EQUIVALENCE OF GENERALIZED JOINT SIGNAL REPRESENTATIONS OF ARBITRARY VARIABLES

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Abstract

Joint signal representations (JSRs) of arbitrary variables generalize time-frequency representations (TFRs) to a much broader class of nonstationary signal characteristics. Two main distributional approaches to JSRs of arbitrary variables have been proposed by Cohen and Baraniuk. Cohen's method is a direct extension of his original formulation of TFRs, and Baraniuk's approach is based on a group theoretic formulation; both use the powerful concept of associating variables with operators.

One of the main results of the paper is that despite their apparent differences, the two approaches to generalized JSRs are completely equivalent. Remarkably, the JSRs of the two methods are simply related via *axis warping* transformations, with the broad implication that JSRs with radically different covariance properties can be generated efficiently from JSRs of Cohen's method via simple pre- and post-processing. The development in this paper, illustrated with examples, also illuminates other related issues in the theory of generalized JSRs. In particular, we derive an explicit relationship between the Hermitian operators in Cohen's method and the unitary operators in Baraniuk's approach, thereby establishing the relationship between the two types of operator correspondences.

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1 Introduction

Time-frequency representations (TFRs) provide a description of signal characteristics jointly in terms of time and frequency by measuring the time-varying spectral energy in the signal [1, 2]. Whereas TFRs are well-suited for representing a fairly rich class of nonstationary signal characteristics, they are inadequate in other situations, such as those involving a nonlinear chirping behavior. To encompass a wider variety of signal characteristics, recently there has been significant interest in the development of generalized joint signal representations (JSRs) which analyze signals in terms of physical quantities other than time and frequency [3, 4, 5, 1, 6, 7, 8, 9, 10, 11]. The wavelet transform and generalizations are the best known, which analyze signals in terms of time and scale content [12, 3, 4, 5, 7].

Owing to the recent interest in generalized JSRs, there has been substantial progress in the development of a general theory for generalized JSRs with respect to arbitrary variables. The most comprehensive theory to date is due to Cohen [1, 6] who has developed the generalization first proposed by Scully and Cohen [13] in direct extension of Cohen's original method for generating TFRs [14]. Baraniuk and Jones developed a general procedure for generating a wide class of JSRs from existing ones via the principle of unitary equivalence [9, 15]. More recently, Baraniuk has proposed a more general theory which is similar in principle to Cohen's general method but is based on group theoretic arguments [16, 17]. Other generalizations have also been proposed by Hlawatsch and Bölcskei [18, 19] and Sayeed and Jones [20] which characterize generalized JSRs that are covariant to certain unitary transformations, and which complement the distributional approaches developed by Cohen and Baraniuk. Our main interest in this paper is in the general methods developed by Cohen [1, 6] and Baraniuk [16, 17].

Fundamental to both Cohen's and Baraniuk's methods is the idea of associating variables of interest with operators. Cohen's method associates variables with *Hermitian* (self-adjoint) operators, whereas Baraniuk's approach associates them with families of *unitary* operators that are unitary representations of certain one-parameter groups. Both methods use analogues of the characteristic function *operator method*, originally introduced by Cohen [14], in which the characteristic function of the joint distribution is first computed via the *characteristic function operator*,[1] and then the distribution is recovered from it. Baraniuk's method appears to be more general than Cohen's since the latter can be recovered from the former by basing the construction on the group of real numbers.[2] Cohen's method, on the other hand, is generally more attractive computationally since it is based on the Fourier transform as opposed to arbitrary group transforms. Moreover, in some situations, it is more natural to associate unitary operators with variables, as is done in Baraniuk's approach, whereas in others the Hermitian operator correspondence is more straightforward. Thus, an understanding of the relationship between these two powerful methods is essential from the viewpoint of both the theory and application of generalized JSRs.

One of the main results of this paper is that, despite the apparent differences between Cohen's and Baraniuk's methods, the two approaches to generalized JSRs are completely equivalent. We prove that there is a one-to-one and onto mapping which relates the JSRs constructed via Baraniuk's method to those generated by Cohen's approach. In addition to explicitly characterizing this mapping, we also derive the explicit relationship between the unitary and Hermitian operators used in the two methods.

Remarkably, the JSRs in the two methods are simply related via *axis warping* transformations,

[1] In general, there are infinitely many possibilities (correspondence rules) for the characteristic function operator, resulting in infinitely many JSRs.

[2] As pointed out by one of the reviewers, and as will be explained later in the paper, certain correspondence rules for the characteristic function operator cannot be defined in the general group setting of Baraniuk's approach. Cohen's approach, on the other hand, does not have this drawback, and our development of the equivalence results in this paper enables us to define the equivalents of all the correspondence rules in Cohen's approach for Baraniuk's recipe. This yields an extension to Baraniuk's method that encompasses all the different correspondence rules, and throughout the paper, "Baraniuk's method" implicitly refers to this "extended Baraniuk's method."

implying that group transforms in Baraniuk's method, which are not computationally efficient in general, may be replaced with the Fourier transform as in Cohen's approach. The broad implication of the results is that JSRs with radically different characteristics can be generated efficiently from JSRs in Cohen's method by simple pre- and post-processing.

The development in this paper also allows us to address some related issues that have not been addressed adequately in existing treatments. An example is the relationship between Hermitian and unitary operator correspondences, which is fundamental to the understanding of JSRs of arbitrary variables. Using Stone's theorem [21] and the notion of *duality*, we characterize the relationship between the two types of correspondences.

In the next section, after providing a brief primer on relevant operator- and group-theoretic concepts, we describe the two distributional approaches to generalized JSRs. In Section 3, we briefly outline an equivalent description of Baraniuk's method in terms of Hermitian operators which is useful in interpreting the main results presented in Section 4 regarding the equivalence of the methods. Section 5 discusses some related issues in light of the presented results, and Section 6 concludes the paper with a summary of the results and their implications.

2 Preliminaries

In order to present the results of the paper, we need a description of the two methods for generalized JSRs. For simplicity, we will consider joint distributions of two variables only; extension to multiple variables is straightforward. Table 1 specifies the notation that we adopt in the two methods. Dual operators are defined in the next section on background material.

Objects	Cohen's Method	Baraniuk's Method
Distribution	P	$\widehat{P}$
Characteristic Function	M	$\widehat{M}$
Kernel	ϕ	$\widehat{\phi}$
Variables	a, b	k, l
Hermitian Operators	$\mathcal{A}$, $\mathcal{B}$	$\mathcal{K}$, $\mathcal{L}$
Unitary Operators	$\mathbf{A}_a$, $\mathbf{B}_b$	$\mathbf{K}_k$, $\mathbf{L}_l$
Dual Variables	α, β	κ, λ
Dual Hermitian Operators	$\mathcal{A}^\diamond$, $\mathcal{B}^\diamond$	$\mathcal{K}^\diamond$, $\mathcal{L}^\diamond$
Dual Unitary Operators	$\mathbf{A}^\diamond_\alpha$, $\mathbf{B}^\diamond_\beta$	$\mathbf{K}^\diamond_\kappa$, $\mathbf{L}^\diamond_\lambda$

Table 1: Notation used for the various objects in the two methods.

