogonal multiwavelets, we assume that scaling vectors and multiwavelets satisfy the following conditions.

- Both Φ and Ψ have compact support. (This implies that the sums in Eqns. (2.1) and (2.2) are finite).

- The scaling vectors and multiwavelets are fully L^2-orthogonal in the following sense:

$$\begin{aligned} \langle \Phi(2^j\ \cdot - k), \Phi(2^j\ \cdot - \ell)\rangle &= \delta_{k\ell} I_r, &(2.3)\\ \langle \Psi(2^j\ \cdot - k), \Psi(2^m\ \cdot - \ell)\rangle &= \delta_{jm,k\ell} I_r, &(2.4)\\ \langle \Phi(2^j\ \cdot - k), \Psi(2^m\ \cdot - \ell)\rangle &= O_r, &(2.5)\end{aligned}$$

  where I and O denotes the identity and zero matrix, respectively. Here we defined the inner product of two vector-valued functions F and G by $\langle F, G\rangle := \int_{\mathbb{R}} F(x)\, G^T(x)\, dx$. (The transpose of a matrix is denoted by ^{T}.)
  In terms of the filter coefficient matrices the above orthogonality conditions read

$$\begin{aligned} \sum_{k\in\mathbb{Z}} P(k)P^T(k+2\ell) &= \delta_{0\ell} I_r &(2.6)\\ \sum_{k\in\mathbb{Z}} Q(k)Q^T(k+2\ell) &= \delta_{0\ell} I_r &(2.7)\\ \sum_{k\in\mathbb{Z}} P(k)Q^T(k+2\ell) &= O_r, \quad \text{for all } \ell \in \mathbb{Z}. &(2.8)\end{aligned}$$

Since $V_{j+1} = V_j \oplus W_j$, every function $f_{j+1} \in V_{j+1}$ can be *decomposed* into an “averaged” or “blurred” component $f_j \in V_j$ and a “difference” or “fine-structure” component $g_j \in W_j$:

$$f_{j+1} = f_j + g_j.$$

This decomposition can be continued until f_{j+1} is decomposed into a coarsest component f_0 and j difference components g_m, $m = 1, \ldots, j$:

$$f_{j+1} = f_0 + g_1 + \ldots + g_j. \tag{2.9}$$

This *decomposition algorithm* can be reversed to give a *reconstruction algorithm*: Given the coarse components together with the fine structure components one reconstructs any $f_j \in V_j$ via reversal of Eqn. (2.9). Let us note that both algorithms are usually applied to the expansion coefficients (in terms of the underlying basis) of f and g and that they involve the matrices $P(k)$ and $Q(k)$. More precisely, the decomposition algorithm applied to $f \in V_j$ gives

$$\begin{aligned} f_j &= \sum_k \langle f, \Phi_{jk} \rangle \, \Phi_{jk} \\ &= \sum_k \langle f, \Phi_{j-1,k} \rangle \, \Phi_{j-1,k} + \sum_k \langle f, \Psi_{j-1,k} \rangle \, \Psi_{j-1,k}, \end{aligned}$$

where the inner products $\langle f, \Phi_{jk} \rangle$, $\langle f, \Phi_{j-1,k} \rangle$, and $\langle f, \Psi_{j-1,k} \rangle$ are related via

$$\langle f, \Phi_{jk} \rangle = \sum_m P(m-2k) \, \langle f, \Phi_{j-1,k} \rangle \tag{2.10}$$

and

$$\langle f, \Psi_{j-1,k} \rangle = \sum_m Q(m-2k) \, \langle f, \Phi_{jk} \rangle. \tag{2.11}$$

The reconstruction algorithm applied to a function $f_{j-1} \in V_{j-1}$, $f_{j-1} = \sum_k \langle f_{j-1}, \Phi_{j-1,k} \rangle \, \Phi_{j-1,k}$ and $g_{j-1} \in W_{j-1}$, $g_{j-1} = \sum_k \langle f_{j-1}, \Phi_{j-1,k} \rangle \, \Psi_{j-1,k}$ yields

$$\langle f_j, \Phi_{jk} \rangle = \sum_m P(m-2k)^T \, \langle f_{j-1}, \Phi_{j-1,k} \rangle + Q(m-2k)^T \, \langle f_{j-1}, \Psi_{j-1,k} \rangle. \tag{2.12}$$

