INTEGRAL TRANSFORMS COVARIANT TO UNITARY OPERATORS AND THEIR IMPLICATIONS FOR JOINT SIGNAL REPRESENTATIONS

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Abstract

Fundamental to the theory of joint signal representations is the idea of associating a variable, such as time or frequency, with an operator, a concept borrowed from quantum mechanics. Each variable can be associated with a Hermitian operator, or equivalently and consistently, as we show, with a parameterized unitary operator. It is well-known that the eigenfunctions of the unitary operator define a signal representation which is *invariant* to the effect of the unitary operator on the signal, and is hence useful when such changes in the signal are to be ignored. However, for detection or estimation of such changes, a signal representation *covariant* to them is needed. Using well-known results in functional analysis, we show that there always exists a *translationally covariant* representation; that is, an application of the operator produces a corresponding translation in the representation. This is a generalization of a recent result in which a transform covariant to dilations is presented.

Using Stone's theorem, the "covariant" transform naturally leads to the definition of another, unique, *dual* parameterized unitary operator. This notion of duality, which we make precise, has important implications for joint distributions of arbitrary variables and their interpretation. In particular, joint distributions of dual variables are structurally equivalent to Cohen's class of time-frequency representations, and our development shows that, for two variables, the Hermitian and unitary operator correspondences can be used consistently and interchangeably if and only if the variables are dual.

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1 Introduction

Time-frequency analysis provides a very useful framework for the analysis and processing of signals whose characteristics change with time [1]. Time-frequency representations (TFRs), one of the main tools of time-frequency analysis, map the one-dimensional signal onto the two-dimensional time-frequency plane, and provide a measure of the signal energy at a particular time and frequency. Thus, in a sense, (bilinear) TFRs are a distribution of the signal energy as a function of time and frequency. However, the strict interpretation as distributions is not correct, since they are not always non-negative [1].

Many theoretical results in time-frequency analysis depend on associating an operator with a variable such as time or frequency, a concept borrowed from quantum mechanics. Traditionally, variables have been associated with Hermitian (self-adjoint) operators. For example, Cohen's Class of TFRs [1, 2, 3] was derived using the "characteristic function operator" method, by associating time and frequency with the Hermitian operators $\mathcal{T}$ and $\mathcal{F}$, respectively, defined as [1, 2][1]

$$(\mathcal{T}s)(t) = ts(t) \leftrightarrow (\mathcal{T}S)(f) = \frac{i}{2\pi}\dot{S}(f) , \tag{1}$$

$$(\mathcal{F}s)(t) = -\frac{i}{2\pi}\dot{s}(t) \leftrightarrow (\mathcal{F}S)(f) = fS(f) , \tag{2}$$

where $\dot{s}(t) = \frac{d}{dt}s(t)$ and $S(f)$ is the Fourier transform of s. Cohen's method uses the fact that the characteristic function of a TFR (distribution), being an average, can be directly computed from the signal, and then the TFR can be obtained from the characteristic function. More precisely, let $P_s(t,f)$ be a TFR of the signal s. The characteristic function $M_s(\theta,\tau)$ is defined as

$$M_s(\theta,\tau) = \int\int P_s(t,f)e^{i2\pi\theta t}e^{i2\pi\tau f}dtdf ; \tag{3}$$

that is, it is the average value of $e^{i(2\pi\theta t+2\pi\tau f)}$ as a function of (θ,τ). P_s can then be recovered as

$$P_s(t,f) = \int\int M_s(\theta,\tau)e^{-i2\pi\theta t}e^{-i2\pi\tau f}d\theta d\tau . \tag{4}$$

The key observation is that M_s can be directly computed from s as [1, 3]

$$M_s(\theta,\tau) = \int s^*(t)(\mathbf{M}(\theta,\tau)s)(t)dt , \tag{5}$$

where $\mathbf{M}(\theta,\tau)$ is a "characteristic function operator" [1, 3], an example being $\mathbf{M}(\theta,\tau) = e^{i2\pi\theta\mathcal{T}+i2\pi\tau\mathcal{F}}$, which is a function of the time and frequency operators corresponding to the function $e^{i2\pi\theta t+i2\pi\tau f}$ of time and frequency variables.[2] However, since the operators $\mathcal{T}$ and $\mathcal{F}$ do not commute, there are an infinitely many ways to associate an operator with the function $e^{i2\pi\theta t+i2\pi\tau f}$ [1]. Thus, correspondingly, infinitely many TFRs (Cohen's class) can be obtained via (4), which were originally characterized in [3].

Recently, Cohen extended his method to joint distributions of arbitrary variables [1, 10, 11], spurred by interest in variables like scale that was inspired by the wavelet transform [12] and the affine class of time-scale representations [13].[3] Such generalized joint signal representations in terms of arbitrary variables

[1] Cohen originally used the (radian) frequency operator $\mathcal{W} = -id/dt$.

[2] The procedure of associating functions of classical variables with operators, commonly referred to as "quantization" in physics, is fundamental to the distributional approach just described. An important quantization rule was first proposed by H. Weyl [4] for functions of position and momentum, which is essentially equivalent to associating the function $e^{j2\pi(\theta t+\tau f)}$ with the operator $e^{j2\pi(\theta\mathcal{T}+\tau\mathcal{F})}$. As one of the reviewers pointed out, the extension of Weyl correspondence to an arbitrary number of variables has been investigated extensively in mathematics and physics such as in [5, 6, 7, 8, 9].

[3] Cohen's generalized method for two arbitrary variables, say a and b, uses the same recipe as outlined above for time and frequency [1]: replace $\mathcal{T}$ and $\mathcal{F}$ with the new operators $\mathcal{A}$ and $\mathcal{B}$ in (4) and (5).

extend the scope of time-frequency analysis to a richer class of nonstationary signals. Baraniuk proposed a distributional approach based on group theoretic arguments [14] that was shown to be equivalent to Cohen's generalized method in [15]. Although other covariance-based generalizations have also been proposed recently [16, 17], our primary interest in this paper is in Cohen's distributional approach [1, 10] (the results apply to Baraniuk's approach [14] too, by virtue of the equivalence results of [15]; see also footnote 6).

The recent developments in the theory for joint distributions of arbitrary variables have raised many fundamental issues which need to be addressed in a general setting. For example, although variables have been associated with both Hermitian (Cohen's method) and unitary (Baraniuk's approach) operators, the relationship between the two correspondences, though of fundamental importance, has not been adequately addressed in existing treatments.[4] Similarly, the notion of covariance of joint distributions to certain unitary transformations is critical from the viewpoint of optimal signal detection based on joint distributions [19, 20, 21]. Although such properties are naturally characterized in covariance-based generalizations of joint signal representations [16, 17], they are neither guaranteed nor easily characterized in Cohen's distributional method [1, 10]. Baraniuk's approach [14] partly addresses such issues in a group theoretic setting and [15] relates them to Cohen's method and bridges some of the gaps. Some related results are presented in [22] in the case when the variables belong to a 2d group. The lack of clear understanding of these issues from a time-frequency viewpoint has resulted in some confusion, as we will elaborate later in the paper. Using well-known results in functional analysis and the theory of group representations, such issues can be addressed precisely and we do so in this paper. Starting with a brief background on operator correspondences, we now specifically discuss the issues addressed in this paper.

A fundamental and important property of the TFRs defined via (4) is that they are *covariant* to time and frequency shifts. That is, if we define the time-shift operator $\mathbf{T}_\mu$ as

$$(\mathbf{T}_\mu s)(t) = s(t-\mu) \ , \tag{6}$$

and the frequency-shift operator $\mathbf{F}_\nu$ as

$$(\mathbf{F}_\nu s)(t) = e^{-i2\pi\nu t} s(t) \ , \quad \text{then} \tag{7}$$

$$P_{\mathbf{T}_\mu \mathbf{F}_\nu s}(t,f) = P_{\mathbf{F}_\nu \mathbf{T}_\mu s}(t,f) = P_s(t-\mu, f+\nu) \ . \tag{8}$$

Note that that there is no coupling between the two shifts; a time-shift in the signal results in a corresponding time-shift in the TFR, leaving the frequency dependence unchanged, and similarly for frequency-shifts. The shift operators $\mathbf{T}_\mu$ and $\mathbf{F}_\nu$ are actually related to $\mathcal{T}$ and $\mathcal{F}$ as [1]

$$\mathbf{T}_\mu = e^{-i2\pi\mu\mathcal{F}} \quad \text{and} \quad \mathbf{F}_\nu = e^{-i2\pi\nu\mathcal{T}} \ . \tag{9}$$

Thus, the variables time and frequency are *dual* in the sense that the time-shift operator $\mathbf{T}_\mu$ is an exponentiated version of the frequency operator $\mathcal{F}$, and the frequency-shift operator $\mathbf{F}_\nu$ is an exponentiated version of the time operator $\mathcal{T}$. In Section 5, using well-known results in functional analysis, we will make this notion of duality precise for arbitrary variables.[5]

As we will see, by Stone's theorem [24], the Hermitian operators $\mathcal{T}$ and $\mathcal{F}$ can be recovered from the shift operators $\mathbf{F}_\nu$ and $\mathbf{T}_\mu$, respectively. Thus, we can equivalently associate the variables of time and frequency with the families $\{\mathbf{T}_\mu\}$ and $\{\mathbf{F}_\nu\}$, respectively, as was done in [25]. We will be more interested

[4] Operator-based formulations of joint distributions have also been discussed by other authors, for example [18] and [11], but they do not address the relationship between the two correspondences.

[5] Duality of operators is related to the notion of duality in the theory of groups [23, 15].

in this correspondence of variables with operators; that is, associating a variable with a corresponding (unitary) shift operator, because the parameter of the operator directly corresponds to a change in the value of the variable. For time, frequency and scale, the notion of the shift operators is self-evident because of our familiarity with the concepts, and many authors have explicitly or implicitly discussed it from a group theoretic perspective; see for example [26, 27, 28, 29, 30, 13, 31, 18]. However, it is not clear what "shift" means for an arbitrary variable. We will precisely define the concept of shift operators for arbitrary variables in Section 6, which is intimately related to the notion of duality (developed in Section 5).

To discuss some of the other issues addressed in this paper, consider a family of unitary operators $\{\mathbf{A}_\alpha\}$ parameterized by $\alpha \in \mathbb{R}$ and satisfying [6]

$$\mathbf{A}_{\alpha_1+\alpha_2} = \mathbf{A}_{\alpha_1}\mathbf{A}_{\alpha_2} \ , \quad \mathbf{A}_0 = \mathbf{I} \ , \ \text{and} \quad \mathbf{A}_\alpha \to \mathbf{A}_{\alpha_1} \ \text{ as } \ \alpha \to \alpha_1 \ , \tag{10}$$

where $\mathbf{I}$ is the identity operator, and convergence is understood with respect to an appropriate operator norm. It is well-known that the eigenfunctions of $\mathbf{A}_\alpha$ (independent of α) define a unitary transformation $\mathbf{S_A}$ which is *invariant* to $\mathbf{A}_\alpha$ in the sense that $|(\mathbf{S_A}\mathbf{A}_\alpha s)(a)| = |(\mathbf{S_A} s)(a)|$; that is, it ignores the changes in the variable corresponding to $\mathbf{A}_\alpha$. We will refer to $\mathbf{S_A}$ as the $\mathbf{A}$-invariant transform. On the other hand, if $\{\mathbf{A}_\alpha\}$ corresponds to a variable we want to detect or estimate changes in, then a different transform is needed which is, in some sense, *covariant* to changes in α. In Section 3, we will show that, given a family $\{\mathbf{A}_\alpha\}$, there always exists a transform, say $\mathbf{S}_B$, which satisfies $(\mathbf{S}_B\mathbf{A}_\alpha s)(b) = (\mathbf{S}_B s)(b \pm \alpha)$; we will refer to it as the $\mathbf{A}$-covariant transform. This is a generalization of the result presented in [31] in which a transform covariant to the scale changes (dilation operator) is presented.