2.1 Operator- and Group-Theoretic Background

Baraniuk's approach is based on associating variables with parameterized unitary operators which are related to certain one-parameter locally compact (abelian) (LCA) groups. Let G be the underlying one-parameter LCA group with group operation $\bullet$.[3] We will use the symbols k, l for elements of G (the variables in Baraniuk's method; see Table 1).

Characters and the dual group. A complex-valued function κ on G is called a *character* of G if $|\kappa(k)| = 1$, for all $k \in G$, and if it satisfies the functional equation $\kappa(k \bullet l) = \kappa(k)\kappa(l)$, for all $k, l \in G$ [22]. Note that a character κ maps the group G into the complex unit circle. The set

[3] An abelian group is a set G in which a binary operation $\bullet$ is defined, with the following properties: 1) $x \bullet y = y \bullet x$, for all $x, y \in G$, 2) $x \bullet (y \bullet z) = (x \bullet y) \bullet z$, for all $x, y, z \in G$, 3) there exists an identity element $0 \in G$ such that $x \bullet 0 = x$ for all $x \in G$, and 4) for each $x \in G$ there exists an inverse $x^{-1} \in G$ such that $x \bullet x^{-1} = 0$ [22].

of all *continuous* characters of G itself forms a one-parameter LCA group Γ, called the *dual* group of G, with the group operation $\circ$ defined by $(\kappa \circ \lambda)(k) \equiv \kappa(k)\lambda(k)$, $k \in G$, $\kappa, \lambda \in \Gamma$ [22]. Because of this duality, it is convenient to use the following symmetric notation for characters $\kappa(k) \equiv (k, \kappa)$. In case of ambiguity, we will explicitly use the group as the subscript; for example, $(k, \kappa)_G$.

The natural signal spaces and the group transform. The natural signal space associated with the group G is $\mathcal{H}_1 = L^2(G, d\mu_G)$, the space of square-integrable functions defined on G, where μ_G is the Haar measure associated with G, which is invariant to group translation: $\mu_G(E \bullet k) = \mu_G(E)$, for all measurable sets $E \subset G$ and all $k \in G$. Similarly, the space associated with the dual group is $\mathcal{H}_2 = L^2(\Gamma, d\mu_\Gamma)$. The natural analogue of the Fourier transform is the group transform, $\mathbb{F}_G : \mathcal{H}_1 \to \mathcal{H}_2$, which is an isometry[4] between $\mathcal{H}_1$ and $\mathcal{H}_2$ and defined as

$$(\mathbb{F}_G s)(\kappa) \equiv \int_G s(k)(k, \kappa)^* d\mu_G(k) \ , \quad s \in \mathcal{H}_1 \ , \tag{1}$$

with the inverse transform $\mathbb{F}_G^{-1} : \mathcal{H}_2 \to \mathcal{H}_1$ defined by

$$(\mathbb{F}_G^{-1} h)(k) \equiv \int_\Gamma h(\kappa)(k, \kappa) d\mu_\Gamma(\kappa) \ , \quad h \in \mathcal{H}_2. \tag{2}$$

Unitary group representations. A parameterized unitary operator $\mathbf{K}_k : \mathcal{H}_1 \to \mathcal{H}_1$, $k \in G$, is a unitary representation of G on $\mathcal{H}_1$ if it satisfies[5] $\mathbf{K}_k \mathbf{K}_l = \mathbf{K}_{k \bullet l}$, $k, l \in G$. Two fundamental unitary representations of G, one defined over $\mathcal{H}_1$ and the other over $\mathcal{H}_2$, are the group translation operator $\Upsilon_l^G : \mathcal{H}_1 \to \mathcal{H}_1$ defined as

$$(\Upsilon_l^G s)(k) \equiv s(k \bullet l) \tag{3}$$

and the "diagonal" operator $\mathbf{\Lambda}_l^G : \mathcal{H}_2 \to \mathcal{H}_2$ defined as

$$(\mathbf{\Lambda}_l^G h)(\kappa) \equiv (l, \kappa) h(\kappa) \tag{4}$$

which is simply multiplication by a character. The corresponding fundamental unitary representations of Γ are $\Upsilon_\lambda^\Gamma : \mathcal{H}_2 \to \mathcal{H}_2$ defined as

$$(\Upsilon_\lambda^\Gamma h)(\kappa) \equiv g(\kappa \circ \lambda) \tag{5}$$

and the diagonal operator $\mathbf{\Lambda}_\lambda^\Gamma : \mathcal{H}_1 \to \mathcal{H}_1$ given by

$$(\mathbf{\Lambda}_\lambda^\Gamma s)(k) \equiv (k, \lambda)^* s(k) \ . \tag{6}$$

Note that for $G = \mathbb{R}$ and identifying it with time, the translation operator Υ_x^G essentially reduces to the time-shift operator, $\mathbf{T}_\tau$, $(\mathbf{T}_\tau s)(t) \equiv s(t - \tau)$, and the diagonal operator $\mathbf{\Lambda}_x^G$ reduces to the frequency-shift operator, $\mathbf{F}_\theta$, $(\mathbf{F}_\theta s)(t) \equiv e^{j2\pi\theta t} s(t)$. It is well-known that the group translation operators and the diagonal operators satisfy the following fundamental relationships [22, 23]

$$\mathbf{\Lambda}_k^G = \mathbb{F}_G \Upsilon_{k^{-1}}^G \mathbb{F}_G^{-1} \quad \text{and} \quad \mathbf{\Lambda}_\kappa^\Gamma = \mathbb{F}_G^{-1} \Upsilon_{\kappa^{-1}}^\Gamma \mathbb{F}_G \ , \tag{7}$$

which essentially state that group translation in one domain is equivalent to multiplication with characters in the dual domain.

Spectral representation of operators and energy densities. By Stone's theorem [21], a group of unitary operators, $\mathbf{K}_k$, admits the spectral representation

$$\mathbf{K}_k = \mathbf{S}_{\mathbf{K}}^{-1} \mathbf{\Lambda}_k^G \mathbf{S}_{\mathbf{K}} \tag{8}$$

[4] That is, $\langle \mathbb{F}_G s, \mathbb{F}_G s \rangle_{\mathcal{H}_2} = \langle s, s \rangle_{\mathcal{H}_1}$ for all $s \in \mathcal{H}_1$, where $\langle \cdot, \cdot \rangle$ denotes the inner product.