As a consequence of the decomposition algorithm, any function $f \in L^2(\mathbb{R})$ may be represented as a *multiwavelet series* of the form

$$\begin{aligned} f &= \sum_{k \in \mathbb{Z}} \sum_{i=1}^{r} \langle f, \phi_{i,jk} \rangle \, \phi_{i,jk} \; + \sum_{m \geq j} \sum_{k \in \mathbb{Z}} \sum_{i=1}^{r} \langle f, \psi_{i,mk} \rangle \, \psi_{i,mk} && (2.13) \\ &= \mathcal{P}^j[f] + \sum_{m \geq j} \mathcal{Q}^m[f], && (2.14) \end{aligned}$$

where $\mathcal{P}^j$ and $\mathcal{Q}^j$ denote the orthogonal projectors of $L^2(\mathbb{R})$ onto V_j and W_j, $j \in \mathbb{Z}$, respectively. Here we defined $\phi_{i,jk} = 2^{j/2}\phi_i(2^j \cdot -k)$ and $\psi_{i,jk} = 2^{j/2}\psi_i(2^j \cdot -k)$.

For approximation-theoretic purposes, the spaces V_j are usually required to reproduce polynomials up to a certain degree $D-1$. As the multiwavelet space W_j is orthogonal to V_j, the first D moments of the multiwavelet Ψ vanish

$$\int_{\mathbb{R}} x^p \Psi(x)\, dx = 0, \qquad p = 0, \ldots, D-1. \tag{2.15}$$

Such a multiwavelet system will be called a *multiwavelet system of order* D. For the remainder of this paper, we assume that we always deal with a multiwavelet system of order $D > 0$. Note that if f is a polynomial of degree at most $D-1$, then its representation (2.14) reduces to $f = \mathcal{P}^j[f]$. In the case $r = 1$ this in particular implies that the span of ϕ contains all polynomials of degree $< D$. For $r > 1$, the span of each individual scaling function ϕ_i may in general not contain all such polynomials. (See [17, 23] for examples and details.)

In general, the projection $\mathcal{P}^j[f]$ is at least as smooth as the most irregular component of the scaling vector Φ. In particular, if ϕ_i is in the Sobolev space $H^\alpha(\mathbb{R})$ for $i = 1, \ldots, r$ then $\mathcal{P}^j[f] \in H^\alpha(\mathbb{R})$ for each $j \in \mathbb{Z}$.

It is well-known that a multiwavelet system of order D satisfies the following Jackson-type inequality.

Lemma 2.1 *Suppose that $f \in C^n(\mathbb{R})$, $1 \le n \le D$, is compactly supported. Then*

$$\| f - \mathcal{P}^j[f] \|_{L^2} \le C\, 2^{-jn}, \tag{2.16}$$

for a positive constant C independent of j and n.

The exact relationship between the reproduction of polynomials by the integer shifts of Φ and the L^p-approximation order of $\mathcal{P}^j$ is discussed in [20].

3 Hidden basis multiwavelet representations

In this section we study autocorrelation functions of multiwavelet systems to extend results presented in [3]. In particular, we consider properties such as translation invariance, regularity, and vanishing moments.

3.1 Translation Invariance

A disadvantage of the orthogonal wavelet and multiwavelet transform is its lack of translation invariance. In the case $r = 1$, this fact is overcome by considering all continuous shifts of the orthogonal wavelet transform. This naturally leads to redundant representations of $L^2(\mathbb{R})$ functions [3, 6]. Here we extend the approach presented in [3] to the case $r > 1$, i.e., multiwavelets.

For $h \in \mathbb{R}$, we denote the *continuous shift operator (by h)* on $L^2(\mathbb{R})$ by $\mathcal{S}_h$. Associated with $\mathcal{S}_h$, we introduce *redundant projectors* for functions $f \in L^2(\mathbb{R})$ by

$$\mathcal{P}_R^j[f] := 2^j \int_0^{2^{-j}} \mathcal{S}_{-h}\mathcal{P}^j\mathcal{S}_h[f]\,dh \tag{3.1}$$

$$\mathcal{Q}_R^j[f] := 2^j \int_0^{2^{-j}} \mathcal{S}_{-h}\mathcal{Q}^j\mathcal{S}_h[f]\,dh, \tag{3.2}$$

where $\mathcal{P}^j$ and $\mathcal{Q}^j$ are the orthogonal projectors defined in (2.14).