Note that we have not assumed that $\{\mathbf{A}_\alpha\}$ corresponds to some meaningful variable like time, frequency or scale. However, Stone's theorem guarantees the existence of a unique Hermitian operator, say $\mathcal{B}$, for which $\mathbf{A}_\alpha$ and $\mathcal{B}$ are related just as $\mathbf{T}_\mu$ and $\mathcal{F}$ are related in (9). Thus, in a mathematically consistent sense, $\{\mathbf{A}_\alpha\}$ defines a variable b corresponding to the Hermitian operator $\mathcal{B}$. Moreover, using the concept of duality, we will also show that there exists a unique variable a (and a corresponding Hermitian operator $\mathcal{A}$) for which $\mathbf{A}_\alpha$ is the shift operator. The question of whether the variables a and b have any physical correspondence is hard to answer in complete generality. Nevertheless, in some cases a meaningful signal transformation may lend itself to a description in terms of a unitary family like $\{\mathbf{A}_\alpha\}$, in which case the question of finding the associated variable is important.[7] For example, we will see that the family of unitary operators associated with scale in [25] is not consistent with the Hermitian operator associated with scale in [1, 10, 34]; this inconsistency is intimately related to the notion of duality developed in this paper, which helps to clarify it.

Using Stone's theorem, the $\mathbf{A}$-covariant transform naturally leads to the definition of another, unique, family of unitary operators, $\{\mathbf{B}_\beta\}$, satisfying (10) and *dual* to $\{\mathbf{A}_\alpha\}$. Thus, the variables corresponding to $\{\mathbf{A}_\alpha\}$ and $\{\mathbf{B}_\beta\}$, say a and b, are dual variables just as time and frequency. This concept of duality has

[6] Thus, $\{\mathbf{A}_\alpha\}_{\alpha\in\mathbb{R}}$ is a unitary representation of the group $\mathbb{R}$ on $L_2(\mathbb{R})$ [24]. It can be easily verified that $\{\mathbf{T}_\mu\}$ and $\{\mathbf{F}_\nu\}$ also satisfy the properties (10). We note at the outset that all the issues addressed in this paper can formulated in a more general setting where $\{\mathbf{A}_\alpha\}_{\alpha\in G}$ is a unitary representation of an arbitrary locally compact abelian (LCA) group G [23, 32, 15]. A method for joint distributions based on LCA groups was proposed in [14] and was shown to be equivalent to Cohen's (to which this paper directly applies) in [15]; the equivalence is in fact simply via *axis warping* transformations [15]. Thus, from the viewpoint of joint distributions, the discussion in this paper based on $\mathbb{R}$ suffices. Moreover, the transforms and distributions discussed in this paper, being based on the group $\mathbb{R}$, are intimately related to the Fourier transform, and thus provide a computationally efficient alternative for generating the transforms and distributions in Baraniuk's method (based on the underlying group transform) via the simple pre- and post-processing (axis warping) determined in [15, 33]. A final motivation for basing the development on $\mathbb{R}$, in this paper, is that it is much more accessible since it does not require any group theoretic machinery. Some issues in the general setting are discussed in [15, 33].

[7] This is another reason for preferring a unitary operator correspondence rather than a Hermitian operator correspondence.

important implications for joint representations of arbitrary variables, which have received much attention lately [1, 25, 10, 35, 14, 34, 15, 16, 17]. For example, we will use it to characterize the exact relationship between Hermitian and unitary operator correspondences, an important issue which is not adequately addressed in existing treatments. Another important implication, as we will see in Section 6, is that the joint distributions of a and b share analogues of all the properties possessed by time-frequency distributions. This ties in intimately with the notion of unitary equivalence introduced in [25], and we will make that connection as well.

The outline of the rest of the paper is as follows. In the next section we review the necessary operator theoretic concepts to be used in the paper. Section 3 describes the **A**-invariant transform associated with $\{\mathbf{A}_\alpha\}$. In Section 4, we derive the **A**-covariant transform, and in Section 5 we develop the notion of duality in terms of the family of dual unitary operators $\{\mathbf{B}_\beta\}$ defined by the covariant transform. Section 6 discusses the implications of the results for joint distributions of arbitrary variables, and Section 7 concludes the paper with a summary and discussion of the results presented.

2 Preliminaries and Notation

We will denote signals in the time domain by lowercase letters and their Fourier transforms by the corresponding uppercase letters. That is, for example, s and S are related by

$$S(f) = (\mathbb{F}s)(f) = \int_{\mathbb{R}} s(t) e^{-i2\pi f t} dt \ , \tag{11}$$

where $\mathbb{F}$ denotes the Fourier transform operator. Throughout the paper, the default domain of signal representation is the time domain unless explicitly stated otherwise. We assume that all signals of interest belong to $L_2(\mathbb{R})$, the Hilbert space of square-integrable functions, equipped with the usual inner product $\langle \cdot, \cdot \rangle$,

$$\langle x, y \rangle = \int_{\mathbb{R}} x(t) y^*(t) dt \ , \ x, y \in L_2(\mathbb{R}) \ , \tag{12}$$

where the superscript * denotes complex conjugation. In some cases, the signal space of interest will be a closed subspace of $L_2(\mathbb{R})$ of the form $L_2(\Theta)$, where $\Theta \subseteq \mathbb{R}$ is an interval. From now on, we will denote the signal space of interest by $\mathcal{H}_1$. All the operators introduced so far are linear mappings, and we will only be dealing with linear operators in this paper; thus, henceforth, "operator" refers to a linear operator. An operator $\mathcal{A}$ is said to be self-adjoint if it satisfies $\mathcal{A} = \mathcal{A}^\dagger$, where $\mathcal{A}^\dagger$ is the adjoint operator of $\mathcal{A}$ defined by $\langle \mathcal{A}x, y \rangle = \langle x, \mathcal{A}^\dagger y \rangle$ for all $x, y \in \mathcal{H}_1$. The operators $\mathcal{T}$ and $\mathcal{F}$ are easily seen to be self-adjoint. An operator $\mathbf{U}$ is said to be unitary if it satisfies $\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{I}$ and thus preserves inner products; that is, $\langle \mathbf{U}x, \mathbf{U}y \rangle = \langle x, y \rangle$. It follows that a unitary operator maps the Hilbert space onto itself and is invertible, with the inverse being the adjoint operator. The shift operators $\mathbf{T}_\mu$ and $\mathbf{F}_\nu$ are easily verified to be unitary for all values of μ and ν.[8]

An isometry is a distance preserving transformation between two Hilbert spaces; that is, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces (closed subspaces of $L_2(\mathbb{R})$ in our case), then $\mathbf{S} : \mathcal{H}_1 \to \mathcal{H}_2$ is an isometry if $\langle \mathbf{S}x, \mathbf{S}y \rangle = \langle x, y \rangle$ rm for $x, y \in \mathcal{H}_1$. If, in addition, the range space of $\mathbf{S}$ is the entire $\mathcal{H}_2$, then we have $\mathbf{S}\mathbf{S}^\dagger = \mathbf{I}_{\mathcal{H}_2}$ and $\mathbf{S}^\dagger\mathbf{S} = \mathbf{I}_{\mathcal{H}_1}$.

Let $\{\mathbf{A}_\alpha : \mathcal{H}_1 \to \mathcal{H}_1\}$ be a family of unitary operators satisfying (10). By Stone's theorem [24], there

[8]Throughout the paper, we will use script letters to denote Hermitian operators and boldfaced letters to denote unitary operators.

exists a unique, generally unbounded, self-adjoint operator $\mathcal{B}$, defined on $\mathcal{H}_1$, such that

$$\mathbf{A}_\alpha = e^{-i2\pi\alpha\mathcal{B}} = \sum_{k=0}^{\infty} \frac{(-i2\pi\alpha\mathcal{B})^k}{k!} \tag{13}$$

$$\mathcal{B} = \frac{i}{2\pi} \lim_{\alpha\to 0} \frac{\mathbf{A}_\alpha - \mathbf{I}}{\alpha} , \tag{14}$$

where the limit is taken with respect to the usual linear operator norm on $L_2(\mathbb{R})$. It follows from (9) that $\mathcal{T}$ and $\mathcal{F}$ can be similarly recovered from the families $\{\mathbf{F}_\nu\}$ and $\{\mathbf{T}_\mu\}$, respectively. Thus, the family $\{\mathbf{A}_\alpha\}$ uniquely defines a variable b corresponding to the Hermitian operator $\mathcal{B}$. In all cases of interest to us, the operator $\mathcal{B}$ will be an unbounded operator, and thus, being a Hermitian operator, it possesses a complete set of orthonormal (generalized) eigenfunctions $\{e_{\mathcal{B}}(b,\cdot)\}$ which satisfy[9]

$$\mathcal{B}e_{\mathcal{B}}(b,\cdot) = b e_{\mathcal{B}}(b,\cdot) \ , \ b \in \mathbb{R} \ . \tag{15}$$

It follows from (13) that, for each α, $\mathbf{A}_\alpha$ satisfies the eigenequation

$$\mathbf{A}_\alpha e_{\mathcal{B}}(b,\cdot) = e^{-i2\pi\alpha b} e_{\mathcal{B}}(b,\cdot) \ , \ b \in \mathbb{R} \ . \tag{16}$$

That is, the family $\{\mathbf{A}_\alpha\}$ possesses the common set of eigenfunctions $\{e_{\mathcal{B}}(b,\cdot) : b \in \mathbb{R}\}$ (independent of α) and the corresponding family of sets of eigenvalues $\{e^{-i2\pi\alpha b} : b \in \mathbb{R}\}_\alpha$. From now on, since we will primarily be dealing with $\{\mathbf{A}_\alpha\}$, we will denote the set $\{e_{\mathcal{B}}(b,\cdot)\}$ by $\{e_{\mathbf{A}}(b,\cdot)\}$, the two sets being identical.

