Covariant Time-Frequency Representations Through Unitary Equivalence

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Abstract—**We propose a straightforward characterization of all quadratic time-frequency representations covariant to an important class of unitary signal transforms (namely, those having two continuous-valued parameters and an underlying group structure). Thanks to a fundamental theorem from the theory of Lie groups, we can describe these representations simply in terms of unitary transformations of the well-known Cohen's and affine classes.**

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I. Introduction

Quadratic time-frequency representations (TFRs) have found wide application in problems requiring time-varying spectral analysis [1,2]. Since the distribution of signal energy jointly over time and frequency coordinates does not have a unique representation, there exist many different TFRs and many different ways to obtain them. Up to the present, two classes of TFRs have predominated: *Cohen's class* [1] and the *affine class* [2–4]. Cohen's class TFRs are covariant to time and frequency shifts in the signal, whereas affine class TFRs are covariant to time shifts and scale changes in the signal.

Since time shifts, frequency shifts, and scale changes are not the only important signal transformations occurring in nature, several new TFR classes matching different transformations have been proposed recently. The TFRs of the *hyperbolic class* [5] are covariant to "hyperbolic time shifts" and scale changes, while the TFRs of the *power classes* [6] are covariant to "power time shifts" and scale changes. Unitarily transformed Cohen's and affine classes furnish even more TFRs [7,8]. While extremely simple both in concept and in application, the unitary equivalence or "warping" procedure that generates these TFRs leads at once to an infinite number of new TFR classes covariant to a broad class of signal transformations. Coping with this veritable explosion of new TFR classes demands a comprehensive theory for time-frequency analysis.

In this paper, we propose a simple characterization of covariant TFRs based on the theory of unitary equivalence. The sufficiency of unitary equivalence for this task comes as somewhat of a surprise, since this method proves more powerful than previously surmised.

II. Cohen's Class and the Affine Class

We denote the TFR of a signal $s \in L^2(\mathbb{R})$ using the operator notation $(\mathbf{P}s)(t,f)$, where $\mathbf{P} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2)$ is the TFR mapping and t and f are the time and frequency coordinates, respectively. We will emphasize only quadratic TFRs in this paper.

Each TFR in *Cohen's class* [1,2] can be expressed as

$$(\mathbf{C}s)(t,f) \;=\; \iint (\mathbf{A}s)(\theta,\tau)\, \phi(\theta,\tau)\, e^{-j2\pi(\theta t+\tau f)}\, d\theta\, d\tau,$$

in terms of the narrowband ambiguity function of the signal $(\mathbf{A}s)(\theta,\tau) = \int s(t+\frac{\tau}{2})\, s^*(t-\frac{\tau}{2})\, e^{j2\pi\theta t}\, dt$ and a kernel function $\phi(\theta,\tau)$. TFRs generated by fixed kernels are covariant to the time shift operator $(\mathbf{T}_n s)(x) \equiv s(x-n)$ and the frequency shift operator $(\mathbf{F}_m s)(x) \equiv e^{j2\pi m x} s(x)$

$$(\mathbf{C}\mathbf{F}_m\mathbf{T}_n s)(t,f) \;=\; (\mathbf{C}s)(t-n,\; f-m) \tag{1}$$

with $m, n \in \mathbb{R}$. Conversely, all quadratic TFRs covariant in this way must belong to Cohen's class [2]. Covariance by translation is natural for Cohen's class TFRs, because $\mathbf{T}$ and $\mathbf{F}$ comprise the heart of the unitary representation on $L^2(\mathbb{R})$ of the *Weyl-Heisenberg group*, with

$$(\mathbf{F}_{m_1}\mathbf{T}_{n_1})(\mathbf{F}_{m_2}\mathbf{T}_{n_2}) \;=\; e^{-j2\pi m_2 n_1}\,\mathbf{F}_{m_1+m_2}\mathbf{T}_{n_1+n2}.$$

(See [4,9,10] for more details on the role of group theory in time-frequency analysis.)