Lemma 3.1 *The redundant projectors $\mathcal{P}_R^j$ and $\mathcal{Q}_R^j$, $j \in \mathbb{Z}$, are translation invariant, i.e.,*

$$\mathcal{P}_R^j[f(\cdot + \delta)](y) = \mathcal{P}_R^j[f](y + \delta) \text{ and} \tag{3.3}$$

$$\mathcal{Q}_R^j[f(\cdot + \delta)](y) = \mathcal{Q}_R^j[f](y + \delta), \tag{3.4}$$

for all $\delta \in \mathbb{R}$, and yield the following representation of a function $f \in L^2(\mathbb{R})$.

$$\begin{aligned} f &= 2^j \sum_{i=1}^{r} \int_{\mathbb{R}} \varrho_{ii}(2^j(x - \cdot))\, f(x)\,dx \\ &\quad + \sum_{m\geq j} \sum_{i=1}^{r} 2^m \int_{\mathbb{R}} \tau_{ii}(2^m(x - \cdot))\, f(x)\,dx \qquad (3.5) \\ &= \mathcal{P}_R^j[f] + \sum_{m \geq j} \mathcal{Q}_R^m[f], \qquad (3.6) \end{aligned}$$

where ϱ_{ii} and τ_{ii} are the autocorrelation functions of the components of the scaling vector Φ and the multiwavelet Ψ, i.e.,

$$\varrho_{ii}(x) := \int_{\mathbb{R}} \phi^i(y)\phi^i(y + x)\,dx \tag{3.7}$$

$$\tau_{ii}(x) := \int_{\mathbb{R}} \psi^i(y)\psi^i(y + x)\,dx. \tag{3.8}$$

Proof. The proof proceeds along the same lines as in [3]. The details are left to the reader. □

Remark 3.1 *In the next subsection we show that, in order to obtain a refinement equation for the autocorrelation functions ϱ_{ii} and τ_{ii}, the cross-correlation functions ϱ_{ij} and τ_{ij} are needed. For this reason, we refer to the above representation (3.5) as a* autocorrelation transformation *or a* hidden basis multiwavelet representation.

3.2 Refinement Equation

In the case $r = 1$, the autocorrelation functions satisfy a refinement equation where the filter coefficients are the so-called *à-trous filters* [2, 15, 22, 24]. In the following lemma we derive a refinement equation for the correlation functions of a scaling vector Φ.

Theorem 3.1 *Let*

$$\Theta(x) := \int_{\mathbb{R}} \Phi(y)\Phi^T(y+x)\, dy \tag{3.9}$$

be the $r \times r$ correlation matrix of the scaling vector Φ. The elements of Θ are the correlation functions of the component functions of Φ, i.e.,

$$\varrho_{ij}(x) := \int_{\mathbb{R}} \phi^i(y)\phi^j(y+x)\, dy, \quad i,j = 1,\ldots,r. \tag{3.10}$$

Then Θ satisfies a matrix-valued refinement equation *of the form*

$$\Theta(x) = \frac{1}{2} \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} P(k+\ell)\, \Theta(2x-\ell)\, P^T(k). \tag{3.11}$$

Proof. Employing the refinement equation (2.1) for Φ, we obtain

$$\begin{aligned} \Phi(x)\Phi^T(x) &= \left[\sum_m P(m)\Phi(2x-m)\right]\left[\sum_\ell P(m)\Phi(2x-\ell)\right]^T \\ &= \sum_{m,\ell} P(m)\, \Phi(2x-m)\, \Phi^T(2x-\ell)\, P^T(\ell). \end{aligned}$$

Therefore,

$$
\begin{aligned}
\Theta(x) &= \int_{\mathbb{R}} \Phi(y)\Phi^T(y+x)\,dy \\
&= \int_{\mathbb{R}} \sum_{m,\ell} P(m)\,\Phi(2x-m)\,\Phi^T(2x+2y-\ell)\,P^T(\ell)\,dy \\
&= \sum_{m,\ell} P(m)\cdot\frac{1}{2}\left[\int_{\mathbb{R}} \Phi(x-m)\Phi^T(x+2y-\ell)\,dy\right] P^T(\ell) \\
&= \sum_{\ell,k} P(\ell+k)\cdot\frac{1}{2}\left[\int_{\mathbb{R}} \Phi(x)\Phi^T(x+2y-k)\,dy\right] P^T(\ell) \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}} P(k+\ell)\,\Theta(2x-\ell)\,P^T(k).
\end{aligned}
$$