The set of eigenfunctions $\{e_{\mathbf{A}}(b,\cdot)\}$ itself defines an isometry $\mathbf{S}_{\mathbf{A}} : \mathcal{H}_1 \to L_2(\mathbb{R})$ as

$$(\mathbf{S}_{\mathbf{A}} s)(b) = \langle s, e_{\mathbf{A}}(b,\cdot)\rangle = \int s(t) e^*_{\mathbf{A}}(b,t)dt \ , \quad b \in \mathbb{R} \ , \tag{17}$$

which is the representation of the signal with respect to the basis $\{e_{\mathbf{A}}(b,\cdot)\}$. Note that the $\mathbf{S}_{\mathbf{A}}$ and $\mathbf{S}_{\mathcal{B}}$ transforms are identical. The inverse transform $\mathbf{S}^{-1}_{\mathbf{A}} : L_2(\mathbb{R}) \to \mathcal{H}_1$, is given by

$$(\mathbf{S}^{-1}_{\mathbf{A}} y)(t) = \langle y, e^*_{\mathbf{A}}(\cdot,t)\rangle = \int_{\mathbb{R}} y(b) e_{\mathbf{A}}(b,t)db \ . \tag{18}$$

This implies that

$$s(t) = \langle (\mathbf{S}_{\mathbf{A}} s)(\cdot), e^*_{\mathbf{A}}(\cdot,t)\rangle = \int_{\mathbb{R}} e_{\mathbf{A}}(b,t)(\mathbf{S}_{\mathbf{A}} s)(b)db \ , \text{ and} \tag{19}$$

$$(\mathbf{A}_\alpha s)(t) = \int_{\mathbb{R}} e_{\mathbf{A}}(b,t) e^{-i2\pi\alpha b}(\mathbf{S}_{\mathbf{A}} s)(b)db \ , \tag{20}$$

with a similar spectral decomposition holding for $\mathcal{B}$. Moreover, since $\{\mathbf{A}_\alpha\}$ is a unitary family satisfying (10), we have, $\mathbf{A}^\dagger_\alpha = \mathbf{A}^{-1}_\alpha = \mathbf{A}_{-\alpha}$. For simplicity of notation, we will denote the spectral decomposition of Hermitian and unitary operators more compactly as

$$\mathcal{B} = \mathbf{S}^{-1}_{\mathcal{B}} \mathbf{\Lambda} \mathbf{S}_{\mathcal{B}} = \mathbf{S}^{-1}_{\mathbf{A}} \mathbf{\Lambda} \mathbf{S}_{\mathbf{A}} \tag{21}$$

$$\mathbf{A}_\alpha = \mathbf{S}^{-1}_{\mathbf{A}} \mathbf{\Lambda}_\alpha \mathbf{S}_{\mathbf{A}} = \mathbf{S}^{-1}_{\mathcal{B}} \mathbf{\Lambda}_\alpha \mathbf{S}_{\mathcal{B}} \tag{22}$$

where the "diagonal" operators $\mathbf{\Lambda}$ and $\mathbf{\Lambda}_\alpha$ are defined as

$$(\mathbf{\Lambda} s)(a) = a s(a) \ , \quad a \in \mathbb{R} \tag{23}$$

$$(\mathbf{\Lambda}_\alpha s)(a) = (e^{-i2\pi\alpha\mathbf{\Lambda}} s)(a) = e^{-i2\pi\alpha a} s(a) \ , \quad a \in \mathbb{R} \ . \tag{24}$$

[9] For simplicity of exposition, our discussion includes generalized eigenfunctions, like complex exponentials, which do not strictly belong to $L_2(\mathbb{R})$ but form a "continuous orthonormal basis" [36]. A rigorous approach is to use a spectral decomposition in terms of projection-valued measures [24].

Another useful operator is the translation operator, $\mathbf{\Gamma}_\alpha$, $\alpha \in \mathbb{R}$, defined as

$$(\mathbf{\Gamma}_\alpha s)(a) = s(a-\alpha) \ , \quad a \in \mathbb{R} \ . \tag{25}$$

Note that both $\mathbf{\Lambda}_\alpha$ and $\mathbf{\Gamma}_\alpha$ are unitary representations of $\mathbb{R}$ on $L_2(\mathbb{R})$. It is well-known that $\mathbf{\Gamma}_\alpha$ and $\mathbf{\Lambda}_\alpha$ are related as [23, 32, 15]

$$\mathbb{F}\mathbf{\Gamma}_\alpha = \mathbf{\Lambda}_\alpha \mathbb{F} \quad \Leftrightarrow \quad \mathbb{F}\mathbf{\Gamma}_\alpha \mathbb{F}^{-1} = \mathbf{\Lambda}_\alpha \ , \quad \text{or} \tag{26}$$

$$\mathbb{F}^{-1}\mathbf{\Gamma}_\alpha = \Lambda_{-\alpha} \mathbb{F}^{-1} \quad \Leftrightarrow \quad \mathbb{F}^{-1}\mathbf{\Gamma}_\alpha \mathbb{F} = \Lambda_{-\alpha} \ , \tag{27}$$

which essentially states the shift property of the Fourier transform.[10] These fundamental Fourier transform relationships, and the spectral representation (22) of unitary operators, will be used repeatedly throughout the paper. It is also worth noting that in the time-domain, $\mathbf{T}_\mu$ is identical to $\mathbf{\Gamma}_\alpha$ and $\mathbf{F}_\nu$ is identical to $\mathbf{\Lambda}_\alpha$.

To illustrate the concepts with concrete examples, we will primarily use four specific families of unitary operators. Two of them are the familiar time and frequency shift operators $\{\mathbf{T}_\mu : L_2(\mathbb{R}) \to L_2(\mathbb{R})\}$ and $\{\mathbf{F}_\nu : L_2(\mathbb{R}) \to L_2(\mathbb{R})\}$, defined in (6) and (7), respectively. The other two are the family of "dilation" operators $\{\mathbf{D}_\sigma : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)\}$ ($\mathbb{R}_+ = [0,\infty)$), and the family of "exponentiated inverse frequency" operators $\{\mathbf{Q}_\kappa : L_2(\mathbb{R}) \to L_2(\mathbb{R})\}$[11] defined by

$$(\mathbf{D}_\sigma s)(t) \;=\; e^{-\sigma/2} s(e^{-\sigma} t) \ , \quad s \in L_2(\mathbb{R}_+) \ \text{ and} \tag{28}$$

$$(\mathbf{Q}_\kappa S)(f) \;=\; e^{-i2\pi\kappa f_o/f} S(f) \ \text{for some constant } f_o \neq 0 \ . \tag{29}$$

Note that $\mathbf{Q}_\kappa$ is defined in the frequency domain. It can be readily verified that the eigenfunctions and the corresponding eigenvalues for the four families are:

$$\{\mathbf{T}_\mu\} : \quad \{e_{\mathbf{T}}(f',t) = e^{i2\pi f' t} : (f',t) \in \mathbb{R}^2\} \ , \ \{e^{-i2\pi\mu f'} : f' \in \mathbb{R}\}_\mu \tag{30}$$

$$\{\mathbf{F}_\nu\} : \quad \{e_{\mathbf{F}}(t',t) = \delta(t-t') : (t',t) \in \mathbb{R}^2\} \ , \ \{e^{-i2\pi\nu t'} : t' \in \mathbb{R}\}_\nu \tag{31}$$

$$\{\mathbf{D}_\sigma\} : \quad \{e_{\mathbf{D}}(c,t) = \frac{e^{i2\pi c \ln(t)}}{\sqrt{t}} : (c,t) \in \mathbb{R} \times \mathbb{R}_+\} \ , \ \{e^{-i2\pi\sigma c} : c \in \mathbb{R}\}_\sigma \tag{32}$$

$$\{\mathbf{Q}_\kappa\} : \quad \{e_{\mathbf{Q}}(r,t) = \frac{\sqrt{|f_o|}}{|r|} e^{i2\pi \frac{f_o}{r} t} : (r,t) \in \mathbb{R}\backslash\{0\} \times \mathbb{R}\} \ , \ \{e^{-i2\pi\kappa r} : r \in \mathbb{R}\backslash\{0\}\}_\kappa \tag{33}$$

where it should be noted that the eigenfunctions of $\{\mathbf{D}_\sigma\}$ are defined on $\mathbb{R}_+$.

3 The Invariant Transform

Consider a family of unitary operators, $\{\mathbf{A}_\alpha : \mathcal{H}_1 \to \mathcal{H}_1\}$, corresponding to a variable a; that is, the parameter α corresponds to a change in the variable a (this correspondence will be made precise in Section 6). Suppose that we are interested in a signal representation which is independent of changes in the signal corresponding to the changes in the variable a; that is, we want to ignore the effects of changes in the variable a. For example, in frequency estimation the phase (time-delay) is usually not of concern. In other words, we are interested in a signal representation which is *invariant* to the transformation of the signal by the operator $\mathbf{A}_\alpha$, for all α. It is well-known, and directly follows from (22) and (24), that the unitary transform $\mathbf{S}_\mathbf{A}$ defined in (17), which is simply the signal representation with respect to the eigenfunctions of $\{\mathbf{A}_\alpha\}$, is invariant to the operator $\mathbf{A}_\alpha$ in the following sense

$$(\mathbf{S}_\mathbf{A}\mathbf{A}_\alpha s)(b) = (\mathbf{\Lambda}_\alpha \mathbf{S}_\mathbf{A} s)(b) = e^{-i2\pi\alpha b}(\mathbf{S}_\mathbf{A} s)(b) \ . \tag{34}$$

[10]The same relation holds for arbitrary LCA groups with $\mathbf{\Gamma}_\alpha$ denoting group translation and $\mathbf{\Lambda}_\alpha$ denoting multiplication by characters of the group, which are complex exponentials in the case of $\mathbb{R}$ [23, 32, 15].

[11]The exponentiated version of the inverse frequency operator $\mathcal{R}$ defined in [1].

That is, the $\mathbf{S_A}$ transforms of s and $\mathbf{A}_\alpha s$ have the same magnitude. Thus, if we want to suppress the effect of changes in the variable a, the ideal signal representation is $|\mathbf{S_A} s|$.

Examples. To start with, using (30) and (31), it can be easily seen that the $\mathbf{S_T}$ transform is the usual Fourier transform $\mathbb{F}$ defined in (11), which is invariant to time-shifts, and the $\mathbf{S_F}$ transform is the identity transform $\mathbf{I}$, which is invariant to frequency-shifts. Similarly, $\mathbf{S_D} : L_2(\mathbb{R}_+) \to L_2(\mathbb{R})$ is the Mellin transform [37, 38, 34] defined by

$$(\mathbf{S_D} s)(c) = M(c) = \int_{\mathbb{R}_+} s(t) \frac{e^{-i2\pi c \ln(t)}}{\sqrt{t}} dt \tag{35}$$

which is invariant to dilations ($\mathbf{D}_\sigma$), and the $\mathbf{S_Q}$ transform is given by

$$(\mathbf{S_Q} s)(r) = \frac{\sqrt{|f_o|}}{|r|} S(f_o/r) = \frac{\sqrt{|f_o|}}{|r|} (\mathbb{F} s)(f_o/r) \ , \tag{36}$$

which is invariant to the $\mathbf{Q}_\kappa$ operator; that is, to changes in the parameter κ.

4 The Covariant Transform

In some cases, instead of ignoring changes in the variable a, we might be interested in detecting and/or estimating them, which is equivalent to estimating the value of the parameter α [20, 19, 21]. From (34) we note that the $\mathbf{A}$-invariant transform, $\mathbf{S_A}$, does contain information about α, buried in the phase of $\mathbf{S_A} s$, which is difficult to extract. Clearly, the $\mathbf{S_A}$ representation is not the right one. However, as shown next, using the spectral representation (22) and the fundamental relation (26) (or (27)), we can always find a unitary signal representation that is *translationally covariant* to changes in the variables a; that is, a change of α in the variable (transformation of the signal by $\mathbf{A}_\alpha$) corresponds to a translation in the signal representation by α.

Substituting the relation (26) in the spectral representation (22) for $\mathbf{A}_\alpha$ we get

$$\mathbf{A}_\alpha = \mathbf{S_A}^{-1} \mathbf{\Lambda}_\alpha \mathbf{S_A} = \mathbf{S_A}^{-1} \mathbb{F} \mathbf{\Gamma}_\alpha \mathbb{F}^{-1} \mathbf{S_A} = \mathbf{S_B}^{-1} \mathbf{\Gamma}_\alpha \mathbf{S_B} \tag{37}$$

where $\mathbf{S_B} : \mathcal{H}_1 \to L_2(\mathbb{R})$ is another isometry given by[12]

$$\mathbf{S_B} = \mathbb{F}^{-1} \mathbf{S_A} \ . \tag{38}$$

From (37) it follows that $\mathbf{S_B} \mathbf{A}_\alpha = \mathbf{\Gamma}_\alpha \mathbf{S_B}$ which, using (25), can be written more explicitly as

$$(\mathbf{S_B} \mathbf{A}_\alpha s)(a) = (\mathbf{\Gamma}_\alpha \mathbf{S_B} s)(a) = (\mathbf{S_B} s)(a - \alpha) \ . \tag{39}$$

Thus, from (39) we see that the unitary transform $\mathbf{S_B}$ defined in (38) is precisely the required $\mathbf{A}$-covariant transform.[13] That is, a change of α in the variable a associated with $\{\mathbf{A}_\alpha\}$ produces a corresponding translation in the $\mathbf{S_B}$ representation of the signal.