Each TFR in the *affine class* [2–4] can be expressed as

$$(\mathbf{Q}s)(t,f) \;=\; \iint (\mathbf{A}s)(\theta,\tau)\,\psi(\theta/f, f\tau)\,e^{-j2\pi(\theta t+\tau f)}\,d\theta\,d\tau,$$

with kernel $\psi(\theta,\tau)$. TFRs generated by fixed kernels are covariant to the time shift operator and the scale change operator $(\mathbf{D}_d s)(x) \equiv |d|^{-1/2}s(x/d)$

$$(\mathbf{Q}\mathbf{T}_n\mathbf{D}_d s)(t,f) \;=\; (\mathbf{Q}s)\left(\frac{t-n}{d},\, df\right). \tag{2}$$

Conversely, all quadratic TFRs covariant in this way must belong to the affine class [2–4]. Affine covariance is natural for these TFRs, because $\mathbf{T}$ and $\mathbf{D}$ comprise the unitary representation on $L^2(\mathbb{R})$ of the *affine group*, with

$$(\mathbf{T}_{n_1}\mathbf{D}_{d_1})(\mathbf{T}_{n_2}\mathbf{D}_{d_2}) \;=\; \mathbf{T}_{n_1+d_1 n_2}\mathbf{D}_{d_1 d_2}.$$

III. Unitarily Equivalent Time-Frequency Representations

To match signal transformations different from time shifts, frequency shifts, and scale changes, new classes of TFRs have been developed, including the hyperbolic class of TFRs covariant to scale changes and "hyperbolic time shifts" [5] and the power classes of TFRs covariant to scale changes and "chirp time shifts" [6]. While both of these classes can be derived from first principles, they can also be obtained directly by transforming Cohen's class and the affine class.

In particular, each hyperbolic class TFR can be expressed as $\mathbf{VCU}$, where $\mathbf{C}$ is a Cohen's class TFR, $\mathbf{U} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a unitary signal transformation and $\mathbf{V} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a unitary change of variables [5]. Each power class TFR can be similarly expressed as $\mathbf{VQU}$, where $\mathbf{Q}$ is an affine class TFR [6]. (See [5,6,8] for the exact form of the transformations $\mathbf{U}$ and $\mathbf{V}$.) The primary advantage of this transformation-based derivation is its conceptual and computational simplicity: To compute a hyperbolic or power class TFR, we simply preprocess the signal by the unitary transform $\mathbf{U}$, compute a Cohen's class TFR $\mathbf{C}$ or affine class TFR $\mathbf{Q}$ of the transformed signal, and then warp the axes of the resulting distribution by $\mathbf{V}$.

Transformation of the Cohen's and affine classes does not have to stop with the hyperbolic and power classes. By varying $\mathbf{U}$ and $\mathbf{V}$, we can generate an *infinite number* of transformed Cohen's and affine classes. We now summarize the salient features of the resulting theory of unitarily equivalent TFRs developed in detail in [8].

Transformed Cohen's and affine class distributions are covariant, not to time and frequency shifts and scale changes, but to the unitarily equivalent operators

$$\widetilde{\mathbf{T}}_n = \mathbf{U}^{-1}\mathbf{T}_n\mathbf{U}, \quad \widetilde{\mathbf{F}}_m = \mathbf{U}^{-1}\mathbf{F}_m\mathbf{U}, \quad \widetilde{\mathbf{D}}_d = \mathbf{U}^{-1}\mathbf{D}_d\mathbf{U}.$$

To see this, ignore $\mathbf{V}$ for the moment and note that

$$(\mathbf{CU}\widetilde{\mathbf{F}}_m\widetilde{\mathbf{T}}_n s)(a,b) \;=\; (\mathbf{CU}s)(a-n,\, b-m) \tag{3}$$

$$(\mathbf{QU}\widetilde{\mathbf{T}}_n\widetilde{\mathbf{D}}_d s)(a,b) \;=\; (\mathbf{QU}s)\left(\frac{a-n}{d},\, db\right). \tag{4}$$

The preprocessed distributions $\mathbf{CU}$ and $\mathbf{QU}$ maintain the same translation and affine covariances exhibited by the distributions from which they are derived (compare (3) to (1) and (4) to (2)), because the operator pairs $\widetilde{\mathbf{F}}\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{T}}\widetilde{\mathbf{D}}$ remain unitary representations of the Weyl-Heisenberg and affine groups, respectively. Thus, transformed Cohen's and affine classes are *unitarily equivalent* to the original Cohen's and affine classes.

