# NEARLY SYMMETRIC ORTHOGONAL WAVELETS WITH NON-INTEGER DC GROUP DELAY*

*Ivan W. Selesnick, Jan E. Odegard and C. Sidney Burrus*
*Department of Electrical and Computer Engineering - MS 366*
*Rice University, 6100 Main St., Houston, TX 77005-1892, USA*
*selesi@ece.rice.edu, odegard@ece.rice.edu, csb@ece.rice.edu*

## ABSTRACT

This paper investigates the design of Coiflet-like nearly symmetric compactly supported orthogonal wavelets. The group delay is used as the main vehicle by which near symmetry is achieved. By requiring a specified degree of flatness of the group delay at $\omega = 0$ (equivalent to appropriate moment conditions), near symmetry is achieved. Gröbner bases are used to obtain the solutions to the defining nonlinear equations. It is found that the DC group delay that maximizes the group delay flatness at $\omega = 0$ is irrational – and for a length 10 orthogonal wavelet with three vanishing moments, the solution is presented.

## 1. INTRODUCTION

One of the interesting problems in the design of wavelets is the problem of symmetry. Coiflets are an example of wavelets with excellent symmetry properties [2, 3]. This paper discusses the design of nearly symmetric compactly supported orthogonal wavelets. The criterion used is the degree of flatness at $w = 0$ and $w = \pi$ of the scaling filter frequency response magnitude and its group delay (equivalent to appropriate moment conditions). The way in which the group delay approximates a constant is a traditional measure of symmetry in filter design.

Gröbner bases are used to obtain the solutions to the nonlinear equations. There are multiple solutions, and the group delay of all the real solutions are shown in the figures of the example below.

## 2. NOTATION

The transfer function of a length $N$ FIR filter is denoted by $H(z) = \sum_{n=0}^{N-1} h(n) z^{-1}$. The real and imaginary parts of the frequency response are denoted by $R(\omega) = \Re\{H(e^{\jmath\omega})\}$ and $I(\omega) = \Im\{H(e^{\jmath\omega})\}$. The frequency response square magnitude is then given by $F(\omega) = R^2(\omega) + I^2(\omega)$ and the group delay by $G(\omega) = (I(\omega)R'(\omega) - R(\omega)I'(\omega))/F(\omega)$.

To obtain an orthogonal wavelet, it is necessary to impose the constraints:

$$\sum_n h(n)h(n+2k) = c \cdot \delta_k. \tag{1}$$

*THIS WORK HAS BEEN SUPPORTED BY BNR AND BY NSF GRANT MIP-9316588.

To obtain a flat frequency response magnitude behavior, $H(z)$ is required to have a multiple degree zero at $z = -1$, so that $H(z)$ is of the form $H(z) = (z+1)^K P(z)$, see [2, 3].

In this paper, instead of spectrally factoring a symmetric polynomial to obtain the coefficients $h(n)$, the nonlinear equations are expressed in the coefficients $h(n)$ directly.

The moments of $H(z)$ will be useful. They are denoted and defined here by: $m(k) = \sum_{n=0}^{N-1} n^k h(n)$.

Note that because $F(\omega)$ and $G(\omega)$ are even functions of $\omega$, for odd $l$, $F^{(l)}(0)$ and $G^{(l)}(0)$ equal zero. The degree of flatness of the response magnitude at $\omega = 0$ is denoted by $M$:

$$F^{(2i)}(0) = 0 \qquad i = 1, \ldots, M. \tag{2}$$

The degree of flatness of the group delay at $\omega = 0$ is denoted by $L$:

$$G^{(2i)}(0) = 0 \qquad i = 1, \ldots, L. \tag{3}$$

The square magnitude derivatives at $\omega = 0$ are given by:

$$F^{(2n)}(0) = \binom{2n}{n} m^2(n) + 2\sum_{i=0}^{n-1} \binom{2n}{i} (-1)^{i+n} m(i)m(2n-i) \tag{4}$$

When $F(0) = 1$ and $F^{(2i)}(0) = 0$ for $i = 1, \ldots, n$, the group delay derivatives at $\omega = 0$ are given by:

$$G^{(2n)}(0) = \sum_{i=0}^{n} \frac{2n+1-2i}{2n+1} \binom{2n+1}{i} (-1)^{i+n} m(i)m(2n+1-i). \tag{5}$$

From (4), the first few derivatives of the square magnitude at $\omega = 0$ are:

$$F(0) = m_0^2 \tag{6}$$
$$F^{(2)}(0) = 2\,m_1^2 - 2\,m_0 m_2 \tag{7}$$
$$F^{(4)}(0) = 6\,m_2^2 + 2\,m_0 m_4 - 8\,m_1 m_3 \tag{8}$$
$$F^{(6)}(0) = 20\,m_3^2 - 2\,m_0 m_6 + 12\,m_1 m_5 - 30\,m_2 m_4. \tag{9}$$