Similarly, if we use the relation (27), instead of (26), in (37), the covariant transform is given by

$$\mathbf{S_B} = \mathbb{F} \mathbf{S_A} \tag{40}$$

with the covariance relation

$$(\mathbf{S_B} \mathbf{A}_\alpha s)(a) = (\Gamma_{-\alpha} \mathbf{S_B} s)(a) = (\mathbf{S_B} s)(a + \alpha) \ . \tag{41}$$

[12] Note that the subscript of $\mathbf{S_B}$ is an operator; this notation is in anticipation of the family of operators that we will associate with $\mathbf{S_B}$ in the next section.

[13] In the more general setting based on LCA groups, the covariant transform is again given by (38) with the Fourier transform $\mathbb{F}$ replaced by the group Fourier transform [23, 15, 33].

Thus, the covariant transform is unique up to the sign of the translation.

Finally, from (38) we note that if we define another complete set of orthonormal basis functions $\{e_{\mathbf{B}}(a,\cdot) : a \in \mathbb{R}\}$ as

$$e_{\mathbf{B}}(a,t) = \int_{\mathbb{R}} e_{\mathbf{A}}(b,t) e^{-i2\pi ba} db \ , \quad a \in \mathbb{R} \ , \tag{42}$$

then the transform $\mathbf{S_B}$ can also be interpreted as the projection onto $\{e_{\mathbf{B}}(a,\cdot) : a \in \mathbb{R}\}$; that is,

$$(\mathbf{S_B} s)(a) = \langle s, e_{\mathbf{B}}(a,\cdot)\rangle = \int s(t) e^*_{\mathbf{B}}(a,t) dt \ , \tag{43}$$

and a similar interpretation holds for the alternative definition (40). This interpretation yields a simple constructive procedure which is illustrated in the following examples.

We believe that this derivation of the covariant transform for arbitrary variables is new in the time-frequency literature and generalizes the results of [31] in which a transform covariant to dilations is derived.

Examples. As a first example, let $\{\mathbf{A}_\alpha\}$ be the family of time-shift operators, $\{\mathbf{T}_\mu\}$. It can be easily verified that the set $\{e_{\mathbf{B}}(a,\cdot)\}$ is exactly the set $\{e_{\mathbf{F}}(t',\cdot)\}$ defined in (30), the set of eigenfunctions of the frequency-shift operators. In this case, the $\mathbf{S_B} = \mathbf{S_F} = \mathbf{I}$ transform is trivially covariant to the time-shift operator; $(\mathbf{I}\mathbf{T}_\mu s)(t) = s(t-\mu) = (\mathbf{I}s)(t-\mu)$.

Now consider the family $\{\mathbf{D}_\sigma\}$. Denote the new set of basis functions by $\{e_{\mathbf{C}}(d,\cdot)\}$ which can be computed (using (42)) as

$$e_{\mathbf{C}}(d,t) = \int_{\mathbb{R}} e_{\mathbf{D}}(c,t) e^{-i2\pi cd} dc = \int \frac{e^{i2\pi c \ln(t)}}{\sqrt{t}} e^{-i2\pi cd} dc = \delta(d - \ln(t))/\sqrt{t} = e^{-d/2}\delta(d-\ln(t)) \ . \tag{44}$$

The $\mathbf{S_C} : L_2(\mathbb{R}_+) \to L_2(\mathbb{R})$ transform then is

$$(\mathbf{S_C} s)(d) = e^{d/2} s(e^d) \ , \tag{45}$$

which can be easily verified to be covariant to dilations as defined by the family $\{\mathbf{D}_\sigma\}$; that is, $(\mathbf{S_C}\mathbf{D}_\sigma s)(d) = (\mathbf{S_C} s)(d-\sigma)$. We note that $\mathbf{S_C}$ is exactly the transform covariant to scaling obtained in [31].

Finally, the covariant transform for $\{\mathbf{Q}_\kappa\}$, which we denote by $\mathbf{S_R}$, is given by

$$(\mathbf{S_R} s)(q) = (\mathbf{S_R} S)(q) = \int_{\mathbb{R}} S(f) \frac{\sqrt{|f_o|}}{|f|} e^{i2\pi \frac{f_o}{f} q} df \ , \tag{46}$$

which is just the projection (in the frequency domain) onto the basis functions $\{e_{\mathbf{R}}(q,f)\}$:

$$e_{\mathbf{R}}(q,f) = \int_{\mathbb{R}} e_{\mathbf{Q}}(r,f) e^{-i2\pi rq} dr = \int_{\mathbb{R}} \frac{\sqrt{|f_o|}}{|r|} \delta(f - f_o/r) e^{-i2\pi rq} dr = \frac{\sqrt{|f_o|}}{|f|} e^{-i2\pi \frac{f_o}{f} q} \ . \tag{47}$$

It can be easily verified that $\mathbf{S_R}$ is covariant to $\{\mathbf{Q}_\kappa\}$; that is $(\mathbf{S_R}\mathbf{Q}_\kappa s)(q) = (\mathbf{S_R} s)(q-\kappa)$.

5 Dual Operators

We know that given a family of unitary operators $\{\mathbf{A}_\alpha\}$, the unitary transform $\mathbf{S_A}$, based on the eigenfunctions of $\{\mathbf{A}_\alpha\}$, is $\mathbf{A}$-invariant, and in the previous section we derived the unitary transform $\mathbf{S_B}$ which is $\mathbf{A}$-covariant. A natural question, which addresses the essence of the notion of "duality", is that whether or not there exists another family of unitary operators on $\mathcal{H}_1$, say $\{\mathbf{B}_\beta\}$, which satisfies (10), and for which $\mathbf{S_B}$ is the *invariant* transform, and $\mathbf{S_A}$ the *covariant* transform. In this section, we show that indeed such a family always exists and our development enables us to define the concept of duality in a precise manner.

Guided by the spectral representation of a unitary operator as dictated by Stone's theorem (see (22)), the $\mathbf{A}$-covariant transform $\mathbf{S_B}$ allows to us to naturally define another family of unitary operators on $\mathcal{H}_1$. More precisely, consider a family of operators $\{\mathbf{B}_\beta : \mathcal{H}_1 \to \mathcal{H}_1\}_\beta$ defined as

$$\mathbf{B}_\beta = \mathbf{S}_\mathbf{B}^{-1} \mathbf{\Lambda}_\beta \mathbf{S}_\mathbf{B} \ . \tag{48}$$

It follows that the family $\{\mathbf{B}_\beta\}$ shares $\{e_\mathbf{B}(a,\cdot)\}$ as the common set of eigenfunctions, with $\{e^{-i2\pi\beta a}\}$ as the corresponding set of eigenvalues, and satisfies the properties given in (10) for $\{\mathbf{A}_\alpha\}$. Thus, again by Stone's theorem, $\{\mathbf{B}_\beta\}$ defines a new variable corresponding to the Hermitian operator, say $\mathcal{A}$, defined by $\{\mathbf{B}_\beta\}$ (see (14)).

By definition, the transform $\mathbf{S_B}$ is $\mathbf{B}$-invariant. As we mentioned earlier, a question that naturally arises is that, since the transform $\mathbf{S_B}$ is covariant to operators $\{\mathbf{A}_\alpha\}$, is the transform $\mathbf{S_A}$ covariant to $\{\mathbf{B}_\beta\}$? It is indeed the case as is simply shown next. Substituting the covariant transform (38) in (48) we get

$$\mathbf{B}_\beta = \mathbf{S}_\mathbf{A}^{-1} \mathbb{F} \mathbf{\Lambda}_\beta \mathbb{F}^{-1} \mathbf{S}_\mathbf{A} = \mathbf{S}_\mathbf{A}^{-1} \Gamma_{-\beta} \mathbf{S}_\mathbf{A} \tag{49}$$

where the last equality uses the fundamental relationship (27). It follows from (49) that

$$(\mathbf{S_A B}_\beta s)(b) = (\Gamma_{-\beta} \mathbf{S_A} s)(b) = (\mathbf{S_A} s)(b+\beta) \ , \tag{50}$$

for all $s \in \mathcal{H}_1$, and thus the transform $\mathbf{S_A}$ is $\mathbf{B}$-covariant. Similarly, if we use the definition (40) for the covariant transform, the covariance relationship becomes

$$(\mathbf{S_A B}_\beta s)(b) = (\Gamma_{\beta} \mathbf{S_A} s)(b) = (\mathbf{S_A} s)(b-\beta) \ . \tag{51}$$

Hence, if the eigenfunctions of the families $\{\mathbf{A}_\alpha\}$ and $\{\mathbf{B}_\beta\}$ are related by (38) or (40), then the $\mathbf{S_B}$ transform is $\mathbf{A}$-covariant and the $\mathbf{S_A}$ transform is $\mathbf{B}$-covariant. In the appendix it is shown that the converse is also true; that is, if $\{\mathbf{A}_\alpha\}$ and $\{\mathbf{B}_\beta\}$ satisfy such covariance relations, their eigenfunctions must necessarily be related by (38) (or (40)). We summarize the results in the following proposition.

Proposition 1. Let $\{\mathbf{A}_\alpha : \mathcal{H}_1 \to \mathcal{H}_1\}$ and $\{\mathbf{B}_\beta : \mathcal{H}_1 \to \mathcal{H}_1\}$ be two families of unitary operators satisfying (10), and admitting the formal spectral representations

$$\mathbf{A}_\alpha = \mathbf{S}_\mathbf{A}^{-1} \mathbf{\Lambda}_\alpha \mathbf{S}_\mathbf{A} \ , \quad \text{and} \tag{52}$$

$$\mathbf{B}_\beta = \mathbf{S}_\mathbf{B}^{-1} \mathbf{\Lambda}_\beta \mathbf{S}_\mathbf{B} \ . \tag{53}$$

Then, $\mathbf{S_A}$ is $\{\mathbf{A}_\alpha\}$-invariant and $\mathbf{S_B}$ is $\{\mathbf{B}_\beta\}$-invariant; that is,

$$|(\mathbf{S_A A}_\alpha s)(b)| = |(\mathbf{S_A} s)(b)| \ \text{ for all } b, \ \alpha \in \mathbb{R} \ , \ s \in \mathcal{H}_1 \ \text{ and} \tag{54}$$

$$|(\mathbf{S_B B}_\beta s)(a)| = |(\mathbf{S_B} s)(a)| \ \text{ for all } a, \ \beta \in \mathbb{R} \ , \ s \in \mathcal{H}_1 \ . \tag{55}$$

Moreover, $\mathbf{S_A}$ is $\mathbf{B}$-covariant and $\mathbf{S_B}$ is $\mathbf{A}$-covariant; that is, for all $s \in \mathcal{H}_1$

$$(\mathbf{S_B A}_\alpha s)(a) = (\mathbf{S_B} s)(a \mp \alpha) \ \text{ for all } a, \ \alpha \in \mathbb{R} \ , \ \text{and} \tag{56}$$

$$(\mathbf{S_A B}_\beta s)(b) = (\mathbf{S_A} s)(b \pm \beta) \ \text{ for all } b, \ \beta \in \mathbb{R} \ , \tag{57}$$

if and only if the unitary transforms $\mathbf{S_A}$ and $\mathbf{S_B}$ are related as

$$\mathbf{S_B} = \mathbb{F}^{\mp 1} \mathbf{S_A} \ , \tag{58}$$

which is equivalent to the eigenfunctions of $\{\mathbf{A}_\alpha\}$ and $\{\mathbf{B}_\beta\}$ being related as

$$e_{\mathbf{B}}(a,t) = \int_{\mathbb{R}} e_{\mathbf{A}}(b,t) e^{\mp i2\pi ba} db \ . \tag{59}$$

We are now in a position to define the notion of duality.