The elements $(\Theta(x))_{ij}$ of the matrix $\Theta(x)$ are the inner products $\rho_{ij}(x) = \int_{\mathbb{R}} \phi^i(y)\phi^j(y+x)\,dy$, $i,j = 1,\ldots,r$. $\square$

The lack of commutativity in the algebra of matrices does not allow us to write Eqn. (3.11) in the usual form (2.1). However, regarding the $r\times r$ matrix Θ as a column vector Γ of length r^2, we may obtain such a form. More precisely, let

$$
\Gamma = \begin{pmatrix} \varrho_{11} & \ldots & \varrho_{1r} & \ldots & \varrho_{r1} & \ldots & \varrho_{rr} \end{pmatrix}^T. \tag{3.12}
$$

Note that the right-hand side of Eqn. (3.11) may be interpreted as a linear operator acting on Θ. Hence, there exists a finite sequence of $r^2\times r^2$ matrices $A(k)$ such that

$$
\Gamma(x) = \sum_k A(k)\Gamma(2x-k). \tag{3.13}
$$

Remark 3.2 *In the case $r=1$, the refinement equations (3.11) and (3.12) reduce to the same classical refinement equation. For $r>1$ the dimension of the second refinement equation increases quadratically in r.*

In this context, the purpose of choosing the expression *hidden basis multiwavelet representation* in Remark (3.1) becomes transparent. Although only the autocorrelation functions ϱ_{ii} appear explicitly in the representation of a function f, the cross-correlation functions enter implicitly through the above matrix-valued refinement equation. The incorporation of hidden functions into a recursive definition of the autocorrelation functions is reminiscent of the construction of so-called *hidden-variable fractal functions* [1].

Analog to the definition of Θ, we define a correlation matrix associated with the multiwavelet Ψ

$$\Xi(x) := \int_{\mathbb{R}} \Psi(y)\Psi^T(y+x)\,dy \tag{3.14}$$

satisfying a refinement equation of the form

$$\Xi(x) = \frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}} Q(k+\ell)\,\Xi(2x-\ell)\,Q^T(k). \tag{3.15}$$

The pair (Θ, Ξ) is called a *translation invariant multiwavelet system.*

An important feature of autocorrelation functions $\alpha_\phi(s) = \langle\phi, \phi(\cdot + s)\rangle$ in classical wavelet theory is their *interpolation property* [10, 12]. This property is equivalent to $\alpha_\phi(n) = \delta_{0n}$, $n \in \mathbb{Z}$, which implies that the function values of α_ϕ at dyadic rationals may be computed exactly using the refinement equation for α_ϕ. The following proposition shows that the interpolation property also holds for $r > 1$.

Proposition 3.1 *The correlation matrix $\Theta(x)$ is skewsymmetric in x and interpolatory, i.e.,*

$$\begin{aligned} \Theta_{ij}(x) &= \Theta_{ji}(-x), \\ \Theta(n) &= \delta_{0n} I_r, \qquad n \in \mathbb{Z}. \end{aligned}$$

Proof. The skewsymmetry of $\Theta(x)$ follows directly from the skewsymmetry of the cross-correlation functions $\varrho_{ij}(x)$.
The orthogonality condition (2.3) implies that $\Theta(0) = I_r$ and $\Theta(n) = O_r$, for all $0 \neq n \in \mathbb{Z}$. $\square$

Remark 3.3 *It is worthwhile pointing out that the elements ϱ_{ij} of Θ are in general not interpolatory.*

3.3 Regularity and Moments

The regularity of classical wavelet systems and their property to reproduce polynomials are fundamental to many applications such as compression and denoising. For that reason we like to investigate the regularity and approximation properties of translation invariant multiwavelet systems. For this purpose, we first review some regularity properties of multiwavelet systems.

The regularity of a multiwavelet systems depends on the matrices $P(k)$ in the refinement equation (2.1). More precisely, in the Fourier domain this refinement equation is by

$$\widehat{\Phi}(u) = \boldsymbol{P}\left(\frac{u}{2}\right) \widehat{\Phi}\left(\frac{u}{2}\right),$$

where the *symbol* or *refinement mask* $\boldsymbol{P}$ is defined by

$$\boldsymbol{P}(u) = \frac{1}{2}\sum_k P(k)\, e^{-iuk}. \tag{3.16}$$

The regularity of Φ depends on the decay rate of the infinite product

$$\widehat{\Upsilon}(u) = \lim_{n\to\infty}\left[\boldsymbol{P}\left(\frac{u}{2}\right)\cdots\boldsymbol{P}\left(\frac{u}{2^n}\right)\widehat{\Phi}(0)\right]. \tag{3.17}$$

