NONLINEAR-PHASE MAXIMALLY-FLAT LOWPASS FIR FILTER DESIGN *

Ivan W. Selesnick and C. Sidney Burrus
Department of Electrical and Computer Engineering - MS 366
Rice University, 6100 Main St., Houston, TX 77005-1892, USA
selesi@ece.rice.edu, csb@ece.rice.edu
tel: (713) 527-8750 x3508

ABSTRACT

This paper reports a new analytic technique for the design of nonlinear-phase maximally-flat lowpass FIR filters. By subjecting the response magnitude and the group delay (individually) to differing numbers of flatness constraints, a new family of filters is obtained. With these filters, the delay can be reduced while maintaining relatively constant group delay in the passband, without significantly altering the response magnitude.

1. INTRODUCTION

A fair amount of attention has been focused on the design of non-symmetric digital FIR filters. Laakso et al. discuss in [6] the design of non-symmetric FIR filters for fractional delay applications. Other recent work has given attention to the design of FIR filters that approximate, in the Chebyshev norm, a given response magnitude and group delay [3, 13] or a given response magnitude and phase [2, 7]. Baher [1] gives an analytic technique for obtaining non-symmetric FIR filters having a maximally-flat behavior (see also [9, 10]). However, in [1], the degrees of flatness of the response magnitude and group delay at $\omega = 0$ can differ by no more than one. In the proposed paper, an analytic technique is given for the same problem Baher addresses in [1], but the degree of flatness of the response magnitude and group delay need not be approximately equal.

Like the above cited papers, this paper considers the problem of giving up exactly linear phase for approximately linear phase in exchange for a smaller delay and improvement in the response magnitude. The family of new FIR filters described below are obtained by subjecting the frequency response magnitude and the group delay (individually) to differing numbers of flatness constraints. The approach taken in this paper is appropriate when:

1. Exactly linear phase is not required.
2. Some degree of phase linearity is desired.
3. A maximally-flat frequency response is desired.

2. NOTATION

The transfer function of a length N FIR filter is denoted by $H(z) = \sum_{n=0}^{N-1} h(n)z^{-1}$. The real and imaginary parts of the frequency response are denoted by $R(\omega) = \Re\{H(e^{\jmath\omega})\}$ and $I(\omega) = \Im\{H(e^{\jmath\omega})\}$. The frequency response square magnitude is then given by $F(\omega) = R^2(\omega) + I^2(\omega)$ and the group delay by $G(\omega) = (I(\omega)R'(\omega) - R(\omega)I'(\omega))/F(\omega)$.

The number of zeros of $H(z)$ at $z = -1$ is denoted by K. Note that because $F(\omega)$ and $G(\omega)$ are even functions of ω, for odd l, $F^{(l)}(0)$ and $G^{(l)}(0)$ equal zero. The degree of flatness of the response magnitude at $\omega = 0$ is denoted by M:

$$F^{(2i)}(0) = 0 \qquad i = 1, \ldots, M. \tag{1}$$

The degree of flatness of the group delay at $\omega = 0$ is denoted by L:

$$G^{(2i)}(0) = 0 \qquad i = 1, \ldots, L. \tag{2}$$

A filter satisfying these conditions will be said to have flatness parameters (K, L, M).

The moments of $H(z)$ will be useful. They are denoted and defined by: $m(k) = \sum_{n=0}^{N-1} n^k h(n)$.

The square magnitude derivatives at $\omega = 0$ are given by:

$$F^{(2n)}(0) = \binom{2n}{n} m^2(n) + 2\sum_{i=0}^{n-1} \binom{2n}{i} (-1)^{i+n} m(i)m(2n-i) \tag{3}$$

When $F(0) = 1$ and $F^{(2i)}(0) = 0$ for $i = 1, \ldots, n$, the group delay derivatives at $\omega = 0$ are given by:

$$G^{(2n)}(0) = \sum_{i=0}^{n} \frac{2n+1-2i}{2n+1} \binom{2n+1}{i} (-1)^{i+n} m(i)m(2n+1-i). \tag{4}$$

From (3), the first few derivatives of the square magnitude at $\omega = 0$ are:

$$F(0) = m_0^2 \tag{5}$$

$$F^{(2)}(0) = 2\,m_1^2 - 2\,m_0 m_2 \tag{6}$$

$$F^{(4)}(0) = 6\,m_2^2 + 2\,m_0 m_4 - 8\,m_1 m_3 \tag{7}$$

$$F^{(6)}(0) = 20\,m_3^2 - 2\,m_0 m_6 + 12\,m_1 m_5 - 30\,m_2 m_4. \tag{8}$$

The derivatives of $G(\omega)$ are similar in appearance.