While the coordinates (a,b) of the preprocessed distributions $\mathbf{CU}$ and $\mathbf{QU}$ do not correspond to time and frequency (in fact, they correspond to the physical quantities associated with the operators $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{F}}$), the postprocessing transformation $\mathbf{V}$ can warp (a,b) to new coordinates providing correct time-frequency localization [8]. Given a fixed $\mathbf{U}$, the procedure to determine the corresponding $\mathbf{V}$ is straightforward: We simply warp the axes of the distributions by functions $A(t,f)$ and $B(t,f)$ that describe the group delay and instantaneous frequency of the transformed eigenfunctions $\mathbf{U}^{-1}\delta(x-a)$ and $\mathbf{U}^{-1}e^{j2\pi bx}$ of $\widetilde{\mathbf{F}}$ and $\widetilde{\mathbf{T}}$, respectively.[1] For transformed Cohen's class distributions, we set

$$(\mathbf{VCU}s)\,(t,f) \;=\; (\mathbf{CU}s)[\,A(t,f),\, B(t,f)\,].$$

The affine class case is similar.

The unitarily equivalent TFRs $\mathbf{VCU}$ and $\mathbf{VQU}$ remain covariant to the operators $\widetilde{\mathbf{F}}\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{T}}\widetilde{\mathbf{D}}$, respectively, although $\mathbf{V}$ warps the group actions (3), (4) along the group delay and instantaneous frequency curves of the transformed eigenfunctions [8].

The simplicity of the unitary equivalence principle makes the study of the properties of unitarily equivalent TFR classes essentially trivial, since the attributes of any transformed class can

[1] The group delay of $\mathbf{U}^{-1}\delta(x-a)$ lies along the curve $a = A(t,f)$. The instantaneous frequency of $\mathbf{U}^{-1}e^{j2\pi bx}$ lies along the curve $b = B(t,f)$ [8].

be obtained immediately from those of Cohen's class or the affine class by a simple translation procedure. (We simply replace s by $\mathbf{U}s$ throughout and warp the axes.) This translation has been performed in detail in [8].

IV. A Unified Theory for Covariant Time-Frequency Representations

Unitary equivalence provides a simple means for developing an infinite number of different TFR classes. However, it is not *a priori* obvious that this theory encompasses all possible covariant TFRs. For instance, unitarily equivalent TFRs are bound (within warping) to the Weyl-Heisenberg group and affine group covariances they inherit from Cohen's class and the affine class. (In [7,8], this was viewed as a limitation of the theory.) A comprehensive theory of covariant TFRs thus seems a worthy goal.

A. Covariant Time-Frequency Representations

Our formulation will characterize TFRs covariant to a class of two-parameter unitary signal transformations $\mathbf{G}_{(p,q)}$ that generalize the time-frequency shift and time-scale change operators so natural for Cohen's class and affine class TFRs. Physical considerations (invertibility, composition, etc.) dictate that each of these transformations be a unitary group representation with group law "$\bullet$" [4,9,10]

$$\mathbf{G}_{(p_1,q_1)}\mathbf{G}_{(p_2,q_2)} \;=\; \mathbf{G}_{(p_1,q_1)\bullet(p_2,q_2)}. \tag{5}$$

We say a TFR $(\mathbf{P}s)(t,f)$ is *covariant* to $\mathbf{G}_{(p,q)}$ if

$$(\mathbf{P}\mathbf{G}_{(p,q)}s)(t,f) \;=\; (\mathbf{P}s)(t,f)\diamond(p,q),$$

where "$\diamond$" is the representation of $\mathbf{G}_{(p,q)}$ on the time-frequency plane (specifically, the coadjoint representation [9,10]). Note that the group property (5) of $\mathbf{G}_{(p,q)}$ yields immediately that

$$(\mathbf{P}\mathbf{G}_{(p_1,q_1)}\mathbf{G}_{(p_2,q_2)}s)(t,f) \;=\; (\mathbf{P}s)(t,f)\diamond[(p_1,q_1)\bullet(p_2,q_2)].$$