[5] More precisely, we will be dealing with *continuous* unitary representations; that is, $k \to l \Rightarrow \mathbf{K}_k \to \mathbf{K}_l$.

where $\mathbf{S_K} : \mathcal{H}_1 \to \mathcal{H}_2$ is a unitary transform (isometry) determined by the (generalized) eigenfunctions of the operator $\mathbf{K}_k$.[6] Note that $\mathbf{S_K}$ "diagonalizes" $\mathbf{K}_k$. Define a new unitary transform $\mathbf{S}_{\mathbf{K}^\diamond} : \mathcal{H}_1 \to \mathcal{H}_1$ via[7]

$$\mathbf{S}_{\mathbf{K}^\diamond} \equiv \mathbb{F}_G^{-1} \mathbf{S_K} \ . \tag{9}$$

Using (7) and (8) we arrive at the following alternative representation for $\mathbf{K}_k$

$$\mathbf{K}_k = \mathbf{S_K}^{-1} \mathbf{\Lambda}_k^G \mathbf{S_K} = \mathbf{S_K}^{-1} \mathbb{F}_G \Upsilon_{k^{-1}}^G \mathbb{F}_G^{-1} \mathbf{S_K} = \mathbf{S}_{\mathbf{K}^\diamond}^{-1} \Upsilon_{k^{-1}}^G \mathbf{S}_{\mathbf{K}^\diamond} \ . \tag{10}$$

From (8) it follows that the transform $\mathbf{S_K}$ is $\mathbf{K}$-*invariant* [16, 11] in the sense that $|(\mathbf{S_K}\mathbf{K}_k s)(\kappa)| = |(\mathbf{\Lambda}_k^G \mathbf{S_K} s)(\kappa)| = |(\mathbf{S_K} s)(\kappa)|$, for all $s \in \mathcal{H}_1$, and all $k \in G$. On the other hand, from (10) it follows that the transform $\mathbf{S}_{\mathbf{K}^\diamond}$ is $\mathbf{K}$-*covariant* [16, 11] in the sense that $(\mathbf{S}_{\mathbf{K}^\diamond}\mathbf{K}_k s)(l) = (\Upsilon_{k^{-1}}^G \mathbf{S}_{\mathbf{K}^\diamond} s)(l) = (\mathbf{S}_{\mathbf{K}^\diamond} s)(l \bullet k^{-1})$, for all $s \in \mathcal{H}_1$, and all $k \in G$. That is, $|(\mathbf{S_K} s)(\kappa)|$ is invariant to the effect of the operator $\mathbf{K}_k$, whereas applying the operator $\mathbf{K}_k$ to the signal produces a group translation of k^{-1} in the signal representation $(\mathbf{S}_{\mathbf{K}^\diamond} s)(l)$. $|(\mathbf{S_K} s)(\kappa)|^2$ and $|(\mathbf{S}_{\mathbf{K}^\diamond} s)(l)|^2$ are referred to as the "invariant energy density (IED)" and the "covariant energy density (CED)", respectively, in [16].

Duality. Define a new unitary operator $\mathbf{K}_\kappa^\diamond : \mathcal{H}_1 \to \mathcal{H}_1$, $\kappa \in \Gamma$, via the spectral representation[8]

$$\mathbf{K}_\kappa^\diamond = \mathbf{S}_{\mathbf{K}^\diamond}^{-1} \mathbf{\Lambda}_\kappa^\Gamma \mathbf{S}_{\mathbf{K}^\diamond} \ , \tag{11}$$

which is a unitary representation of Γ on $\mathcal{H}_1$. In fact, by Stone's theorem, any unitary representation of Γ on $\mathcal{H}_1$ admits a spectral representation of the form (11). It immediately follows that the transform $\mathbf{S}_{\mathbf{K}^\diamond}$ is $\mathbf{K}^\diamond$-invariant, and using (7) and (11) it can be easily verified that $\mathbf{S_K}$ is $\mathbf{K}^\diamond$-covariant; that is, $(\mathbf{S_K}\mathbf{K}_\kappa^\diamond s)(\lambda) = (\mathbf{S_K} s)(\lambda \circ \kappa^{-1})$.

Given two groups of unitary operators, $\mathbf{K}_k$ and $\mathbf{L}_\lambda^\diamond$, there exist, again by Stone's theorem [21], two unique Hermitian operators, $\mathcal{K}$ and $\mathcal{L}^\diamond$, respectively, such that, formally,

$$\mathbf{K}_k \equiv (k, \mathcal{K}) \quad \text{and} \quad \mathbf{L}_\kappa^\diamond \equiv (\mathcal{L}^\diamond, \lambda)^* \tag{12}$$

and $\mathcal{K}$ and $\mathcal{L}^\diamond$ admit the spectral representations

$$\mathcal{K} = \mathbf{S}_\mathcal{K}^{-1} \mathbf{\Lambda}^\Gamma \mathbf{S}_\mathcal{K} = \mathbf{S_K}^{-1} \mathbf{\Lambda}^\Gamma \mathbf{S_K} \quad \text{and} \quad \mathcal{L}^\diamond = \mathbf{S}_{\mathcal{L}^\diamond}^{-1} \mathbf{\Lambda}^G \mathbf{S}_{\mathcal{L}^\diamond} = \mathbf{S}_{\mathbf{L}^\diamond}^{-1} \mathbf{\Lambda}^G \mathbf{S}_{\mathbf{L}^\diamond} \tag{13}$$

where the diagonal operators $\mathbf{\Lambda}^\Gamma$ and $\mathbf{\Lambda}^G$ are defined on appropriate subspaces of $\mathcal{H}_2$ and $\mathcal{H}_1$, respectively, as

$$(\mathbf{\Lambda}^\Gamma x)(\kappa) \equiv \kappa x(\kappa) \ , \quad \kappa \in \Gamma \tag{14}$$

$$(\mathbf{\Lambda}^G x)(k) \equiv k x(k) \ , \quad k \in G \ . \tag{15}$$

Note that both $\mathbf{K}$ and $\mathcal{K}$ share the same eigenfunctions, that is, $\mathbf{S_K} = \mathbf{S}_\mathcal{K}$, and similarly for $\mathbf{L}^\diamond$ and $\mathcal{L}^\diamond$. The diagonal operators $\mathbf{\Lambda}^\Gamma$ and $\mathbf{\Lambda}^G$ are Hermitian and are related to the parameterized diagonal unitary operators $\mathbf{\Lambda}_k^G$ and $\mathbf{\Lambda}_\lambda^\Gamma$, defined in (4) and (6), as $\mathbf{\Lambda}_k^G = (k, \mathbf{\Lambda}^\Gamma)$ and $\mathbf{\Lambda}_\lambda^\Gamma = (\mathbf{\Lambda}^G, \lambda)^*$. We are now in a position to define the notion of *dual operators*.