The half-magnitude frequency, denoted by ω_o, is that frequency at which the response magnitude equals one half. Like the 3 dB point, it indicates the location of the cut-off point.

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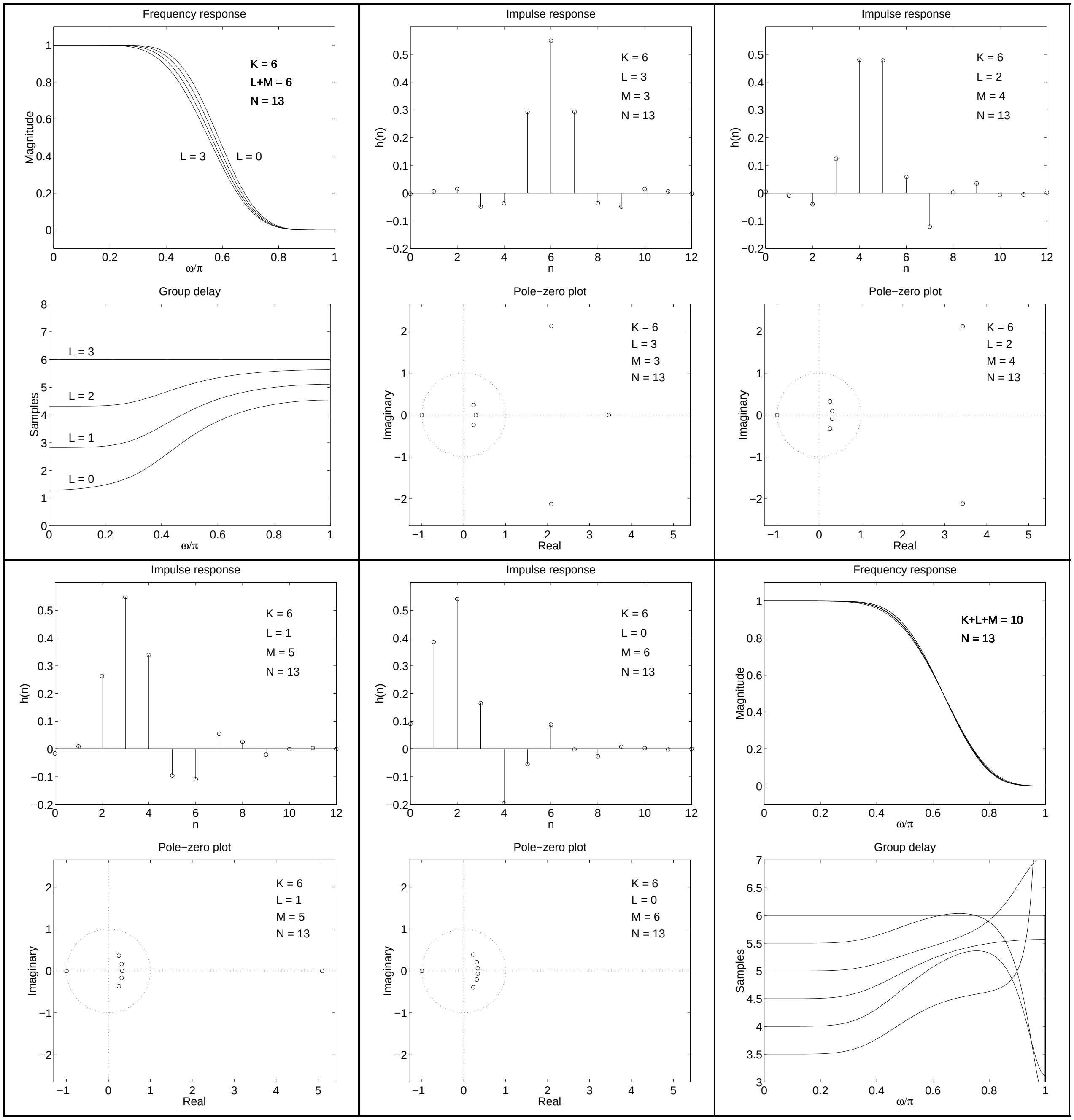


Figure 1. A selection of nonlinear-phase maximally-flat filters of length 13. The four filters shown for which $K + L + M = 12$ are solutions to the problem formulation of Section 3. For each of these four filters, the zero at $z = -1$ is of multiplicity 6. The length 13 filters, the frequency responses of which are shown in the lower right panel, are obtained by giving up two degrees of flatness and by specifying that the half-magnitude frequency be 0.636π — and that the specified DC group delay be $3.5 : 0.5 : 6$. See [11, 12].