In [7] it is shown that the above limit exists and that the finite product $\Pi_n(u)\widehat{\Phi}(0) = \boldsymbol{P}\left(\frac{u}{2}\right)\cdots\boldsymbol{P}\left(\frac{u}{2^n}\right)\widehat{\Phi}(0)$ converges pointwise for all $u \in \mathbb{R}$ and uniformly on compact sets. In particular, under the assumptions

$$\|\boldsymbol{P}(u) - \boldsymbol{P}(0)\| \le C\,|u|^\alpha \quad \text{and} \quad \|\boldsymbol{P}(0)\| < 2^\alpha$$

for some $\alpha > 0$, the following estimate is obtained

$$\|\Pi_k(u)\| \le e^{C_\alpha |u|^\alpha}\,\|\boldsymbol{P}(0)\|^k \tag{3.18}$$

and implies that

$$\|\Pi_{N+k}(u)\widehat{\Phi}(0) - \Pi_N(u)\widehat{\Phi}(0)\| \le C'\,e^{C_\alpha|u|^\alpha(1+2^{-N\alpha})}\,|u|^\alpha\left(\frac{\|\boldsymbol{P}(0)\|}{2^\alpha}\right). \tag{3.19}$$

The latter estimate is used to prove the decay of $\widehat{\Upsilon}(u)$ [7].

Theorem 3.2 *Let* $\boldsymbol{P}$ *be an* $r\times r$ *matrix of the form*

$$\boldsymbol{P}(u) = \frac{1}{2^m}\,C_0(2u)\cdots C_{m-1}(2u)\,\boldsymbol{P}^{(m)}(u)\,C_{m-1}(u)^{-1}\cdots C_0(u)^{-1}, \tag{3.20}$$

where the C_i are certain $r \times r$ matrices and $\boldsymbol{P}^{(m)}$ is an $r \times r$ matrix with trigonometric polynomials as entries. Suppose that the spectral radius of $\boldsymbol{P}^{(m)}(0)$ is strictly less than two. For $k \geq 1$, let

$$\gamma_k = \frac{1}{k} \log_2 \sup_u \left\| \boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \cdots \boldsymbol{P}^{(m)}\left(\frac{u}{2^k}\right) \right\|. \tag{3.21}$$

Then there exists a positive constant C such that for all $u \in \mathbb{R}$

$$\|\widehat{\Upsilon}(u)\| \leq C\,(1 + |u|)^{-m+\gamma_k}. \tag{3.22}$$

The precise matrix factorization (3.20) may be found in [7, 20]. As in the case $r = 1$ the rate of decay m determines the smoothness of the components of Φ in terms of Sobolev norms (see [7] for more details).

Next we use the above results to obtain estimates on the regularity of translation invariant multiwavelet systems. For this purpose, note that the Fourier transform of Eqn. (3.11) is given by

$$\widehat{\Theta}(u) = \boldsymbol{P}\left(\frac{u}{2}\right)\,\widehat{\Theta}\left(\frac{u}{2}\right)\boldsymbol{P}\left(\frac{u}{2}\right)^T. \tag{3.23}$$

Lemma 3.2 *Suppose that $\boldsymbol{P}$ satisfies*

$$\|\boldsymbol{P}(u) - \boldsymbol{P}(0)\| \leq C\,|u|^\alpha \tag{3.24}$$

for some $\alpha > 0$ and that

$$\|\boldsymbol{P}(0)\| < 2^\alpha. \tag{3.25}$$

Then the infinite product

$$\widehat{\Omega}(u) = \lim_{n\to\infty} \Pi_n(u)\,\widehat{\Theta}(0)\,\Pi_n(u)^T \tag{3.26}$$

converges pointwise for all $u \in \mathbb{R}$ and uniformly on compact subsets.