From (5), the first few derivatives of the group delay at $\omega = 0$ are:

$$G(0) = m_0 m_1 \tag{10}$$

$$G^{(2)}(0) = -m_0 m_3 + m_1 m_2 \tag{11}$$

$$G^{(4)}(0) = m_0 m_5 - 3\, m_1 m_4 + 2\, m_2 m_3 \tag{12}$$

$$G^{(6)}(0) = -m_0 m_7 + 5\, m_1 m_6 - 9\, m_2 m_5 + 5\, m_3 m_4. \tag{13}$$

(The equation $m_0 = 1$ is used in this paper.) By inspecting $F^{(2i)}(0)$ and $G^{(2i)}(0)$, it can be seen that when an equal number of the derivatives are made to vanish, then these DC flatness constraints are equivalent to the moment equations: $m_i = m_1^i$ for an appropriate range of $i$. However, if the number of group delay equations used is less than the number of magnitude equations, then not enough of the flatness constraints can be written as $m_i = m_1^i$ to obtain the filter coefficients $h(n)$, making the equations more difficult to solve.

## 3. PROBLEM FORMULATION

To obtain orthogonal wavelets with a specified degree of flatness of the magnitude and group delay, the following problem formulation is suggested. Given $N$, $K$, and $A$ (the specified DC group delay), find $N$ real filter coefficients $h(0), \ldots, h(N-1)$ such that:

1. $H(1) = 1$.
2. $\sum_n h(n) h(2k+n) = \frac{1}{2}\delta_k$.
3. $H(z)$ has a root at $z = -1$ of order $K$.
4. $G(0) = A$.
5. $G^{(2i)}(0) = 0$ for $i = 1, \ldots, N/2 - K - 1$.

## 4. GRÖBNER BASES

For this paper, Gröbner bases [1] were used to solve this problem formulation. Given a system of multivariate polynomials, a Gröbner basis (GB) is a new set of multi-variable polynomial equations, having the same set of solutions. When the lexicographic ordering of monomials is used, and there is a finite number of solutions, the 'last' equation of the GB will be a polynomial in a single variable – so its roots can be computed. These roots can be substituted into the remaining equations, etc – like back substitution in Gaussian elimination for linear equations. Unfortunately, this use of GBs is only practical for small problem sizes. We were unable to obtain wavelets of longer lengths for this paper using GBs due to the high computation needed. (There are methods using GBs with other monomial orderings for solving multi-variable polynomial systems, which we have not yet examined [5].)

## 5. DISCUSSION

Our goal is to obtain a set of coefficients for which the group delay is very flat. In Figure 1, the group delays for a set of length 10 filters (scaling vectors for orthogonal wavelets) are shown. In each figure, the group delay is incremented by one quarter. Each of these filters possess a zero of multiplicity 3 at $z = -1$. Because there are multiple solutions to the problem formulated above, for each case there are several group delay curves shown, each of which correspond to a filter satisfying the requirements listed above.

The problem formulated above can be modified so that the value of the group delay at $\omega = 0$ is not specified. With the extra degree of freedom so obtained, an extra derivative of the group delay can be made to vanish. The Gröbner basis for the resulting set of polynomial equations was found for $N = 10$, $K = 3$ using the software 'Singular' [4]. (The Gröbner basis for this system contained a polynomial of degree 20.) The scaling filter coefficients are given in Table 1. The Hölder exponent was found to be 1.0441. Figure 2 shows the scaling function, the zero plot, and the group delay. $G^{(i)}(0) = 0$ for $i = 1, \ldots, 5$. Here, $G(0)$ was not specified, but was obtained by extracting the appropriate root from the following polynomial of degree 20:

$$\begin{aligned} &64A^{20} - 5760A^{19} + 241280A^{18} - 6246720A^{17} + \\ &111944784A^{16} - 1473690240A^{15} + 14760732960A^{14} - \\ &114954798240A^{13} + 705334664064A^{12} - \\ &3434246514720A^{11} + 13301429369040A^{10} - \\ &40913515089360A^{9} + 99378635912264A^{8} - \\ &188777678032800A^{7} + 276459291710920A^{6} - \\ &306048229816680A^{5} + 249409993446324A^{4} - \\ &144246954606480A^{3} + 55971826143300A^{2} - \\ &13095607284000A + 1405452313125 \end{aligned} \tag{14}$$

This polynomial has only two real roots. The two real roots, 3.75501 and 5.24498, are located symmetrically about 4.5. The solution corresponding to one of these roots is the time-reversal of the solution obtained by using the other one of these roots. All the roots of this polynomial are shown in Figure 3. Due to the symmetry about 4.5, the change of variables $y^2 = x - 4.5$ gives a new polynomial of degree 10 the roots of which can be mapped back.