Definition: Duality. Two families of unitary operators, $\{\mathbf{A}_\alpha = e^{-i2\pi\alpha\mathcal{B}}\}$ and $\{\mathbf{B}_\beta = e^{-i2\pi\beta\mathcal{A}}\}$, both satisfying (10) and admitting spectral representations (52) and (53), respectively, are **dual** if and only if $\mathbf{S_B}$ is $\mathbf{A}$-covariant and $\mathbf{S_A}$ is $\mathbf{B}$-covariant (see (56) and (57)). The corresponding Hermitian operators, $\mathcal{B}$ and $\mathcal{A}$, respectively, and the corresponding variables, are also **dual**. Finally, two complete orthonormal sets, $\{e_{\mathbf{A}}(b,\cdot) : b \in \mathbb{R}\}$ and $\{e_{\mathbf{B}}(a,\cdot) : a \in \mathbb{R}\}$, are **dual** bases for $\mathcal{H}_1$ if and only if they are related by (59).

Proposition 1 states that the families $\{\mathbf{A}_\alpha\}$ and $\{\mathbf{B}_\beta\}$ are dual if and only if their corresponding eigenfunctions are dual bases for $\mathcal{H}_1$. Moreover, Proposition 1 implies that given a family of unitary operators $\{\mathbf{A}_\alpha\}$, and the definition of covariance as in (56) and (57), there is a unique dual family $\{\mathbf{B}_\beta\}$ which can be obtained via (58) and (53).

Examples. We have already seen in Section 4 that the sets $\{e_{\mathbf{T}}(f',\cdot)\}$ and $\{e_{\mathbf{F}}(t',\cdot)\}$ are dual bases for $L_2(\mathbb{R})$ and $\mathbf{S_F}$ is covariant to $\{\mathbf{T}_\mu\}$. Moreover, it can be easily verified that the dual family defined via (53), with $\{e_{\mathbf{F}}(t',\cdot)\}$ as the basis functions, is precisely $\{\mathbf{F}_\nu\}$. Thus, $\{\mathbf{T}_\mu\}$ and $\{\mathbf{F}_\nu\}$ are dual families of unitary operators, the corresponding Hermitian operators $\mathcal{F}$ and $\mathcal{T}$, respectively, are dual operators, and hence time and frequency are dual variables.

Now consider the operators $\{\mathbf{D}_\sigma\}$ with the basis $\{e_{\mathbf{D}}(c,\cdot)\}$. From Section 4 we know that the dual basis is $\{e_{\mathbf{C}}(d,\cdot)\}$ defined in (44). Using (44) and (45), the corresponding dual family of unitary operators, $\{\mathbf{C}_\rho : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)\}$, is given by

$$\begin{aligned}(\mathbf{C}_\rho s)(t) &= \int_{\mathbb{R}_+} e_{\mathbf{C}}(x,t) e^{-i2\pi\rho x} (\mathbf{S_C} s)(x) dx = \int_{\mathbb{R}_+} e^{-x/2} \delta(x - \ln(t)) e^{-i2\pi\rho x} e^{x/2} s(e^x) dx \\ &= e^{-i2\pi \ln(t)\rho} s(t) \ ,\end{aligned} \tag{60}$$

which is the family of "hyperbolic-modulation" operators [25, 39, 40]. It can be easily verified that the $\mathbf{S_D}$ (Mellin) transform defined in (35) is $\mathbf{C}$-covariant; that is $(\mathbf{S_D}\mathbf{C}_\rho s)(c) = (\mathbf{S_D} s)(c+\rho)$. Thus, the family of hyperbolic-modulation operators $\{\mathbf{C}_\rho\}$ is the dual of dilation operators $\{\mathbf{D}_\sigma\}$.

Finally, using (53) for the basis $\{e_{\mathbf{R}}(q,\cdot)\}$ defined in (47), the family of operators $\{\mathbf{R}_\zeta\}$, dual to $\{\mathbf{Q}_\kappa\}$, can be easily computed to be

$$(\mathbf{R}_\zeta S)(f) = \int_{\mathbb{R}} e_{\mathbf{R}}(q,f) e^{-i2\pi q\zeta} (\mathbf{S_R} S)(q) dq = \frac{|f_o|}{|f||\frac{f_o}{f} + \zeta|} S\left(\frac{f_o}{\frac{f_o}{f} + \zeta}\right) \ , \tag{61}$$

where it should be noted that the operator is represented in the frequency domain. It is readily verified that the $\mathbf{S_Q}$ transform defined in (36) is $\mathbf{R}$-covariant; that is, $(\mathbf{S_Q}\mathbf{R}_\zeta s)(r) = (\mathbf{S_Q} s)(r + \zeta)$.

6 Dual Operators and Joint Distributions

Suppose that $\{\mathbf{A}_\alpha\}$ and $\{\mathbf{B}_\beta\}$ are dual families of unitary operators, as defined in the previous section and characterized in Proposition 1. In the context of joint distributions, a natural question is: what kind of properties do the joint distributions of the corresponding dual variables possess? In particular, we are interested in a shift-covariance property analogous to the one possessed by time-frequency representations (see (8)). Such a shift-covariance property is desirable because it enables simultaneous estimation of changes in the variables by estimating the corresponding shifts in the distribution. Moreover, such covariance properties are also crucial from the viewpoint of optimal signal detection based on joint signal representations [20, 19, 21]. Before we address this question, we need to make the concept of a shift operator precise. The concept of a time-, frequency- or scale-shift operator is evident because of our familiarity with time, frequency and scale changes, and as we mentioned in the Introduction, many authors have explicitly or implicitly discussed it from a group theoretic perspective [26, 27, 28, 29, 30, 12, 31, 18]. But what does shift in an arbitrary variable mean? More precisely, for a given variable, how do we define and find the family of shift operators? The answer is intimately tied to the concept of duality as explained next.

6.1 Shift operators

Let $\mathcal{B}$ and $\mathcal{A}$ be the Hermitian operators, defined on $\mathcal{H}_1$, corresponding to the dual families $\{\mathbf{A}_\alpha\}$ and $\{\mathbf{B}_\beta\}$, respectively; that is

$$\mathbf{A}_\alpha = e^{-i2\pi\alpha\mathcal{B}} \quad , \quad \mathbf{B}_\beta = e^{-i2\pi\beta\mathcal{A}} \quad , \quad \text{and} \quad \mathbf{S}_\mathbf{B} = \mathbb{F}^{\mp 1}\mathbf{S}_\mathbf{A} \ , \tag{62}$$

and let a and b be the variables associated with the Hermitian operators $\mathcal{A}$ and $\mathcal{B}$, respectively. Recall that $\mathcal{B}$ and $\{\mathbf{A}_\alpha\}$ share the same set of eigenfunctions $\{e_\mathbf{A}(b,\cdot) = e_\mathcal{B}(b,\cdot)\}$; that is,

$$\mathcal{B}e_\mathbf{A}(b,\cdot) = b e_\mathbf{A}(b,\cdot) \quad \text{and} \quad \mathbf{A}_\alpha e_\mathbf{A}(b,\cdot) = e^{-i2\pi\alpha b} e_\mathbf{A}(b,\cdot) \ , \tag{63}$$

where $b \in \mathbb{R}$. As evident from the spectral representation (52), the operators $\mathcal{B}$ and $\mathbf{A}_\alpha$ are "diagonal" in the $\mathbf{S}_\mathbf{A}(=\mathbf{S}_\mathcal{B})$ representation; that is,

$$(\mathbf{S}_\mathbf{A}\mathcal{B}\mathbf{S}_\mathbf{A}^{-1}x)(b) = bx(b) \quad \text{and} \tag{64}$$

$$(\mathbf{S}_\mathbf{A}\mathbf{A}_\alpha\mathbf{S}_\mathbf{A}^{-1}x)(b) = e^{-i2\pi\alpha b}x(b) \ . \tag{65}$$

Thus, $(\mathbf{S}_\mathbf{A}s)(b)$ is the "natural" representation of the signal s in terms of the variable b; that is, $(\mathbf{S}_\mathbf{A}s)(b) = (\mathbf{S}_\mathcal{B}s)(b)$ is the b-domain representation of s, just as $(\mathbf{S}_\mathbf{T}s)(f) = (\mathbf{S}_\mathcal{F}s)(f) = (\mathbb{F}s)(f)$ is the f-domain (frequency domain) representation of s. Similarly, $(\mathbf{S}_\mathbf{B}s)(a) = (\mathbf{S}_\mathcal{A}s)(a)$ is the a-domain representation of s. These observations allows us to define the shift operator for a variable.

Definition: Shift operator. Corresponding to a variable a, associated with a Hermitian operator $\mathcal{A}$, a shift operator $\mathbf{G}_\epsilon$, with the shift parameter ϵ, is one which produces a translation of $\pm\epsilon$ in the a-domain representation, $(\mathbf{S}_\mathcal{A}s)(a)$, of s; that is, $(\mathbf{S}_\mathcal{A}\mathbf{G}_\epsilon s)(a) = (\Gamma_{\pm\epsilon}\mathbf{S}_\mathcal{A}s)(a) = (\mathbf{S}_\mathcal{A}s)(a \mp \epsilon)$.[14]

This definition is obviously consistent with the definition of time and frequency shift operators. Note that the definition implies that

$$\mathbf{G}_\epsilon = \mathbf{S}_\mathcal{A}^{-1}\Gamma_{\pm\epsilon}\mathbf{S}_\mathcal{A} = \mathbf{S}_\mathcal{A}^{-1}\mathbb{F}^{-1}\Lambda_{\pm\epsilon}\mathbb{F}\mathbf{S}_\mathcal{A} \ , \quad \text{or} \tag{66}$$

$$\mathbf{G}_\epsilon = \mathbf{S}_\mathcal{A}^{-1}\Gamma_{\pm\epsilon}\mathbf{S}_\mathcal{A} = \mathbf{S}_\mathcal{A}^{-1}\mathbb{F}\Lambda_{\mp\epsilon}\mathbb{F}^{-1}\mathbf{S}_\mathcal{A} \tag{67}$$

where in (66) we used (26) and in (67) we used the companion relation (27). Clearly, $\mathbf{G}_\epsilon$ is a unitary operator and, using our convention for the spectral representation of a unitary operator, we can associate the following two spectral representations with it:

$$\mathbf{G}_\epsilon = \mathbf{S}_\mathcal{A}^{-1}\Gamma_{\epsilon}\mathbf{S}_\mathcal{A} = \mathbf{S}_\mathcal{A}^{-1}\mathbb{F}^{-1}\Lambda_{\epsilon}\mathbb{F}\mathbf{S}_\mathcal{A} \ , \quad \text{or} \tag{68}$$

$$\mathbf{G}_\epsilon = \mathbf{S}_\mathcal{A}^{-1}\Gamma_{-\epsilon}\mathbf{S}_\mathcal{A} = \mathbf{S}_\mathcal{A}^{-1}\mathbb{F}\Lambda_{\epsilon}\mathbb{F}^{-1}\mathbf{S}_\mathcal{A} \ . \tag{69}$$

[14]In the more general setting of LCA groups, the appropriate notion of a shift operator is related to the group translation operator which replaces Γ_ϵ [23, 15].