A similar (and equivalent) approach to covariance has been developed independently by Hlawatsch and Bölcskei in [11,12]. In their terminology, $\mathbf{G}$ is a *time-frequency displacement operator* with "$\diamond$" the associated *displacement function.*

Examples of displacement operators and functions include: $\mathbf{G}_{(m,n)} = \mathbf{F}_m \mathbf{T}_n$ and (1) for Cohen's class TFRs;[2] $\mathbf{G}_{(n,d)} = \mathbf{T}_n \mathbf{D}_d$ and (2) for affine class TFRs; $\mathbf{G}_{(m,n)} = \widetilde{\mathbf{F}}_m \widetilde{\mathbf{T}}_n = \mathbf{U}^{-1}\mathbf{F}_m \mathbf{T}_n \mathbf{U}$ and a version of (1) warped by $A(t,f), B(t,f)$ for unitarily equivalent Cohen's class TFRs; and $\mathbf{G}_{(n,d)} = \mathbf{U}^{-1}\mathbf{T}_n \mathbf{D}_d \mathbf{U}$ and a version of (2) warped by $A(t,f), B(t,f)$ for unitarily equivalent affine class TFRs.

B. A Simple Unified Theory

Unitary equivalence generates TFRs covariant to an infinite number of different two-parameter displacement operators $\mathbf{G}$. What we now show is that *there exist no covariant TFRs beyond these.* In other words, the simple unitary equivalence procedure described in Section III proves sufficient for characterizing all covariant TFRs.

The key realization is the following: Since the displacement operator $\mathbf{G}$ determining the covariance properties of a TFR class is constrained by (5) to be a unitary group representation, it is clear that the classification of all covariant TFR classes is equivalent to the classification of all two-parameter Lie groups (within a phase) that can act on the signal space $L^2(\mathbb{R})$. It has been recently pointed out [10] that there exist only *two* such Lie groups: the Weyl-Heisenberg group leading to displacement operators of the form $\mathbf{G} = \mathbf{U}^{-1}\mathbf{FTU}$ and the affine group leading to displacement operators of the form $\mathbf{G} = \mathbf{U}^{-1}\mathbf{TDU}$. Since the TFR classes covariant to these displacements correspond to unitarily equivalent Cohen's and affine classes, we have the following fundamental result.

Theorem: All quadratic TFRs that are covariant in the sense of Section IV-A can be represented in the form $\mathbf{VCU}$ or $\mathbf{VQU}$, with $\mathbf{C}$ a Cohen's class TFR, $\mathbf{Q}$ an affine class TFR, $\mathbf{U}$ a unitary signal transformation, and $\mathbf{V}$ a two-dimensional coordinate transformation as described in Section III.

V. Discussion and Conclusions

An observation regarding the dearth of two-parameter covariances has lead us to an extremely simple characterization of all covariant quadratic TFR classes as unitarily transformed Cohen's or affine classes. The "shortcut" of unitary equivalence gives this theory several advantages. First, we

[2] The Weyl-Heisenberg group actually has three, and not two, parameters; therefore, the displacement operator $\mathbf{G}_{(n,m)} = \mathbf{F}_m \mathbf{T}_n$ does not strictly obey (5). However, since the desired group $\mathbb{R}^2$ of translations in the time-frequency plane does not have a representation on the space $L^2(\mathbb{R})$, we are forced to employ the Weyl-Heisenberg group in signal processing applications. Fortunately, the third parameter plays the role of a phase and can be ignored.

see immediately that the set of all two-parameter time-frequency displacement operators inducing covariant TFRs is limited to operators of the form $\mathbf{U}^{-1}\mathbf{FTU}$ or $\mathbf{U}^{-1}\mathbf{TDU}$. Second, since each covariant TFR class corresponds directly to either Cohen's class or the affine class, we can leverage our years of experience with these classes into new contexts with no additional effort.

It is straightforward to demonstrate the equivalence of our theory to that of Hlawatsch and Bölcskei [11,12], since the axioms they impose on a time-frequency displacement operator constrain it to be (within a phase) a unitary representation of a two-parameter Lie group. The prime advantage of the present theory is its ease of use; in contrast, the Hlawatsch-Bölcskei construction appears quite complicated. Thus, despite its striking simplicity, the power and generality of unitary equivalent time-frequency analysis should not be underestimated.

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