## 6. CONCLUSION

By imposing constraints on the group delay derivatives at $\omega = 0$, and by using Gröbner bases to solve the resulting multivariate polynomial system of equations, Coiflet-like wavelets can be obtained. Currently, however, this technique is only practical for low order filters. Using this approach, for a fixed number of vanishing moments, it is possible to find the DC group delay that maximizes the number of DC group delay derivatives, as was done in the example above.

## REFERENCES


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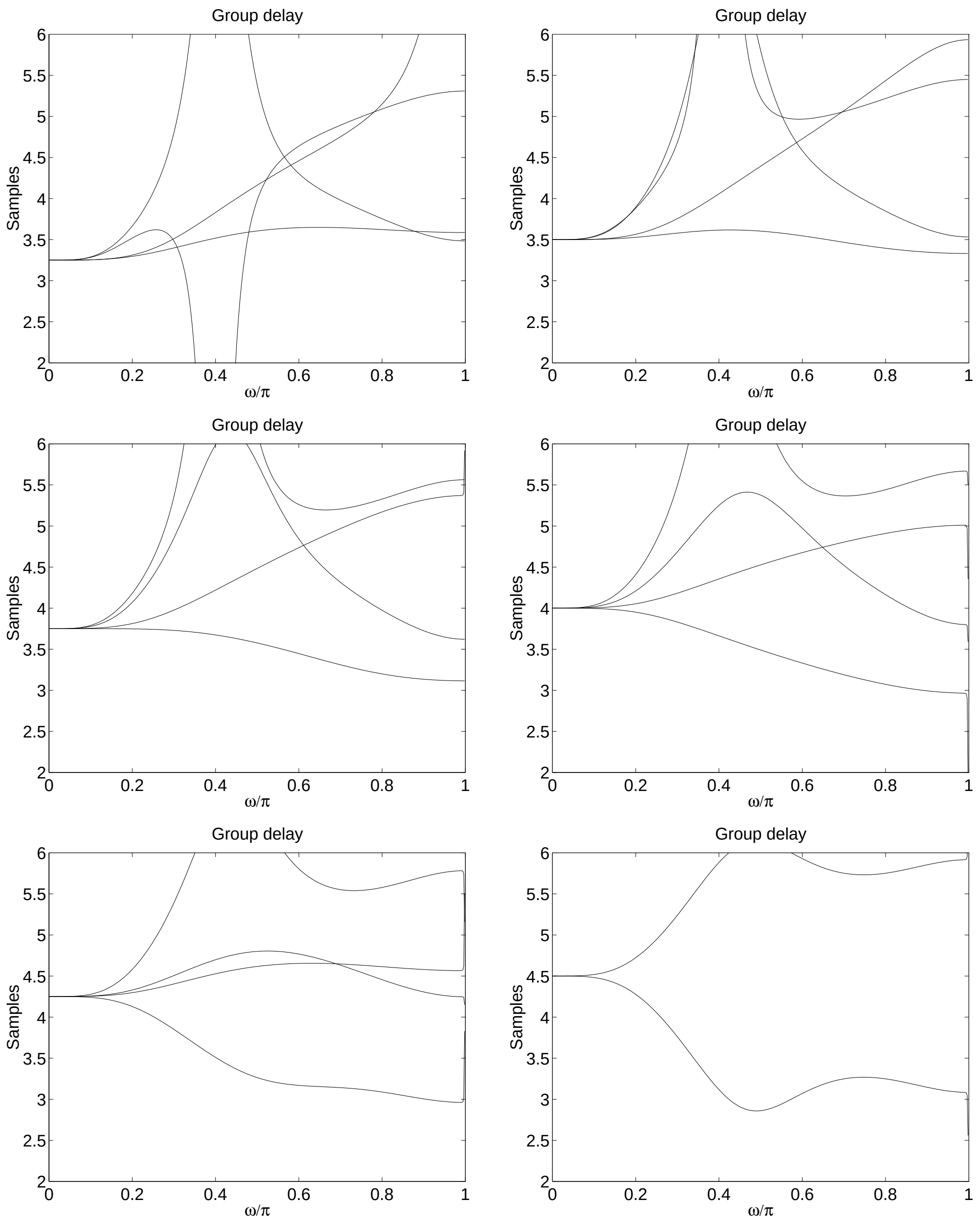


Figure 1. The group delays of all the real solutions to the problem formulated in Section 3. of length $N = 10$ having a root at $z = -1$ of multiplicity 3. For each curve shown, $G^{(i)}(0) = 0$ for $i = 1, 2, 3$.