Recalling (58) which relates the spectral representations of dual operators, we note from (68) and (69) that the concept of a shift operator is intimately related to the notion of duality: a shift operator for the variable a (operator $\mathcal{A}$) is precisely the unitary operator associated, via Stone's theorem, with the *dual* Hermitian operator ($\mathcal{B}$). Note that operators $\mathcal{T}$, $\mathcal{F}$ and the families $\{\mathbf{T}_\mu\}$,$\{\mathbf{F}_\nu\}$, respectively, are related precisely in this sense. Moreover, by uniqueness of the spectral representation, the shift operator is unique up to the sign of the shift.

Thus, since $\mathcal{A}$ and $\mathcal{B}$ are assumed to be dual, we note from (62) that the family $\{\mathbf{A}_\alpha = e^{-ie2\pi\alpha\mathcal{B}}\}$ precisely fits the definition of shift operators for the variable a, with α as the shift parameter (recall that $\mathbf{S}_\mathcal{A} = \mathbf{S}_\mathbf{B}$ is $\mathbf{A}$-covariant). Similarly, $(\mathbf{S}_\mathcal{B} s)(b) = (\mathbf{S}_\mathbf{A} s)(b)$ is the b-domain representation of s, and thus $\{\mathbf{B}_\beta = e^{-i2\pi\beta\mathcal{A}}\}$ is a family of shift operators for the variable b, with β as the shift parameter, which is consistent with the fact that $\mathbf{S}_\mathcal{B} = \mathbf{S}_\mathbf{A}$ is $\mathbf{B}$-covariant. We will refer to $\{\mathbf{A}_\alpha\}$ and $\{\mathbf{B}_\beta\}$ as the families of a-shift and b-shift operators, respectively.

Relationship between Hermitian and unitary operator correspondences. As we mentioned in the Introduction, variables have traditionally been associated with Hermitian operators [1]. More recently, variables have also been associated with families of unitary operators [25, 35, 17]. Stone's theorem, which uniquely relates a Hermitian operator to a family of unitary operators, makes this correspondence of variables with the two types of operators possible. However, existing treatments exclusively consider one type of correspondence or the other, and do not address the relationship between the two. This has resulted in some confusion as demonstrated in the following examples. The concept of duality makes the relationship between the two correspondences precise. As far as associating variables with unitary operators is concerned, in order to be consistent with the unitary operator correspondence for time and frequency variables, a variable must be associated with the family of shift operators for that variable. Thus, if we start with two families $\{\mathbf{A}_\alpha\}$ and $\{\mathbf{B}_\beta\}$, and they necessarily have to be dual in order to be consistent with Hermitian operator correspondence, the family $\{\mathbf{A}_\alpha\}$ should be associated (as shift operators) with precisely the variable corresponding to the Hermitian operator $\mathcal{A}$ defined by the *dual* family $\{\mathbf{B}_\beta\}$ via Stone's theorem (14).

For the remaining sections, it is worth remembering that $\mathbf{S}_\mathbf{A} = \mathbf{S}_\mathcal{B}$ and $\mathbf{S}_\mathbf{B} = \mathbf{S}_\mathcal{A}$; that is, the signal representation with respect to the eigenfunctions of a Hermitian operator is identical to the representation with respect to the eigenfunctions of its exponential operator (*dual* shift operator). We will be using the two representations interchangeably.

Examples. First consider the familiar example of time and frequency operators. The frequency operator $\mathcal{F}$ and the time-shift operators $\{\mathbf{T}_\mu\}$ share the same set of eigenfunctions, the complex exponentials, as given in (30). Thus, the f-domain representation is the familiar Fourier transform, and the frequency-shift operator $\mathbf{F}_\nu$, as given by (9), is precisely the exponentiated dual Hermitian operator, the dual operator being time operator $\mathcal{T}$ in this case. Similarly, the time-shift operator $\mathbf{T}_\mu$ is given by (9).

Now consider the dual families of dilation operators $\{\mathbf{D}_\sigma\}$ and hyperbolic modulation operators $\{\mathbf{C}_\rho\}$ defined in (28) and (60). Let $\mathcal{C}$ and $\mathcal{D}$, respectively, be the corresponding Hermitian operators defined as in (14), and let c and d, respectively, be the variables associated with them. Then, by applying Stone's theorem (14), $\mathcal{C}$ and $\mathcal{D}$ are given by [1]

$$(\mathcal{C}s)(t) = -\frac{i}{2\pi}[s(t)/2 + t\dot{s}(t)] = ([-\frac{i}{4\pi}\mathbf{I} + \mathcal{T}\mathcal{F}]s)(t) \ , \qquad (70)$$

$$(\mathcal{D}s)(t) = \ln(t)s(t) = (\ln(\mathcal{T})s)(t) \ . \qquad (71)$$

Note that $\mathcal{C}$ defined in (70) is precisely the "time scale" operator defined in [1, 10, 34]; that is, the variable c is defined to be the "scale" variable. Now $\mathcal{C}$ and $\{\mathbf{D}_\sigma\}$ share the same eigenfunctions $\{e_\mathbf{D}(c, \cdot)\}$ defined in

(32), and thus the natural c-domain representation is the Mellin transform $(\mathbf{S_D}s)(c)$. Thus, the family of scale-shift (c-shift) operators is $\{\mathbf{C}_\rho = e^{-i2\pi\rho\mathcal{D}}\}$, the family of hyperbolic modulation operators [25, 39, 40], which is consistent with the fact that $\mathbf{S_D}$, defined in (35), is $\mathbf{C}$-covariant. Thus, if we start with the Hermitian operator correspondence for scale used in [1, 10, 34], then $\{\mathbf{C}_\rho\}$, being the family of shift operators, should be assigned to the scale variable. This is in contrast with [25, 31] in which the family $\{\mathbf{D}_\sigma\}$, instead of $\{\mathbf{C}_\rho\}$, is associated with scale. In relation to the Hermitian correspondence used in [1, 10, 34], the family $\{\mathbf{D}_\sigma = e^{i2\pi\sigma\mathcal{C}}\}$ is precisely the family of shift operators for the *dual* variable d associated with the operator $\mathcal{D}$ (Recall that $(\mathbf{S_C}s)(d) = (\mathbf{S}_\mathcal{D}s)(d)$ is $\mathbf{D}$-covariant). However, based on our intuition about scale, the family $\{\mathbf{D}_\sigma\}$ seems to be the more appropriate one (compared to $\{\mathbf{C}_\rho\}$) to be associated with scale as is done (explicitly or implicitly) in [28, 29, 30, 31, 18]. With the family $\{\mathbf{D}_\sigma\}$ associated with scale, the corresponding Hermitian scale operator is $\mathcal{D}$ defined in (71), and the natural scale representation is the transform $(\mathbf{S_C}s)(d) = (\mathbf{S}_\mathcal{D}s)(d) = e^{d/2}s(e^d)$ defined in (45), which is precisely the transform covariant to scale changes derived in [31].

Finally, let $\mathcal{R}$ and $\mathcal{Q}$ be the dual Hermitian operators corresponding to $\{\mathbf{Q}_\kappa\}$ and $\{\mathbf{R}_\zeta\}$, respectively, and let r and q be the corresponding dual variables. Then, $\mathcal{R}$ and $\mathcal{Q}$ are given by

$$(\mathcal{R}S)(f) = \frac{f_o}{f}S(f) = \left(\frac{f_0}{\mathcal{F}}S\right)(f) , \tag{72}$$

$$(\mathcal{Q}S)(f) = \frac{if}{2\pi f_o}[S(f) - f\dot{S}(f)] = \frac{1}{2\pi f_o}([i\mathcal{F} - 2\pi\mathcal{F}^2\mathcal{T}]S)(f) , \tag{73}$$

where $\mathcal{R}$ is the same as the inverse frequency operator defined in [1]. The natural "inverse frequency"-domain representation is $(\mathbf{S}_\mathcal{R}s)(r) = (\mathbf{S_Q}s)(r)$ defined in (36), and the "inverse frequency"-shift operators are precisely the exponentiated dual operators $\{\mathbf{R}_\zeta = e^{-i2\pi\zeta\mathcal{Q}}\}$ defined in (61). Similarly, the q-domain representation is $(\mathbf{S}_\mathcal{Q}s)(q) = (\mathbf{S_R}s)(q)$ defined in (46), and the q-shift operators are the exponentiated inverse frequency operators $\{\mathbf{Q}_\kappa\}$ defined in (29).

6.2 Joint Distributions of Dual Variables

Let a, b be two dual variables with $\mathcal{A}$, $\mathcal{B}$ as the corresponding Hermitian operators, and $\{\mathbf{A}_\alpha\}$, $\{\mathbf{B}_\beta\}$ as the corresponding unitary (shift) operators, respectively, as given by (62). Because of duality, the operators $\mathcal{A}$ and $\mathcal{B}$ are structurally similar to the time and frequency operators $\mathcal{T}$ and $\mathcal{F}$, and this similarity is precisely captured by the notion of unitary equivalence first introduced in the context of joint distributions in [25]. We next define unitary equivalence and characterize the structural similarity of dual operators to the time and frequency operators. By virtue of unitary equivalence, the joint distributions of dual variables can be simply characterized via the methodology developed in [25, 35] as an alternative to Cohen's general method [1] mentioned in the Introduction.

Definition: Unitary Equivalence [25]. Two operators, $\mathcal{A}_1$ and $\mathcal{A}_2$, are **unitarily equivalent** if there exists a unitary transformation (or isometry) $\mathbf{U}$ such that

$$\mathcal{A}_1 = \mathbf{U}\mathcal{A}_2\mathbf{U}^{-1} . \tag{74}$$

It readily follows that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are related by (74) then their eigenfunctions are related by

$$\mathbf{S}_{\mathcal{A}_1}^{-1} = \mathbf{U}\mathbf{S}_{\mathcal{A}_2}^{-1} \tag{75}$$

Using the spectral representation of operators, it is fairly straightforward to show that two operators are dual if and only if they are unitarily equivalent to the time and frequency operators. We demonstrate unitary equivalence of unitary operators only; the equivalence of corresponding Hermitian operators follows directly from the fact that

$$\mathcal{A}_1 = \mathbf{U}\mathcal{A}_2\mathbf{U}^{-1} \quad \Leftrightarrow \quad e^{-i2\pi\gamma\mathcal{A}_1} = \mathbf{U}e^{-i2\pi\gamma\mathcal{A}_2}\mathbf{U}^{-1} \ , \tag{76}$$

which can be easily verified using (13) and (14).