Proof. The proof is very similar to that of Lemma 3.1 in [7]. The orthogonality of Φ (2.3) implies that $\widehat{\Theta}(0) = I_r$ and, therefore, $\Pi_k(u)\widehat{\Theta}(0)\Pi_k^T - \widehat{\Theta}(0) = \Pi_k(u)\Pi_k(u)^T - I_r$. Using the estimate (3.18) we obtain

$$\begin{aligned}\Pi_k(u)\Pi_k(u)^T - I_r \;=\; & \left(\sum_{\ell=1}^{k} \boldsymbol{P}\left(\frac{u}{2}\right)\cdots\boldsymbol{P}\left(\frac{u}{2^{\ell-1}}\right)\left[\boldsymbol{P}\left(\frac{u}{2^\ell}\right) - \boldsymbol{P}(0)\right]\boldsymbol{P}(0)^{k-\ell}\right)\cdot \\ & \left(\sum_{\ell=1}^{k} \boldsymbol{P}\left(\frac{u}{2}\right)\cdots\boldsymbol{P}\left(\frac{u}{2^{\ell-1}}\right)\left[\boldsymbol{P}\left(\frac{u}{2^\ell}\right) - \boldsymbol{P}(0)\right]\boldsymbol{P}(0)^{k-\ell}\right)^T.\end{aligned}$$

It follows that for any $k \in \mathbb{N}$

$$\|\Pi_k(u)\Pi_k(u)^T - I_r\| \leq C\, e^{2C_\alpha\,|u|^\alpha}\, |u|^{2\alpha} \sum_{\ell=1}^{\infty} \left[\frac{\|\boldsymbol{P}(0)\|}{2^\alpha}\right]^{2\ell} \leq C'\, e^{2C_\alpha\,|u|^\alpha} |u|^{2\alpha}.$$

Using the fact that $\Pi_{N+k}(u) = \Pi_N(u)\Pi_k(2^{-N}u)$ we obtain

$$\|\Pi_{N+k}(u)\Pi_{N+k}(u)^T - \Pi_N(u)\Pi_N(u)^T\| \leq C'\, e^{2C_\alpha\,|u|^\alpha(1+2^{-N\alpha})}\, |u|^{2\alpha} \left[\frac{\|\boldsymbol{P}(0)\|}{2^\alpha}\right]^{2N}.$$

□

Next we establish a decay estimate of $\widehat{\Omega}$ in complete analogy to Theorem 3.2.

Theorem 3.3 *Let $\boldsymbol{P}$ be an $r \times r$ matrix of the form*

$$\boldsymbol{P}(u) = \frac{1}{2^m}\, C_0(2u) \cdots C_{m-1}(2u)\, \boldsymbol{P}^{(m)}(u)\, C_{m-1}(u)^{-1} \cdots C_0(u)^{-1},$$

where the C_i are certain $r \times r$ matrices and $\boldsymbol{P}^{(m)}$ is an $r \times r$ matrix with trigonometric polynomials as entries. Suppose that the spectral radius of $\boldsymbol{P}^{(m)}(0)$ is strictly less than two. For $k \geq 1$, let

$$\gamma_k = \frac{1}{k}\, \log_2 \sup_u \left\| \boldsymbol{P}^{(m)}\left(\frac{u}{2}\right) \cdots \boldsymbol{P}^{(m)}\left(\frac{u}{2^k}\right) \right\|.$$

Then there exists a positive constant C such that for all $u \in \mathbb{R}$

$$\|\widehat{\Omega}(u)\| \leq C\,(1 + |u|)^{2(-m+\gamma_k)}. \tag{3.27}$$

Proof. The result follows immediately from the proof of Theorem 4.1 in [7] by observing that instead of products of the form $\Pi_k(u)$ we are dealing with products of the form $\Pi_k(u)\Pi_k(u)^T$. This implies that all norm estimates are squared. □

Note that just like in the case $r = 1$, the decay rate of $\widehat{\Omega}$ is increased by a factor of 2. In particular, the approximation order also increases by a factor of 2. This doubling of the approximation order follows from the form of the Fourier transform as a product $\widehat{\varrho}_{ii}(u) = \widehat{\phi}_i(u)\overline{\widehat{\phi}_i(u)}$. The correlation functions τ_{ii} have an analogous product representation in terms of the wavelet functions ψ_i in the Fourier domain and, therefore, τ_{ii} has vanishing moments up to order $2D$. This implies that the autocorrelation transformation (3.6) has approximation order $2D$. Since the cross-correlation functions τ_{ij}, $i \neq j$, are products of the form $\widehat{\psi}_i \overline{\widehat{\psi}_j}$, they also have vanishing moments up to order $2D$. Therefore, we say that the translation invariant multiwavelet system (Θ, Ξ) has approximation order $2D$.