[6] The unitary transform $\mathbf{S_K}$ is the projection onto the (generalized) eigenfunctions of $\mathbf{K}_k$. Throughout the paper, for brevity of notation, we will specify the relationship between eigenfunctions of operators by the corresponding relationship between the unitary transforms defined by them. See footnote 7 for an interpretation of such a relationship explicitly in terms of eigenfunctions.

[7] If $\mathbf{S}_{\mathbf{K}^\diamond}$ is the projection onto the eigenfunctions $\{e_{\mathbf{K}^\diamond}(k', \cdot) : k' \in G\}$ and $\mathbf{S_K}$ is the projection onto the eigenfunctions $\{e_\mathbf{K}(\lambda, \cdot) : \lambda \in \Gamma\}$, then (9) can be written more explicitly as $(\mathbf{S}_{\mathbf{K}^\diamond} s)(k') = \int_G e_{\mathbf{K}^\diamond}^*(k', k) s(k) d\mu_G(k) \equiv \int_\Gamma (k', \lambda)[\int_G e_\mathbf{K}^*(\lambda, k) s(k) d\mu_G(k)] d\mu_\Gamma(\lambda)$ for all $s \in \mathcal{H}_1$, which is equivalent to the relationship $e_{\mathbf{K}^\diamond}(k', k) = \int_\Gamma (k', \lambda)^* e_\mathbf{K}(\lambda, k) d\mu_\Gamma(\lambda) \equiv (\mathbb{F}_G^{-1} e_{\mathbf{K}^\diamond}(\cdot, k))(k')$ for all $k', k \in G$.

[8] Throughout the paper, we denote unitary representations of the dual group Γ with the superscript $\diamond$. The corresponding Hermitian operators defined by Stone's theorem will be denoted similarly.

Definition: Dual Operators. Let $\mathbf{K}_k = \mathbf{S}_{\mathbf{K}}^{-1}\mathbf{\Lambda}_k^G\mathbf{S}_{\mathbf{K}}$ and $\mathbf{L}_\lambda^\diamond = \mathbf{S}_{\mathbf{L}^\diamond}^{-1}\mathbf{\Lambda}_\lambda^\Gamma\mathbf{S}_{\mathbf{L}^\diamond}$ be unitary representations of G and Γ, respectively, on $\mathcal{H}_1$, and let $\mathcal{K}$ and $\mathcal{L}^\diamond$ be the corresponding Hermitian operators defined by Stone's theorem. Then, the operators $\mathbf{K}_k$ ($\mathcal{K}$) and $\mathbf{L}_\lambda^\diamond$ ($\mathcal{L}^\diamond$) are **dual** if the spectral families of $\mathbf{K}_k$ ($\mathcal{K}$) and $\mathbf{L}_\lambda^\diamond$ ($\mathcal{L}^\diamond$) are related by $\mathbf{S}_{\mathbf{K}} = \mathbb{F}_G^{-1}\mathbf{S}_{\mathbf{L}^\diamond}$ ($\mathbf{S}_{\mathcal{K}} = \mathbb{F}_G^{-1}\mathbf{S}_{\mathcal{L}^\diamond}$).

If $\mathbf{L}_\lambda^\diamond$ and $\mathcal{L}^\diamond$ are dual to $\mathbf{K}$ and $\mathcal{K}$, we denote them by $\mathbf{K}_\kappa^\diamond$ and $\mathcal{K}^\diamond$.[9]

Example. Let $(G, \bullet) = (\mathbb{R}_+, \times)$ where $\mathbb{R}_+ = (0, \infty)$ and $\times$ denotes multiplication. One characterization of the dual group is $(\Gamma, \circ) = (\mathbb{R}, +)$ with the characters given by $(k, \kappa) \equiv e^{j2\pi\kappa \ln(k)}$. In this case $d\mu_G(k) = dk/k$ and $d\mu_\Gamma(\kappa) = d\kappa$. The group Fourier transform is the Mellin transform [24, 25]

$$(\mathbb{F}_{\mathbb{R}_+} s)(\kappa) \equiv \int_0^\infty s(k) e^{-j2\pi\kappa \ln(k)} dk/k \ . \tag{16}$$

2.2 Cohen's Approach

Let a and b be two variables of interest; they could be time and scale, for example. In Cohen's approach, the variables a and b are associated with appropriate Hermitian operators $\mathcal{A}$ and $\mathcal{B}$, respectively. The eigenfunctions of $\mathcal{A}$ and $\mathcal{B}$ define unitary signal representations $\mathbf{S}_{\mathcal{A}}$ and $\mathbf{S}_{\mathcal{B}}$ which yield the a- and b-representations of the signal. Ideally, the joint distribution $(Ps)(a, b)$ should satisfy the a and b marginals; that is [1]

$$\int (Ps)(a,b)db = |(\mathbf{S}_{\mathcal{A}} s)(a)|^2 \ , \tag{17}$$

$$\int (Ps)(a,b)da = |(\mathbf{S}_{\mathcal{B}} s)(b)|^2 \ . \tag{18}$$

The characteristic function M is defined as [1]

$$(Ms)(\alpha, \beta) = \int\int (Ps)(a,b) e^{j2\pi\alpha a} e^{j2\pi\beta b} da db \tag{19}$$

and thus the distribution P can be recovered from M as

$$(Ps)(a,b) = \int\int (Ms)(\alpha, \beta) e^{-j2\pi\alpha a} e^{-j2\pi\beta b} d\alpha d\beta. \tag{20}$$

The key observation in Cohen's method is that the characteristic function, being an *average* of $e^{j2\pi\alpha a} e^{j2\pi\beta b}$, can be directly computed from the signal using a *characteristic function operator* [1], $\mathbf{M}_{(\alpha,\beta)}$, corresponding to the function $e^{j2\pi\alpha a} e^{j2\pi\beta b}$, an example being $\mathbf{M}_{(\alpha,\beta)} = e^{j2\pi\alpha\mathcal{A}} e^{j2\pi\beta\mathcal{B}}$; that is,[10]

$$(Ms)(\alpha, \beta) = \left\langle \mathbf{M}_{(\alpha,\beta)} s, s \right\rangle \equiv \int (\mathbf{M}_{(\alpha,\beta)} s)(t) s^*(t) dt \ . \tag{21}$$

However, in general, there are infinitely many ways to associate an operator with the function $e^{j2\pi\alpha a} e^{j2\pi\beta b}$, and infinitely many corresponding joint distributions. Taken together, the different distributions generated by these infinitely many operator correspondences *define* the class of joint a-b distributions (the *operator method*). As a simple description of all the joint distributions, Cohen has proposed the *kernel method*, which implicitly assumes that all the different characteristic functions can be generated by weighting a particular one, say M_o, with a two-dimensional (2d) kernel [1]. For example, one formulation could be

$$(Ms)(\alpha, \beta; \phi) = \phi(\alpha, \beta)(M_o s)(\alpha, \beta) = \phi(\alpha, \beta)\langle e^{j2\pi\alpha\mathcal{A}} e^{j2\pi\beta\mathcal{B}} s, s\rangle \ , \tag{22}$$

[9] The operators $\mathbf{\Lambda}_k^G$ and $\Upsilon_{\kappa^{-1}}^\Gamma$ are dual operators, and similarly $\mathbf{\Lambda}_\lambda^\Gamma$ and Υ_l^G are dual operators.