3. BASIC PROBLEM FORMULATION

To obtain maximally-flat FIR filters having different degrees of magnitude and group delay flatness, the following problem formulation is suggested. Given the flatness parameters K, L, M, (with $K > 0$, $M \geq 0$, $L \leq M$), find N filter coefficients $h(0), \ldots, h(N-1)$ such that:

1. $N = K + L + M + 1$.
2. $F(0) = 1$.
3. $H(z)$ has a root at $z = -1$ of order K.
4. $F^{(2i)}(0) = 0$ for $i = 1, \ldots, M$.
5. $G^{(2i)}(0) = 0$ for $i = 1, \ldots, L$.

4. DISCUSSION

Note that the problem formulated in Section 3 gives rise to nonlinear equations. Consequently, the existence of multiple solutions should not be surprising, and indeed, that is true here.

To find all the solutions, a Gröbner basis can be used. Given a system of multivariate polynomials, a Gröbner basis (GB) is a new set of multivariate polynomial equations, having the same set of solutions [4]. When the lexicographic ordering of monomials is used, and there is a finite number of solutions, the "last" equation of the GB will be a polynomial in a single variable — so its roots can be computed. These roots can be substituted into the remaining equations, etc — like back substitution in Gaussian elimination for linear equations. Unfortunately, Gröbner basis solutions are only practical for small problems, because computing a Gröbner basis is highly computationally intensive.

It is informative to construct a table indicating the number of solutions as a function of K, L and M. It turns out that the number of solutions is independent of K. The number of solutions as a function of L and M is indicated in Table 1 for the first few L and M. We should note that many solutions have complex coefficients or posses frequency response magnitudes that are unacceptable between 0 and π. For this reason, it is useful to tabulate the number of *real* solutions possessing *monotonic* responses, as is done in Table 2. From Table 2, two distinct regions emerge. Let us define two regions in the (L, M) plane. Define region I as all pairs L, M for which $\lfloor \frac{M-1}{2} \rfloor \leq L \leq M$. Define region II as all pairs (L, M) for which $0 \leq L \leq \lfloor \frac{M-1}{2} \rfloor - 1$. See Table 3. It turns out that for (L, M) in region I, all the variables in the problem formulation, except $G(0)$, are linearly related and can be eliminated, yielding a polynomial in $G(0)$; the details are given in [11, 12]. For region II, we have no similarly simple technique.

4.1. Examples

Panels 1 through 5 of Figure 1 show the frequency responses of four different FIR filters of length 13 for which $K + L + M = 12$. Each of these filters has 6 zeros at $z = -1$ ($K = 6$) and 6 zeros contributing to the flatness of the passband at $z = 1$ ($L + M = 6$). The four filters shown were obtained using the four values $L = 0, 1, 2, 3$.

Table 1. Number of solutions to the problem formulated in Section 3.

		L							
		0	1	2	3	4	5	6	7
M	0	1							
	1	2	3						
	2	4	4	5					
	3	8	6	6	7				
	4	16	8	8	8	9			
	5	32	16	10	10	10	11		
	6	64	26	12	12	12	12	13	
	7	128	48	24	14	14	14	14	15

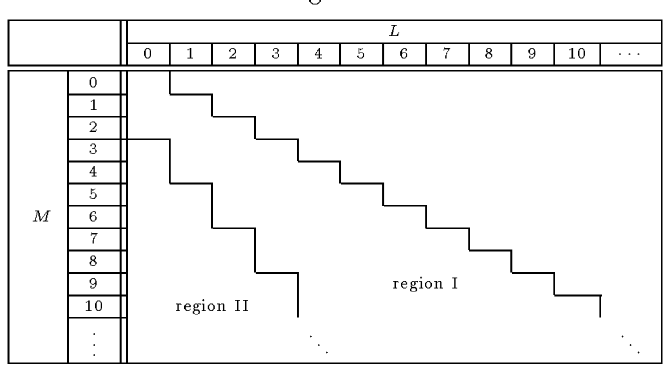
Table 2. Number of real monotonic solutions, not counting time-reversals.