Remark 3.4 *It is worthwhile mentioning that the above results hold for semiorthogonal, biorthogonal, and oblique wavelet systems as long as the approximation spaces V_j contain polynomials and are orthogonal to the wavelet spaces W_j.*

In the case $r = 1$, the translation invariant wavelet system has the *coiflet property of order* $2D$, i.e., in addition to the vanishing moments up to order $2D$ of the autocorrelation functions $\tau(x) := \int_{\mathbb{R}} \psi(y)\psi(y+x)\,dx$, the functions $\varrho(x) := \int_{\mathbb{R}} \phi(y)\phi(y+x)\,dx$ also have vanishing moment of orders one to $2D-1$. This follows directly from the fact that $\widehat{\varrho}(u) = \widehat{\phi}(u)\widehat{\phi}(-u)$. This property transfers to the case $r > 1$ for the functions ϱ_{ii}, but *not necessarily* for the cross-correlation functions ϱ_{ij}. Since $\widehat{\Theta}(u) = \widehat{\Phi}(u)\widehat{\Phi}^T(-u)$, the translation invariant multiwavelet system (Θ, Ξ) has a *multicoiflet property of order* $2D$, only if the off-diagonal terms in the matrix

$$\sum_{\nu=0}^{n} \binom{n}{\nu} \widehat{\Phi}^{(n-\nu)}(0)\widehat{\Phi}^{(\nu)T}(0), \qquad n = 1, \ldots, 2D-1,$$

vanish identically. For example, for $n = 1$ and $r = 2$, it means that the scaling functions ϕ_1 and ϕ_2 must satisfy

$$\widehat{\phi}_1{}'(0)\widehat{\phi}_2(0) - \widehat{\phi}_2{}'(0)\widehat{\phi}_1(0) = 0. \tag{3.28}$$

Similar to orthogonal multiwavelet systems, the smoothness of the projection $\mathcal{P}_R^j[f]$ is, in general, determined by the smoothness of the functions ϱ_{ii}. Suppose $\phi_i \in H^\alpha(\mathbb{R})$ for $i = 1, \ldots, r$ then $\varrho_{ii} \in H^{2\alpha}(\mathbb{R})$. Since the redundant projection $\mathcal{P}_R^j[f]$ is a sum of convolutions, a result in [18] shows that $\mathcal{P}_R^j[f]$ is in the Sobolev space $H^{2\alpha+\beta}(\mathbb{R})$ whenever $f \in H^\beta(\mathbb{R})$.

4 Example: The DGHM Multiwavelet

In [11] a one-parameter family of multiwavelet system based on fractal functions is constructed. If the parameter is chosen to be -1/5, the multiwavelet system is orthogonal and its symbol is given by

$$\boldsymbol{P}(u) = \frac{1}{2}\left(P(0) + P(1)e^{-iu} + P(2)e^{-2iu} + P(3)e^{-3iu}\right), \tag{4.1}$$

where

$$P(0) = \begin{pmatrix} 3\sqrt{2}/10 & 4/5 \\ -1/20 & -3\sqrt{2}/20 \end{pmatrix}, \qquad P(1) = \begin{pmatrix} 3\sqrt{2}/10 & 0 \\ 9/20 & \sqrt{2}/20 \end{pmatrix},$$
$$P(2) = \begin{pmatrix} 0 & 0 \\ 9/20 & -3\sqrt{2}/20 \end{pmatrix}, \qquad P(3) = \begin{pmatrix} 0 & 0 \\ -1/20 & 0 \end{pmatrix}.$$

The DGHM multiwavelet system has approximation order 2 and $\Phi, \Psi \in \mathrm{Lip}^1(\mathbb{R})$ which implies that $\mathcal{P}^j[f] \in \mathrm{Lip}^1$. The calculation of the regularity based on the estimates in Theorem (3.2) is given is [7] and yields $m = 2$ and $\gamma_k < 1$, for k large enough. Moreover, the individual wavelet functions have vanishing moments up to order 2.

The translation invariant multiwavelet system (Θ, Ξ) contains the eight functions ϱ_{11}, ϱ_{12}, ϱ_{21}, ϱ_{22}, τ_{11}, τ_{12}, τ_{21}, and τ_{22} where $\varrho_{12}(x) = \varrho_{21}(-x)$ and $\tau_{12}(x) = \tau_{21}(-x)$. The graphs of these functions are depicted in Figures 1 and 2.