[10] Note that $\mathbf{A}_\alpha^\diamond = e^{j2\pi\alpha\mathcal{A}}$ and $\mathbf{B}_\beta^\diamond = e^{j2\pi\beta\mathcal{B}}$ are unitary representations of $(\mathbb{R}, +)$ on $L^2(\mathbb{R}, dx)$.

where ϕ is the 2d kernel. Another formulation is given by the Weyl correspondence [26, 1, 27, 28]

$$\mathbf{M}^{W}_{(\alpha,\beta)} \equiv e^{j2\pi(\alpha\mathcal{A}+\beta\mathcal{B})} \ . \tag{23}$$

$P(\phi)$ can be then be recovered via (20), and the marginals (17) and (18) are satisfied if the kernel satisfies $\phi(\alpha, 0) = 1$, for all α, and $\phi(0, \beta) = 1$, for all β, respectively. However, for arbitrary pairs of variables (operators), the kernel method does *not* generate all possible operator correspondences for the characteristic function [29, 30], and hence does not generate all possible joint distributions. Thus, in general, starting with a given characteristic function, M_o, the kernel method generates a proper subset of all possible joint distributions of the given variables generated by the operator method [30].

2.3 Baraniuk's Method

In Baraniuk's method, variables are associated with operators which are unitary representations of an LCA group, say G. Suppose we are interested in the joint distributions of two variables associated with the groups of unitary operators $\mathbf{K}_k$ and $\mathbf{L}_l$, $k, l \in G$.[11] Baraniuk's approach allows us to recover either the IED or CED marginal corresponding to an operator; that is, a joint distribution $\widehat{P}$ satisfies

$$\begin{aligned} \int (\widehat{P}s)(u,v)d\mu(v) &= |\overline{\mathbf{S}}_{\mathbf{K}}(u)|^2 \ , & (24)\\ \int (\widehat{P}s)(u,v)d\mu(u) &= |\overline{\mathbf{S}}_{\mathbf{L}}(v)|^2 \ , & (25) \end{aligned}$$

where the measure μ is either μ_G or μ_Γ, and $\overline{\mathbf{S}}_{\mathbf{K}}$ is either $\mathbf{S}_{\mathbf{K}}$ (IED) or $\mathbf{S}_{\mathbf{K}^\diamond}$ (CED) ($\overline{\mathbf{S}}_{\mathbf{L}}$ is defined similarly) [16]. In this approach too, a modification of the characteristic function method is used. If the $\mathbf{K}$-IED marginal is desired, define the unitary operator $\overline{\mathbf{K}}_k = \mathbf{K}_k$, and if the CED marginal is desired define $\overline{\mathbf{K}}_\kappa = \mathbf{K}^\diamond_\kappa$, the dual operator of $\mathbf{K}_k$. Similarly define $\overline{\mathbf{L}}_l$ and $\overline{\mathbf{L}}_\lambda$. Since there are four different combinations, each corresponding to different pairs of marginals, we illustrate with a specific case parallel to [16]; other cases are obvious. Thus, suppose we are interested in $\mathbf{K}$-CED and $\mathbf{L}$-IED marginals. Then, the characteristic function is computed as

$$(\widehat{M}s)(\kappa, l) = \widehat{\phi}(\kappa, l)\langle \overline{\mathbf{K}}_\kappa \overline{\mathbf{L}}_l s, s\rangle = \widehat{\phi}(\kappa, l)\langle \mathbf{K}^\diamond_\kappa \mathbf{L}_l s, s\rangle \ , \tag{26}$$

and the distribution can be recovered (using $\mathbb{F}_G$) as

$$(\widehat{P}(\widehat{\phi})s)(k,\lambda) = \int_G \int_\Gamma (\widehat{M}s)(\kappa, l)(k,\kappa)(l,\lambda)^* d\mu_\Gamma(\kappa) d\mu_G(l) \ , \tag{27}$$

which yields the $\mathbf{K}$-CED and $\mathbf{L}$-IED marginals if $\widehat{\phi}(\kappa, 0) = 1$, for all $\kappa \in \Gamma$, and $\phi(0, l) = 1$, for all $l \in G$ [16].

Note that this method also assumes that all the different characteristic functions can be generated from a particular one, namely $(\widehat{M}_o s)(\kappa, l) = \langle \overline{\mathbf{K}}_\kappa \overline{\mathbf{L}}_l s, s\rangle$, via a weighting kernel. However, this is not true in general [29, 30]. Moreover, in this method, as described in [16], the choices for the characteristic function operator $\widehat{\mathbf{M}}_o$ are rather limited as compared to Cohen's method; in particular, there is no analogue of Weyl correspondence as defined in (23).[12] However, our development will enable us to define equivalents of different correspondence rules in Cohen's method,

[11] The nature of this correspondence with variables is not clear in [16, 17]. Using the concepts of *duality* and *shift operators* [11], we will make this correspondence precise in Section 5.

[12] Since there is no direct analog of the function $e^{j2\pi(a\alpha+b\beta)}$ in terms of the characters of the group G (which define the individual unitary operators, in terms of Hermitian operators, via Stone's theorem). When the two variables jointly generate a 2d Lie group, such exponential forms can be recovered using Lie algebras [31, 32], but Baraniuk's method does not assume any joint group structure [16].

such as Weyl correspondence, for Baraniuk's recipe. This equivalent *operator method* in Baraniuk's approach, based on different correspondence rules, yields an extended form of Baraniuk's method.[13] Moreover, by using different correspondence rules $\widehat{\mathbf{M}}_o$, Baraniuk's kernel method ((26) and (27)) generates subsets of the entire class of distributions.

3 Baraniuk's Approach in Terms of Hermitian Operators

Using Stone's theorem, the unitary operators in Baraniuk's method can also be uniquely associated with Hermitian operators, providing an interpretation of Baraniuk's method in terms of Hermitian operators. In this section, we briefly describe this interpretation which makes the comparison between Cohen's and Baraniuk's methods more explicit and is helpful in understanding the mechanism of the equivalence results of next section.