We first note that in the time domain, the time-shift operator $\mathbf{T}_\mu$ is identical to the translation operator $\mathbf{\Gamma}_\alpha$, and the frequency-shift operator $\mathbf{F}_\nu$ is identical to the diagonal operator $\mathbf{\Lambda}_\alpha$. Now suppose that $\{\mathbf{A}_\alpha\}$ and $\{\mathbf{B}_\beta\}$ are dual unitary operators with their eigenfunctions related by (38); that is $\mathbf{S_B} = \mathbb{F}^{-1}\mathbf{S_A}$. Then, using their spectral representations and the fundamental relation (26) we have

$$\mathbf{A}_\alpha = \mathbf{S}_\mathbf{A}^{-1}\mathbf{\Lambda}_\alpha\mathbf{S_A} = \mathbf{S_A}\mathbb{F}\mathbf{\Gamma}_\alpha\mathbb{F}^{-1}\mathbf{S_A} = \mathbf{S}_\mathbf{B}^{-1}\mathbf{T}_\alpha\mathbf{S_B} \ , \quad \text{and} \tag{77}$$

$$\mathbf{B}_\beta = \mathbf{S}_\mathbf{B}^{-1}\mathbf{\Lambda}_\beta\mathbf{S_B} = \mathbf{S}_\mathbf{B}^{-1}\mathbf{F}_\beta\mathbf{S_B} \ , \tag{78}$$

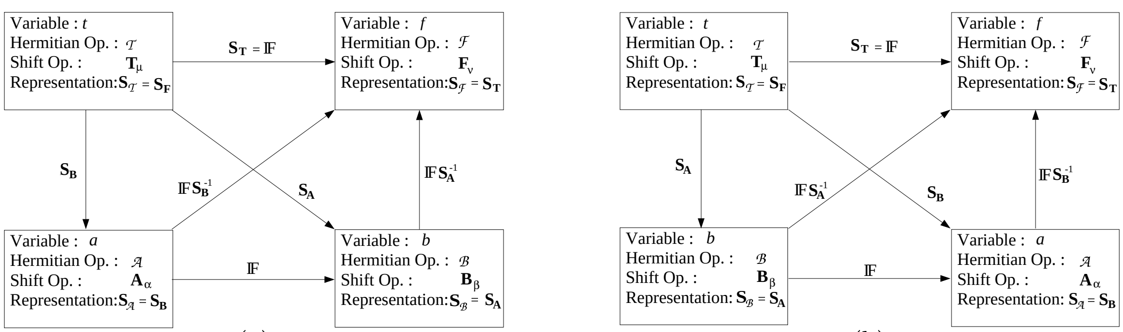
and thus $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are unitarily equivalent to $\mathbf{T}_\mu$ and $\mathbf{F}_\nu$, respectively, via $\mathbf{U} = \mathbf{S}_\mathbf{B}^{-1} = \mathbf{S}_\mathbf{A}^{-1}\mathbb{F}$. This situation is depicted in Figure 1(a). Similarly, if the eigenfunctions of $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are related by (40), that is, $\mathbf{S_B} = \mathbb{F}\mathbf{S_A}$, then $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are unitarily equivalent to $\mathbf{F}_\nu$ and $\mathbf{T}_\mu$, respectively, via $\mathbf{U} = \mathbf{S}_\mathbf{A}^{-1} = \mathbf{S}_\mathbf{B}^{-1}\mathbb{F}$. This situation is depicted in Figure 1(b).

Conversely, if $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are unitarily equivalent to $\mathbf{T}_\mu$ and $\mathbf{F}_\nu$, respectively, via $\mathbf{U}$, then

$$\mathbf{A}_\alpha = \mathbf{U}\mathbf{T}_\alpha\mathbf{U}^{-1} = \mathbf{U}\mathbf{\Gamma}_\alpha\mathbf{U}^{-1} = \mathbf{U}\mathbb{F}^{-1}\mathbf{\Lambda}_\alpha\mathbb{F}\mathbf{U}^{-1} \ , \quad \text{and} \tag{79}$$

$$\mathbf{B}_\beta = \mathbf{U}\mathbf{F}_\beta\mathbf{U}^{-1} = \mathbf{U}\mathbf{\Lambda}_\beta\mathbf{U}^{-1} \ , \tag{80}$$

and thus $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are dual because $\mathbf{S_B} = \mathbf{U}^{-1} = \mathbb{F}^{-1}\mathbf{S_A}$ (Figure 1(a)). Similarly, if $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are unitarily equivalent to $\mathbf{F}_\nu$ and $\mathbf{T}_\mu$, respectively, then $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are dual with $\mathbf{S_B} = \mathbb{F}\mathbf{S_A}$ (Figure 1(b)). The following proposition summarizes the unitary equivalence between dual variables and time and frequency.



(a) (b)

Figure 1: Schematics depicting unitary equivalence. (a) $\{\mathcal{A}, \mathcal{B}\} \Leftrightarrow \{\mathcal{T}, \mathcal{F}\}$. (b) $\{\mathcal{A}, \mathcal{B}\} \Leftrightarrow \{\mathcal{F}, \mathcal{T}\}$.

Proposition 2. Let $\mathcal{A}$ and $\mathcal{B}$ be two Hermitian operators admitting the formal representations

$$\mathcal{A} = \mathbf{S}_\mathcal{A}^{-1}\mathbf{\Lambda}\mathbf{S}_\mathcal{A} \quad \text{and} \quad \mathcal{B} = \mathbf{S}_\mathcal{B}^{-1}\mathbf{\Lambda}\mathbf{S}_\mathcal{B} \ , \tag{81}$$

and let $\{\mathbf{A}_\alpha\}$ and $\{\mathbf{B}_\beta\}$ be two families of unitary operators defined as

$$\mathbf{A}_\alpha = e^{-i2\pi\alpha\mathcal{B}} = \mathbf{S}_\mathcal{B}^{-1}\mathbf{\Lambda}_\alpha\mathbf{S}_\mathcal{B} \quad \text{and} \quad \mathbf{B}_\beta = e^{-i2\pi\beta\mathcal{A}} = \mathbf{S}_\mathcal{A}^{-1}\mathbf{\Lambda}_\beta\mathbf{S}_\mathcal{A} \ . \tag{82}$$

Then, $\mathcal{A}$ and $\mathcal{B}$ are dual operators with $\{\mathbf{A}_\alpha\}$ and $\{\mathbf{B}_\beta\}$ as the corresponding dual families of unitary (shift) operators if and only if $\mathcal{A}$ and $\mathcal{B}$ ($\{\mathbf{A}_\alpha\}$ and $\{\mathbf{B}_\beta\}$) are unitarily equivalent to $\mathcal{T}$ and $\mathcal{F}$ ($\{\mathbf{T}_\mu\}$ and $\{\mathbf{F}_\nu\}$), respectively, via $\mathcal{U} = S_{\mathcal{A}}^{-1}$, or to $\mathcal{F}$ and $\mathcal{T}$ ($\{\mathbf{F}_\nu\}$ and $\{\mathbf{T}_\mu\}$), respectively, via $\mathcal{U} = S_{\mathcal{B}}^{-1}$.

Examples. First consider the dual families of operators $\{\mathbf{D}_\sigma\}$ and $\{\mathbf{C}_\rho\}$ defined in (28) and (60), respectively, where the eigenfunctions are related by $\mathbf{S_C} = \mathbb{F}^{-1}\mathbf{S_D}$. Then, $\{\mathbf{D}_\sigma\}$ is unitarily equivalent to $\{\mathbf{T}_\mu\}$, and thus $\mathcal{D}$ is unitarily equivalent to $\mathcal{T}$. The transformation $\mathbf{U}$ is given by $\mathbf{U}^{-1} = \mathbf{S}_{\mathcal{D}} = \mathbf{S_C}$ as defined in (45). Thus, we have the relations

$$\mathbf{D}_\sigma = \mathbf{S_C}^{-1}\mathbf{T}_\sigma\mathbf{S_C} \ , \quad \mathcal{D} = \mathbf{S_C}^{-1}\mathcal{T}\mathbf{S_C} \ \text{ and} \tag{83}$$

$$\mathbf{C}_\rho = \mathbf{S_C}^{-1}\mathbf{F}_\rho\mathbf{S_C} \ , \quad \mathcal{C} = \mathbf{S_C}^{-1}\mathcal{F}\mathbf{S_C} \ , \tag{84}$$

which can be easily verified.

Similarly, consider the dual operators $\mathcal{R}$ and $\mathcal{Q}$ defined in (72) and (73), respectively, where the eigenfunctions are related by $\mathbf{S}_{\mathcal{R}} = \mathbb{F}\mathbf{S}_{\mathcal{Q}}$. Since $\mathbf{S}_{\mathcal{F}} = \mathbb{F}\mathbf{S}_{\mathcal{T}}$, $\mathcal{Q}$ is unitarily equivalent to $\mathcal{T}$ with the unitary transform being $\mathbf{U}^{-1} = \mathbf{S}_{\mathcal{Q}} = \mathbf{S_R}$ as defined in (46). Similarly, $\mathcal{R} = \mathbf{S_R}^{-1}\mathcal{F}\mathbf{S_R}$, $\mathbf{R}_\zeta = \mathbf{S_R}^{-1}\mathbf{F}_\zeta\mathbf{S_R}$, and $\mathbf{Q}_\kappa = \mathbf{S_R}^{-1}\mathbf{T}_\kappa\mathbf{S_R}$.

Now consider joint distributions of a and b. Using Cohen's method outlined in the Introduction, a joint a-b distribution, $P_s(a,b)$, of a signal s, can be obtained as[15] [1]

$$P_s(a,b) = \int\int M_s(\theta,\tau)e^{-i2\pi\theta a}e^{-i2\pi\tau b}d\theta d\tau \ \ , \tag{85}$$

where the characteristic function $M_s(\theta,\tau)$ can be computed as [1]

$$M_s(\theta,\tau) = \int s^*(t)(\mathbf{M}(\theta,\tau)s)(t)dt = \int (S_{\mathcal{B}}s)^*(b)(\mathbf{M}(\theta,\tau)S_{\mathcal{B}}s)(b)db \tag{86}$$

$$= \int (S_{\mathcal{A}}s)^*(a)(\mathbf{M}(\theta,\tau)S_{\mathcal{A}}s)(a)da \ , \tag{87}$$

since the "average" of a function can be computed in any representation [1]. Proposition 2 implies that all the pairs of dual operators $\{\mathcal{A},\mathcal{B}\}$ can be generated by choosing different unitary transformations $\mathbf{U}$ and transforming the operators $\mathcal{T}$ and $\mathcal{F}$ via (74). Thus, for generating joint distributions of dual variables we can use the simpler method described in [25, 35] for variables unitarily equivalent to time and frequency: all joint a-b representations can be generated by first transforming the signal s by the unitary transform $\mathbf{S}_{\mathcal{A}}$ as in (87), or by the transform $\mathbf{S}_{\mathcal{B}}$ as in (86), and then computing any member of Cohen's class of time-frequency representations for the transformed signal (appropriately interpreting the transformed signal as being in the "time" or "frequency" domain [25, 35]). This is exactly the $\mathbf{U}$-Cohen's class in introduced in [25] with $\mathbf{U} = \mathbf{S}_{\mathcal{A}}$. Note that in some cases, one of the transforms $\mathbf{S}_{\mathcal{A}}$ or $\mathbf{S}_{\mathcal{B}}$ may have a simpler form than the other, and thus may be preferred for transforming the signal. For example, for the dual operators $\{\mathcal{C},\mathcal{D}\}$, the $\mathbf{S}_{\mathcal{D}}$ transform given in (45) has a much simpler form than the $\mathbf{S}_{\mathcal{C}}$ (Mellin) transform.

By virtue of unitary equivalence, in the a and b domains, the operators $\mathcal{A}$ and $\mathcal{B}$ behave exactly like the operators $\mathcal{T}$ and $\mathcal{F}$, respectively, in the time and frequency domains [25, 35]. In particular, the joint a-b distributions are covariant to a-b shifts; that is

$$P_{\mathbf{A}_\alpha\mathbf{B}_\beta s}(a,b) = P_{\mathbf{B}_\beta\mathbf{A}_\alpha s}(a,b) = P_s(a-\alpha, b+\beta) \ , \tag{88}$$

[15] We use the same symbols $P_s(\cdot,\cdot)$ and $M_s(\theta,\tau)$ for joint distributions and characteristic functions of arbitrary variables; the variables will be clear from the context.

which also implies that there is no coupling between a and b shifts; that is, an a-shift in the signal will produce a corresponding shift in $P_s(a,b)$, leaving the b variable unchanged, and similarly for b-shifts. This is a desirable property, particularly in scenarios where joint a-b representations are to be used in simultaneously estimating shifts in the variables a and b; for example, in nonstationary signal detection based on joint distributions of arbitrary variables [20, 19, 21]. The decoupled nature of shifts allows us to estimate the a-shift and the b-shift independently.