For this example the filter matrices $A(k)$ in (3.13) are given by

$$\begin{aligned}
A(0) &= \begin{pmatrix} 9/50 & 6\sqrt{2}/25 & 6\sqrt{2}/25 & 16/25 \\ -\sqrt{2}/10 & -9/100 & -1/25 & -3\sqrt{2}/25 \\ -\sqrt{2}/10 & -1/25 & -9/100 & -3\sqrt{2}/25 \\ 1/400 & 3\sqrt{2}/400 & 3\sqrt{2}/400 & 9/200 \end{pmatrix}, \\
A(1) &= \begin{pmatrix} 9/50 & 0 & 0 & 0 \\ 27\sqrt{2}/200 & 3/100 & 0 & 0 \\ 27\sqrt{2}/200 & 0 & 3/100 & 0 \\ 81/400 & 9\sqrt{2}/400 & 9\sqrt{2}/400 & 1/200 \end{pmatrix}, \\
A(2) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 81/400 & -27\sqrt{2}/400 & -27\sqrt{2}/400 & 9/200 \end{pmatrix}, \\
A(3) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/400 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

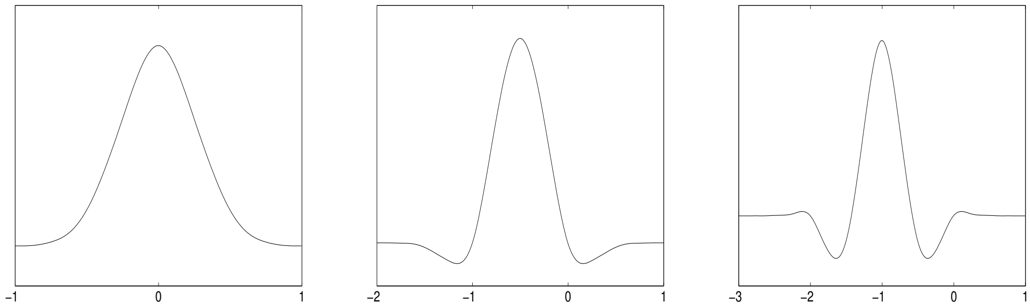

Figure 1: The graphs of $\varrho_{11}, \varrho_{12}, \varrho_{22}$.

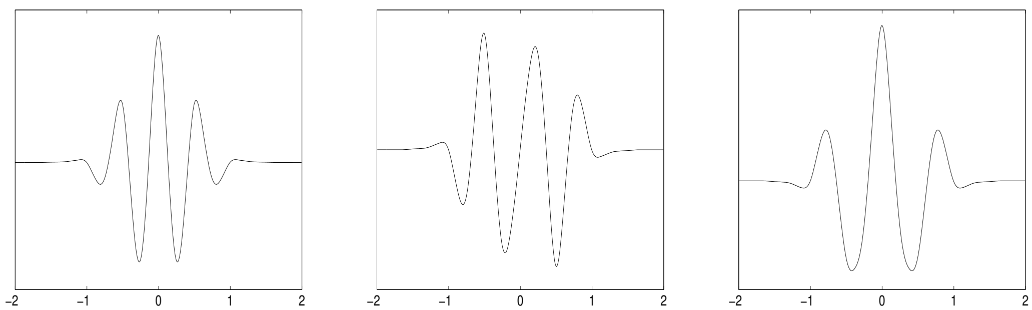

Figure 2: The graphs of $\tau_{11}, \tau_{12}, \tau_{22}$.

Employing the results from the previous section, it follows that Θ has approximation order four and that Ξ has vanishing moments up to order four. Moreover, the first to third moments of Θ vanish. The system (Θ, Ξ) does not have the multicoiflet property of order $2D$, since, e.g., condition (3.28) is not satisfied. The elements of Θ are in Lip^2. As a consequence, the redundant projections $\mathcal{P}_R^j[f]$ are in the Sobolev space $H^{2+\beta}(\mathbb{R})$ whenever $f \in H^\beta(\mathbb{R})$.

5 Concluding remarks

The motivation for studying translation invariant multiwavelet systems was based on the success of translation invariant scalar ($r = 1$) wavelet systems in signal denoising [3, 6, 16]. In the present article we show that most of the characteristics of scalar wavelet systems such as smoothness and approximation order carry over to the $r > 1$ case. However, for $r > 1$ the number of necessary scaling functions increased quadratically in r.

As in denoising orthogonal multiwavelet systems outperform orthogonal wavelet systems [27], it is expected that the introduction of translation invariant multiwavelet systems will further improve the quality of denoised signals. This expectation relies on the increased flexibility in the choice of the basis functions for the reconstruction.

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