We simply mimic Cohen's general recipe (based on the group $(\mathbb{R}, +)$) in the more general setting of LCA groups and show that it naturally leads to Baraniuk's method. Since the natural signal spaces are $L^2(G, d\mu_G)$ and $L^2(\Gamma, d\mu_\Gamma)$, the variables of interest can take on values in G or Γ with the corresponding Hermitian operators of the form $\mathcal{K}^\diamond$ and $\mathcal{L}$ as defined in (13).[14] Suppose we have two variables, $k \in G$ and $\lambda \in \Gamma$, associated with operators $\mathcal{K}^\diamond$ and $\mathcal{L}$, and we are interested in the corresponding JSRs, $(Ps)(k, \lambda)$. The natural definition of the characteristic function, based on the group Fourier transform,[15] is

$$(\widehat{M}s)(\kappa, l) = \mathbb{F}_{G\,k\to\kappa}\mathbb{F}_{G\ \lambda\to l}^{-1}(\widehat{P}s)(k,\lambda) = \int_G\int_\Gamma (\widehat{P}s)(k,\lambda)(k,\kappa)^*(l,\lambda)\,d\mu_G(k)\,d\mu_\Gamma(\lambda) \qquad (28)$$

where $l \in G$, and $\kappa \in \Gamma$. The rule for deciding whether to use $\mathbb{F}_G$ or $\mathbb{F}_G{}^{-1}$ in the definition of $\widehat{M}$ is simple: if the variable is in G (as k above), use the forward group transform $\mathbb{F}_G : \mathcal{H}_1 \to \mathcal{H}_2$, and if the variable is in Γ, use $\mathbb{F}_G{}^{-1}$. Again, using the *characteristic function operator* method, $(Ms)(\kappa, l)$, being an average of $(k, \kappa)^*(l, \lambda)$, can be directly computed from the signal as

$$(\widehat{M}s)(\kappa, l) = \widehat{\phi}(\kappa, l)\,\langle(\mathcal{K}^\diamond, \kappa)^*(l, \mathcal{L})s, s\rangle = \widehat{\phi}(\kappa, l)\,\langle\mathbf{K}_\kappa^\diamond\mathbf{L}_l s, s\rangle \qquad (29)$$

where $\widehat{\phi}(\kappa, l)$ is an arbitrary 2d kernel. The corresponding JSRs can then be recovered by inverting (28). The second equality in (29) shows the relationship with Baraniuk's original approach via the unitary operators $\mathbf{K}_\kappa^\diamond$ and $\mathbf{L}_l$. The resulting JSR $(Ps)(k, \lambda)$ satisfies the following marginals

$$\int_\Gamma (Ps)(k,\lambda)\,d\mu_\Gamma(\lambda) \;=\; |(\mathbf{S}_{\mathbf{K}^\diamond}s)(k)|^2 \text{ if and only if } \phi(\kappa, 0) = 1 \text{ for all } \kappa \in \Gamma \qquad (30)$$

$$\int_G (Ps)(k,\lambda)\,d\mu_G(k) \;=\; |(\mathbf{S}_{\mathbf{L}}s)(\lambda)|^2 \text{ if and only if } \phi(0, l) = 1 \text{ for all } l \in G\ . \qquad (31)$$

Note that if we take $(G, \bullet) = (\mathbb{R}, +)$, we recover Cohen's method outlined in Section 2.2.

4 Equivalence of Cohen's and Baraniuk's Approaches

In Section 2, we described Cohen's and Baraniuk's approaches to generalized JSRs. Baraniuk's generalization, based on unitary representations of certain one-parameter LCA groups, is apparently broader than Cohen's since, as we saw in the previous section, Cohen's method can be recovered

[13] That is, computing the characteristic functions for different correspondence rules, and recovering the corresponding distributions via (27). In Baraniuk's method, for given variables, we define the entire class of joint distributions as those generated by this operator method.

[14] A variable taking on values in G is associated with a Hermitian operator whose diagonal operator (in the spectral representation) is $\mathbf{\Lambda}^G$.

[15] The group Fourier transform is the usual Fourier transform in the case $G = \mathbb{R}$.

as a special case by taking $(G,\bullet) = (\mathbb{R},+)$. In this section, we derive the main results of the paper which prove that, despite the apparent differences between them, the two approaches are completely equivalent. An explicit relationship between the operators of the two methods is also characterized.

As mentioned in Section 2.3, equivalents of certain correspondence rules in Cohen's method, such as Weyl correspondence (23), cannot be directly described in Baraniuk's method. Using the development in this section, we define the equivalent, in Baraniuk's recipe, of any arbitrary correspondence rule in Cohen's method. This yields an extended form of Baraniuk's method which we prove to be equivalent to Cohen's method.

The equivalence of the two methods stems from the fact that Baraniuk's approach is implicitly based on a class of one-parameter LCA groups which are isomorphic to each other.[16] The reason is that, in Baraniuk's method, different groups define different signal spaces, and, thus, in order to construct joint distributions of variables belonging to different groups, it must be possible to relate the corresponding signal spaces. Moreover, in view of the fundamental importance of time and frequency variables (related to $(\mathbb{R},+)$), the class of groups must contain $(\mathbb{R},+)$ (the dual group of $(\mathbb{R},+)$ is also $(\mathbb{R},+)$). It follows that if $(G,\bullet)$ is the underlying group in Baraniuk's construction, then there exist *isomorphisms*[17] $\psi : G \to \mathbb{R}$ (onto $\mathbb{R}$) and $\varphi : \Gamma \to \mathbb{R}$ (onto $\mathbb{R}$), which are not unique in general.[18] However, given ψ, we can explicitly characterize a dual isomorphism, φ, satisfying certain properties, as shown next. These isomorphisms play a central role in all the results of the paper.

4.1 Characterization of a Dual Isomorphism

Let $(G_1,\bullet)$ and $(G_2,*)$ be two isomorphic LCA groups, and let $(\Gamma_1,\circ)$ and $(\Gamma_2,\cdot)$ be the dual groups. Then, there exists an isomorphism $\psi : G_1 \to G_2$ (onto G_2) such that $\psi(k\bullet l) = \psi(k) * \psi(l)$ [22].[19] Without loss of generality we assume that, given ψ, the Haar measures μ_{G_1} and μ_{G_2} are appropriately normalized so that [22]

$$\mu_{G_1}(E) = \mu_{G_2}(\psi(E)) \quad \text{for all measurable sets } E \subset G_1 \ . \tag{32}$$

Similarly, without loss of generality we assume that, given the normalized measures μ_{G_1} and μ_{G_2}, the dual Haar measures μ_{Γ_1} and μ_{Γ_2} are (individually) normalized so that the (corresponding) forward and inverse group Fourier transforms are symmetric without any additional factors [22]. With this setting, the following theorem, proved in the Appendix, explicitly characterizes a dual isomorphism $\varphi : \Gamma_1 \to \Gamma_2$, and, thus, explicitly relates the characters of G_1 and G_2.