		L							
		0	1	2	3	4	5	6	7
M	0	1							
	1	1	1						
	2	1	1	1					
	3	2	1	1	1				
	4	2	1	1	1	1			
	5	4	2	1	1	1	1		
	6	4	2	1	1	1	1	1	
	7	8	4	2	1	1	1	1	1

When $L = 3$, $M = 3$, the symmetric filter shown in the second panel of Figure 1 is obtained. This filter is most easily obtained using formulas for maximally-flat symmetric filters Herrmann presents in [5]. When $L = 0$, $M = 6$, the minimum-phase filter shown in the fifth panel of Figure 1 is obtained. This filter is most easily obtained by spectrally factoring a length 25 maximally-flat symmetric filter. The other two filters shown ($L = 2$, $M = 4$ and $L = 1$, $M = 5$) can *not* be obtained using the formulas of Herrmann. They are new and provide a compromise solution.

Observe that for the filters shown, the way in which the passband zeros are split between the interior of the unit circle and its exterior is given by the values L and

Table 3. Regions I and II.



M. It is interesting to note that the location of these zeros in this regard was not part of the way in which the problem was formulated – so it is very satisfying.

It may be observed that the half-magnitude frequencies of the four filters in Figure 1 are unequal. This is to be expected, because the half-magnitude frequency was not included in the problem formulation above. In the problem formulation of Section 3 both the half-magnitude frequency and the DC group delay can be only indirectly controlled by specifying K, L, and M.

4.2. Continuously Tuning ω_o and $G(0)$

To understand the relationship between ω_o, $G(0)$ and K, L, M, it is useful to consider ω_o and $G(0)$ as coordinates in a plane. Then each solution can be indicated by a point in the ω_o-$G(0)$ plane. For $N = 13$, those region I filters, that are real and posses monotonic responses, appear as the vertices in Figure 2.

To obtain filters of length 13 for which $(\omega_o, G(0))$ lies within one of the sectors, two degrees of flatness must be given up. (Then $K + L + M + 3 = N$, in contrast to item 1 in Section 3.) In this way arbitrary (non-integer) half-magnitude frequencies and DC group delays can be achieved exactly. This is ideally suited for applications requiring fractional delay lowpass filters.

The flatness parameters of a point in the ω_o-$G(0)$ plane are the (component-wise) minimum of the flatness parameters of the vertices of the sector in which the point lies. See [11, 12] for further details.

4.3. Reducing the Delay

To design a set of filters of length 13 for which $\omega_o = 0.636\pi$ and for which $G(0)$ is varied from 3.5 to 6 in increments of 0.5, we use Figure 2 to determine the appropriate flatness parameters — they are tabulated in Table 4. The resulting responses are shown in the lower right panel of Figure 1. It can be seen that *the delay can be reduced while maintaining relatively constant group delay around* $\omega = 0$*, without significantly altering the response magnitude.*

Table 4. The flatness parameters for the filters shown in the lower right panel of Figure 1.

N	ω_o/π	$G(0)$	K	L	M
13	0.636	3.5	3	2	5
		4	3	2	5
		4.5	4	2	4
		5	3	3	4
		5.5	3	3	4
		6	4	3	3

5. CONCLUSION

The problem of simultaneous magnitude and group delay approximation is a classic one in filter design. That the problem is a difficult one for FIR digital filters is made evident by the sophisticated iterative algorithms

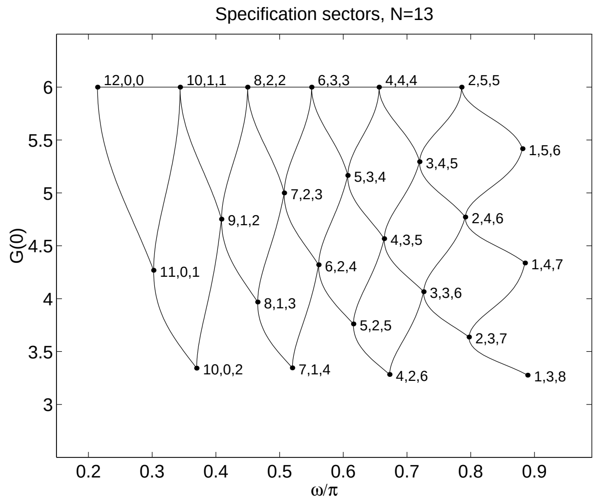


Figure 2. Specification sectors in the ω_o-$G(0)$ plane for length 13 filters in region I. The vertices are points at which $K + L + M + 1 = 13$. The three integers by each point give the flatness parameters (K, L, M).

that are required [3, 7, 13]. However, if a maximally-flat approximation is employed, then, for a class of FIR filters, there exists a straight-forward design technique [11, 12].

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