Moreover, since the commutator relation [1] between operators is independent of the domain of representation, $\mathcal{A}$ and $\mathcal{B}$ satisfy exactly the same commutator relation as $\mathcal{T}$ and $\mathcal{F}$ [1, 35]:

$$[\mathcal{A},\mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} = \frac{i\mathbf{I}}{2\pi} \ . \tag{89}$$

It can be shown, using (85), (86) and (87), that the commutator relation (89) is sufficient to guarantee the decoupled shift-covariance property (88). On the other hand, if the joint distributions of any two variables, a and b, possess the shift-covariance property (88), it can be shown that the $\mathbf{S_A}$ transform is $\mathbf{B}$-covariant and the $\mathbf{S_B}$ transform is $\mathbf{A}$-covariant (within phase factors). This observation, in conjunction with the fact shown in the appendix, implies that if (88) holds then the two operators $\mathcal{A}$ and $\mathcal{B}$ are dual. It then follows from Proposition 2 that the covariance relation (88) implies the commutator relation (89). Thus, the commutator relation (89) and the decoupled shift-covariance relation (88) are equivalent.[16]

Another aspect which highlights the intimate relationship between dual variables can be appreciated by considering the distributions of single variables. The a-distribution of a signal s is [1, 14]

$$P_s(a) = |(\mathbf{S}_{\mathcal{A}}s)(a)|^2 = |\langle x, e_{\mathcal{A}}(a,\cdot)\rangle|^2 = |\langle x, e_{\mathbf{B}}(a,\cdot)\rangle|^2 \ . \tag{90}$$

However, using the "characteristic function operator" approach [1], $P_s(a)$ can also be obtained from its characteristic function, $M_s(\theta)$, which can be directly computed from the signal as

$$M_s(\theta) \;=\; \langle e^{i2\pi\theta\mathcal{A}}s, s\rangle = \langle \mathbf{B}_{-\theta}s, s\rangle = \langle \mathbf{S}_{\mathbf{A}}^{-1}\Gamma_\theta \mathbf{S}_{\mathbf{A}}s, s\rangle \tag{91}$$

$$= \langle \Gamma_\theta \mathbf{S}_{\mathcal{B}}s, \mathbf{S}_{\mathcal{B}}s\rangle = \int (\mathbf{S}_{\mathcal{B}}s)^*(b)(\mathbf{S}_{\mathcal{B}}s)(b-\theta)db \ . \tag{92}$$

Thus, from (92), the characteristic function of $P_s(a)$ can be interpreted as an autocorrelation of the signal in precisely the dual domain (b-domain).

6.3 Discussion

A very important implication of the notion of duality and shift operators is that, in order to be consistent with associating time and frequency variables with the unitary time-shift and frequency-shift operators, we can consistently and interchangeably use the Hermitian operator correspondence or the unitary operator correspondence *if and only if* the two variables (operators) are dual. If the variables are not dual, the exponential operator corresponding to a Hermitian operator will *not* be the shift operator for the other variable, corresponding to the other Hermitian operator. This is precisely the reason behind the inconsistency between the Hermitian operator correspondence for scale used in [34] and the unitary operator correspondence used in [25].

Proposition 2 gives us an alternate way of finding the covariant transform (see section 4) for a family of unitary operators $\{\mathbf{A}_\alpha\}$. By Proposition 2 there exists a unitary transform $\mathbf{U}$ which makes $\{\mathbf{A}_\alpha\}$ and $\{\mathbf{T}_\mu\}$ unitarily equivalent; that is,

$$\mathbf{A}_\alpha = \mathbf{U}\mathbf{T}_\alpha\mathbf{U}^{-1} \ . \tag{93}$$

[16] A related result is given in [35].

Then, by (74) and (75), the eigenfunctions of the dual family $\{\mathbf{B}_\beta\}$ are given by $\mathbf{S}_\mathbf{B}^{-1} = \mathbf{U}\mathbf{S}_\mathbf{F}^{-1} = \mathbf{U}$, and thus the covariant transform is $\mathbf{S}_\mathbf{B} = \mathbf{U}^{-1}$. This is precisely the way in which a transform covariant to scale changes is derived in [31] which is essentially the $\mathbf{D}$-covariant transform $\mathbf{S}_\mathbf{C}$ defined in (45). Thus, given a family $\{\mathbf{A}_\alpha\}$, our approach gives a direct approach for finding the $\mathbf{A}$-covariant transform via Proposition 1 (see (59)), whereas the alternative approach based on unitary equivalence requires finding the unitary transform $\mathbf{U}$ in (93) as an intermediate step.

Finally, note that in most cases we used the time-domain representation as the default representation for the operators. However, we can immediately generate another pair of dual operators by changing the default representation. For example, the operators $\mathcal{C}$ and $\mathcal{D}$ were defined in (70) and (71) in terms of the time-domain representation. However, if we alternatively define them in the frequency-domain (for analytic signals); that is,

$$(\mathcal{C}_1 S)(f) = -\frac{i}{2\pi}[S(f)/2 + f\dot{S}(f)] \quad \text{and} \quad (\mathcal{D}_1 S)(f) = \ln(f)S(f) \ , \tag{94}$$

then we get another pair of dual operators $\{\mathcal{C}_1, \mathcal{D}_1\}$. The pair $\{\mathcal{C}, \mathcal{D}\}$ corresponds to joint representations of scale and hyperbolic-modulation [25, 1], whereas the pair $\{\mathcal{C}_1, \mathcal{D}_1\}$ corresponds to joint representations of scale and hyperbolic time-shift [25, 39, 40]. As mentioned in [25], the joint distributions of $\{\mathcal{C}, \mathcal{D}\}$ are analogous to the joint time-scale distributions derived in [10], whereas the joint representations of $\{\mathcal{C}_1, \mathcal{D}_1\}$ are analogous to the joint frequency-scale distributions derived in [10]. However, there is one difference between the joint distributions of the pairs $\{\mathcal{C}, \mathcal{D}\}$, $\{\mathcal{C}_1, \mathcal{D}_1\}$ and the joint time-scale and frequency-scale representations; the shift-covariance property (88) is not possessed by the joint time-scale or frequency-scale representations derived in [1, 10], the reason being that the operator pairs $\{\mathcal{T}, \mathcal{C}\}$ and $\{\mathcal{F}, \mathcal{C}\}$ are not dual.

7 Summary and Conclusions

Associating variables with operators, traditionally Hermitian operators, is fundamental to time-frequency analysis. More recently, variables have also been associated with parameterized unitary operators since, in some cases, meaningful signal transformations are best described as unitary operators [25, 35, 17]. Existing treatments exclusively adopt one type of correspondence or the other; the relationship between the two types of correspondences, though of fundamental importance, has not been adequately addressed. Stone's theorem shows that for unitary operators satisfying (10), the two types of correspondences are indeed equivalent. Moreover, by developing the notions of *duality* and *shift* operators, we explicitly characterize the relationship between the two correspondences, yielding the interpretation of unitary operators as shift operators, consistent with the unitary operator correspondence for time and frequency.

One of the concepts underlying most of the results in this paper is that of a covariant transform. Projection onto the eigenfunctions of a parameterized unitary operator yields a transform which is invariant to the operator; it ignores the changes in the signal effected by the operator. A covariant transform is needed if such changes in the signal (or the variable) are to be detected or estimated. Using Stone's theorem and fundamental Fourier transform properties, we show in this paper that, given a parameterized unitary operator satisfying (10), we can always find a unitary transform which is translationally covariant to the changes produced by the operator.

The covariant transform corresponding to a unitary operator naturally leads to the definition of another, unique, family of unitary operators. The two families of unitary operators define the notion of duality; the *invariant* transform for one is the *covariant* transform for the other and vice versa. This notion of duality has strong implications for joint representations of arbitrary variables. In particular, given a Hermitian operator corresponding to a variable, the appropriate corresponding unitary (shift) operator is precisely the "exponential" operator of the *dual* Hermitian operator as defined by Stone's theorem. Thus, we can consistently and interchangeably use the Hermitian or unitary operator correspondence if and only if the two variables (operators) are dual.

Joint distributions of dual variables share analogues of all the characteristic properties of joint time-frequency representations. In particular, a translation in one variable produces a corresponding translation in the representation, leaving the dual variable unaffected. In fact, two variables (operators) are dual if and only if they are unitarily equivalent to time and frequency.

Finally, in this paper we restricted our discussion to unitary operators satisfying (10), which are unitary representations of the translation group $\mathbb{R}$ [24], and which are pertinent to Cohen's method for joint distributions of arbitrary variables [1]. What if the unitary operators of interest (representing a meaningful signal transformation, for instance) are representations of some other group?[17] Such operators have been considered by Baraniuk in [14] in which joint distributions based on locally compact abelian (LCA) groups are proposed. In fact, all the issues discussed in this paper can be addressed in the more general setting of arbitrary LCA groups [24, 23, 32], and some of them have been addressed in [15, 33, 14]. As explained in footnote 6 in the Introduction, the motivation for restricting the discussion to $\mathbb{R}$ in this paper comes from the equivalence results of [15, 33].

Appendix

Fact. Let $\{\mathbf{A}_\alpha : \mathcal{H}_1 \to \mathcal{H}_1\}$ and $\{\mathbf{B}_\beta : \mathcal{H}_1 \to \mathcal{H}_1\}$ be two families of unitary operators satisfying (10) and admitting the formal spectral representations

$$\begin{aligned} \mathbf{A}_\alpha &= \mathbf{S}_\mathbf{A}^{-1} \mathbf{\Lambda}_\alpha \mathbf{S}_\mathbf{A} \quad \text{and} && (95) \\ \mathbf{B}_\beta &= \mathbf{S}_\mathbf{B}^{-1} \mathbf{\Lambda}_\beta \mathbf{S}_\mathbf{B} \;, && (96) \end{aligned}$$

respectively. If the unitary transform $\mathbf{S}_\mathbf{A}$ is $\mathbf{B}$-covariant or $\mathbf{S}_\mathbf{B}$ is $\mathbf{A}$-covariant; that is,

$$\begin{aligned} (\mathbf{S}_\mathbf{A} \mathbf{B}_\beta s)(b) &= (\mathbf{S}_\mathbf{A} s)(b+\beta) \quad \text{or} && (97) \\ (\mathbf{S}_\mathbf{B} \mathbf{A}_\alpha s)(a) &= (\mathbf{S}_\mathbf{B} s)(a-\alpha) \;, && (98) \end{aligned}$$

for all $s \in \mathcal{H}_1$, then $\mathbf{S}_\mathbf{A}$ and $\mathbf{S}_\mathbf{B}$ are related as

$$\mathbf{S}_\mathbf{B} = \mathbb{F}^{-1} \mathbf{S}_\mathbf{A} \tag{99}$$

Proof: Suppose that (98) holds. It follows that

$$\mathbf{A}_\alpha = \mathbf{S}_\mathbf{B}^{-1} \mathbf{\Gamma}_\alpha \mathbf{S}_\mathbf{B} \;, \tag{100}$$

which, using the fundamental relationship (26), can be equivalently written as

$$\mathbf{A}_\alpha = \mathbf{S}_\mathbf{B}^{-1} \mathbb{F}^{-1} \mathbf{\Lambda}_\alpha \mathbb{F} \mathbf{S}_\mathbf{B} \;. \tag{101}$$

Comparing (101) with (95) and using the uniqueness of the spectral representation [24], we find that $\mathbf{S}_\mathbf{A} = \mathbb{F}\mathbf{S}_\mathbf{B}$ must hold which is equivalent to the relationship (99). Similarly, it can be shown easily that the covariance relation (97) implies (99). This completes the proof.

[17] Such unitary group representations are fundamental to covariance properties of joint distributions. Many authors have discussed such representations, particularly for the affine and Heisenberg groups, and in relation to time-frequency and time-scale representations [26, 27, 28, 29, 30, 12, 22, 18, 17, 16].

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