Theorem 1. For each $\kappa \in \Gamma_1$, define $\varphi(\kappa) \in \Gamma_2$ as

$$(m, \varphi(\kappa))_{G_2} \equiv (\psi^{-1}(m), \kappa)_{G_1} \ , \quad m \in G_2 \ . \tag{33}$$

Then, the functional equation (33) defines a continuous isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ which is onto Γ_2 with a continuous inverse and satisfies

$$\mu_{\Gamma_1}(F) = \mu_{\Gamma_2}(\varphi(F)) \quad \text{for all measurable sets } F \subset \Gamma_1 \ . \tag{34}$$

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[16]In [16] it is stated that *all* one-parameter LCA groups are isomorphic; this is strictly not true since $(\mathbb{R},+)$ and $(Z,+)$ are clearly not isomorphic, where Z is the set of integers.

[17]An isomorphism is a one-to-one mapping between two sets.

[18] More precisely, the underlying groups are *topologically* isomorphic to $(\mathbb{R},+)$; that is, both ψ and ψ^{-1} are continuous. This assumption is necessary to preserve the continuity of unitary representations in different signal spaces (see foonote 5).

[19]This implies that an isomorphism preserves the identity element; that is $\psi(0) = 0$.

Note that (33) is well-defined; that is, for each character $\kappa \in \Gamma_1$, the right hand side of (33) is well-defined for all $m \in G_2$, and defines a corresponding (continuous)[20] character of G_2, denoted by $\varphi(\kappa)$ on the left hand side. Theorem 1 states that defining characters of G_2 in this way characterizes the entire dual group Γ_2 in terms of Γ_1, and that the measures are preserved by the resulting isomorphism φ.

Example (continued). Let $(G_1, \bullet) = (\mathbb{R}_+, \times)$ and $(G_2, *) = (\mathbb{R}, +) = (\Gamma_2, \cdot)$. One characterization of the characters of G_2 is $(a, \alpha) \equiv e^{j2\pi a\alpha}$, $a, \alpha \in \mathbb{R}$. Since in this case, $(\Gamma_1, \circ) = (\Gamma_2, \cdot) = (\mathbb{R}, +)$, clearly $\varphi(\kappa) = \kappa$, which can also be inferred from (33) starting with $\psi(k) = \ln(k)$:

$$e^{j2\pi a\varphi(\kappa)} = (\psi^{-1}(a), \kappa)_{\mathbb{R}_+} = e^{j2\pi\kappa \ln(e^a)} = e^{j2\pi\kappa a} \text{ for all } \kappa,\ a \in \mathbb{R} . \tag{35}$$

4.2 Relationship Between the Signal Spaces

Now suppose that $(G, \bullet)$ is the underlying group in Baraniuk's method, with $(\Gamma, \circ)$ being the dual group and $\mathcal{H}_1 = L^2(G, d\mu_G)$ the underlying signal space. Then, there exists an isomorphism $\psi : G \to \mathbb{R}$, and let $\varphi : \Gamma \to \mathbb{R}$ be as characterized in Theorem 1. We use the following characterization for the characters of $(\mathbb{R}, +) :\ (a, \alpha) = e^{j2\pi a\alpha}$. Let $\mathbf{K}_k$ and $\mathbf{L}_l$ be two unitary operators in Baraniuk's approach corresponding to the variables whose joint representations are desired. Recall that $\mathbf{K}_k$ and $\mathbf{L}_l$ are unitary representations of G on $\mathcal{H}_1$.

The signal space in Cohen's method is $L^2(\mathbb{R}, dx)$ and the mapping that relates it to $L^2(G, d\mu_G)$ is $T_\psi : L^2(\mathbb{R}, dx) \to L^2(G, d\mu_G)$ defined as

$$(\mathbf{T}_\psi s)(k) \equiv s(\psi(k)) \ , \quad k \in G \ . \tag{36}$$

$\mathbf{T}_\psi$ is an isometry from $L^2(\mathbb{R}, dx)$ onto $L^2(G, d\mu_G)$ since

$$\|\mathbf{T}_\psi s\|^2_{\mathcal{H}_1} = \int_G |g(\psi(k))|^2 d\mu_G(k) = \int_{\mathbb{R}} |g(x)|^2 d\mu_G(\psi^{-1}(x)) = \int_{\mathbb{R}} |g(x)|^2 dx = \|g\|^2_{L^2} \ , \tag{37}$$

where the last equality follows from (32). The mapping $\mathbf{T}_\psi$ also relates the operators defined on $\mathcal{H}_1$ to those defined on $L^2(\mathbb{R}, dx)$. For example, $\mathbf{K}_k$ on $L^2(G, d\mu_G)$ defines a corresponding unitary operator $\mathbf{A}_x$ on $L^2(\mathbb{R}, dx)$ via[21]

$$\mathbf{A}_{\psi(k)} = \mathbf{T}_\psi^{-1} \mathbf{K}_k \mathbf{T}_\psi \ , \tag{38}$$

which is a unitary representation of $(\mathbb{R}, +)$ on $L^2(\mathbb{R}, dx)$ since for all a, b in $\mathbb{R}$

$$\mathbf{A}_a \mathbf{A}_b = \mathbf{T}_\psi^{-1} \mathbf{K}_{\psi^{-1}(a)} \mathbf{K}_{\psi^{-1}(b)} \mathbf{T}_\psi = \mathbf{T}_\psi^{-1} \mathbf{K}_{\psi^{-1}(a) \bullet \psi^{-1}(b)} \mathbf{T}_\psi = \mathbf{T}_\psi^{-1} \mathbf{K}_{\psi^{-1}(a+b)} \mathbf{T}_\psi = \mathbf{A}_{a+b} \ . \tag{39}$$

Similarly, if we have a unitary representation of Γ on $\mathcal{H}_1$, $\mathbf{K}^\diamond_\kappa$, then the operator

$$\mathbf{A}^\diamond_\alpha = \mathbf{T}_\psi^{-1} \mathbf{K}^\diamond_{\varphi^{-1}(\alpha)} \mathbf{T}_\psi \tag{40}$$

is also a representation of $(\mathbb{R}, +)$ (the dual group!) on $L^2(\mathbb{R}, dx)$. Given $\mathbf{A}_a$ and $\mathbf{A}^\diamond_\alpha$, by Stone's theorem [21] there exist unique Hermitian operators $\mathcal{A}$ and $\mathcal{A}^\diamond$ defined on $L^2(\mathbb{R}, dx)$ such that

$$\mathbf{A}_a = e^{j2\pi a\mathcal{A}} \equiv (a, \mathcal{A})_{\mathbb{R}} \quad \text{and} \quad \mathbf{A}^\diamond_\alpha = e^{-j2\pi\alpha\mathcal{A}^\diamond} \equiv (\mathcal{A}^\diamond, \alpha)^*_{\mathbb{R}} \ . \tag{41}$$