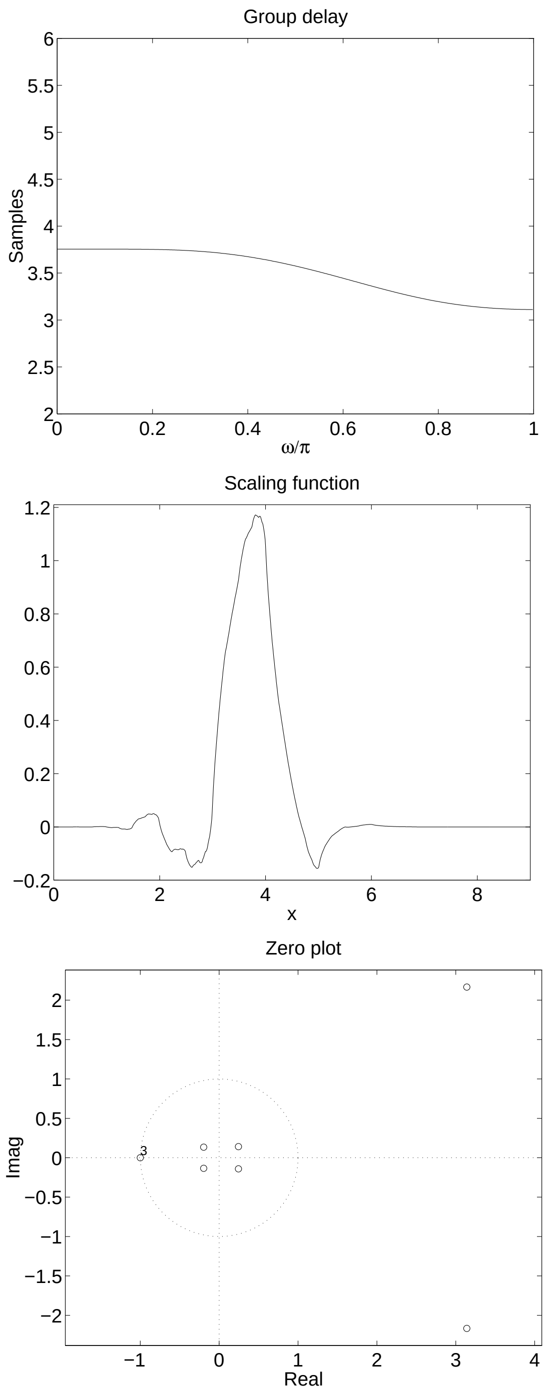


Figure 2. The wavelet with length $N = 10$ having a root at $z = -1$ of multiplicity 3, for which $G^{(i)}(0) = 0$ for $i = 1, 2, 3, 4, 5$. For this wavelet, $G(0)$ was not specified, but was obtained by finding the roots of a polynomial of degree 20. The consequent value of $G(0)$ is 3.75501.

Table 1. Length 10 scaling filter for which $m_1 = G(0) = 3.75501$ and $G^{(i)} = 0$ for $i = 1, \ldots, 5$. The scaling function, zero plot, and group delay are shown in Figure 2.

| $n$ | $h(n)$ |
|---|---|
| 0 | 0.01572426325815 |
| 1 | −0.05303980885284 |
| 2 | −0.01627935203249 |
| 3 | 0.41058417280995 |
| 4 | 0.55011176730188 |
| 5 | 0.15022838452622 |
| 6 | −0.05303701336740 |
| 7 | −0.00880453389945 |
| 8 | 0.00348033483987 |
| 9 | 0.00103178541613 |

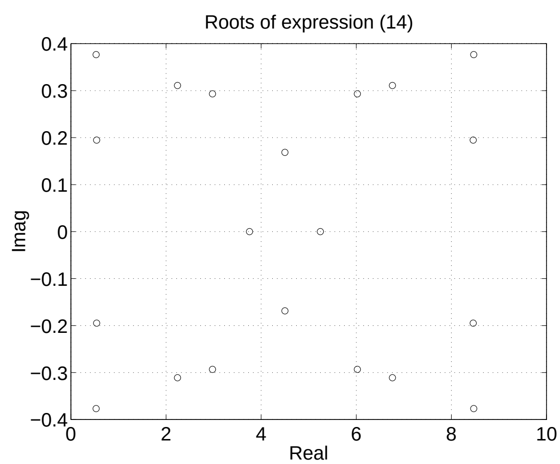


Figure 3. The roots of the polynomial (14).