Extended Baraniuk's Method. Using the mapping $\mathbf{T}_\psi$ we can define the equivalent, in Baraniuk's method, of any arbitrary correspondence rule in Cohen's method. Let $\mathbf{M}_{(\alpha,\beta)}$ be any

[20] Recall from Section 2.1 that the dual group is the set of all *continuous* characters. Continuity of $\varphi(k)$ (as a character) defined in (33) follows from the continuity of $(k, \kappa)_{G_1}$ on $G_1 \times \Gamma_1$ [22, p. 10], and the continuity of ψ^{-1} (see footnote 18).

[21] Due to the continuity of isomorphisms and inverse isomorphisms (see footnote 18), the continuity of unitary representations is preserved (see footnote 5).

arbitrary (characteristic function) operator correspondence rule in Cohen's method. *Define* the equivalent of $\mathbf{M}_{(\alpha,\beta)}$ in Baraniuk's method (corresponding to **K**-CED and **L**-IED marginals) as[22]

$$\widehat{\mathbf{M}}_{(\kappa,l)} \equiv \mathbf{T}_\psi \mathbf{M}_{(\varphi(\kappa),\psi(l))} \mathbf{T}_\psi^{-1} \ . \tag{42}$$

These operator correspondence rules yield an extended Baraniuk's method that encompasses both the *operator* and *kernel methods* just as in Cohen's method.

Example (continued). The mapping relating $L^2(G, d\mu_G) = L^2(\mathbb{R}_+, dk/k)$ and $L^2(\mathbb{R}, dx)$ is $(\mathbf{T}_\psi s)(k) = s(\ln(k))$, $k \in \mathbb{R}_+$. The dilation operator $(\mathbf{D}_{k'} s)(k) \equiv s(k/k')$ is a unitary representation of $(\mathbb{R}_+, \times)$ (the group translation operator) on $L^2(G, d\mu_G)$, and gets mapped to the time-shift operator $(\mathbf{T}_{a'} s)(a) \equiv s(a - a')$, the group translation operator on $L^2(\mathbb{R}, dx)$, via (38).

4.3 Equivalence Results

We are now in a position to prove the main result of the paper. The following theorem proves the equivalence of the two methods for the subsets of JSRs generated, via the kernel method, by the correspondence rules of the form (22) and (26). Analogous arguments, as those used in the proof of the next theorem, then yield the equivalence of the extended Baraniuk's method and Cohen's approach.[23]

Since there are four types of JSRs in Baraniuk's approach, corresponding to the choice of marginals, we characterize the equivalence for representations with one CED and one IED marginal; the equivalence for the remaining types can be readily inferred from the stated result.

Theorem 2. For each $\widehat{P}$ from Baraniuk's class of JSRs corresponding to operators $\mathbf{K}_k$ and $\mathbf{L}_l$ and yielding **K**-CED and **L**-IED marginals, there exists a P in a corresponding Cohen's class (associated with a pair of Hermitian operators) of JSRs (and vice versa) such that

$$(\widehat{P}(\widehat{\phi})s)(k,\lambda) = (P(\phi)\mathbf{T}_\psi^{-1}s)(\psi(k),\varphi(\lambda)) \quad \text{where} \tag{43}$$

$$(\mathbf{T}_\psi s)(k) \equiv s(\psi(k)) \tag{44}$$

is an isometry from $L^2(\mathbb{R}, dx)$ onto $L^2(G, d\mu_G)$ and the kernels are related as

$$\widehat{\phi}(\kappa, l) = \phi(\varphi(\kappa), \psi(l)) \ . \tag{45}$$

Moreover, the equivalence (43) is an isometry; that is,

$$(\widehat{P}(\widehat{\phi})s)(k,\lambda) = (\mathbf{V}_G P(\phi)\mathbf{U}_G s)(k,\lambda) \tag{46}$$

where $\mathbf{U}_G = \mathbf{T}_\psi^{-1}$ and $\mathbf{V}_G$, defined as

$$(\mathbf{V}_G P)(k,\lambda) \equiv P(\psi(k),\varphi(\lambda)) \ , \tag{47}$$

is an isometry from $L^2(\mathbb{R}^2, dx \times dx)$ onto $L^2(G \times \Gamma, d\mu_G \times d\mu_\Gamma)$.

Proof: From (26) and (27) we see that $\widehat{P}(\widehat{\phi})$ is given by

$$(\widehat{P}(\widehat{\phi})s)(k,\lambda) = \int_G \int_\Gamma \widehat{\phi}(\kappa, l) \langle \mathbf{K}_\kappa^\diamond \mathbf{L}_l s, s\rangle (k,\kappa)(l,\lambda)^* d\mu_\Gamma(\kappa) d\mu_G(l) \ , \ s \in L^2(G, d\mu_G) \tag{48}$$

[22] The equivalent correspondence rules for other choices of marginals can be defined similarly: for a CED marginal use φ and for IED marginal use ψ. For example, for **K**-IED and **L**-CED marginals we have $\widehat{\mathbf{M}}_{(k,\lambda)} \equiv \mathbf{T}_\psi \mathbf{M}_{(\psi(k),\varphi(\lambda))} \mathbf{T}_\psi^{-1}$. Moreover, the relationship between individual operators can be recovered from any arbitrary correspondence rule in (42) because $\mathbf{M}_{(\varphi(\kappa),0)} = \mathbf{A}^\diamond_{\varphi(\kappa)}$ and $\mathbf{M}_{(0,\psi(l))} = \mathbf{B}_{\psi(k)}$, and $\widehat{\mathbf{M}}_{(\kappa,0)} = \mathbf{K}_\kappa^\diamond$ and $\widehat{\mathbf{M}}_{(0,k)} = \mathbf{L}_k$.

[23] That is, the equivalence for any arbitrary correspondence rule.

where we have substituted $\overline{\mathbf{K}}_\kappa = \mathbf{K}_\kappa^\diamond$ and $\overline{\mathbf{L}}_l = \mathbf{L}_l$ in (26), since $\mathbf{K}$-CED and $\mathbf{L}$-IED marginals are desired. Using (38) and (40) we get

$$(\widehat{P}(\widehat{\phi})s)(k,\lambda) = \int_G \int_\Gamma \widehat{\phi}(\kappa,l)\langle \mathbf{A}^\diamond_{\varphi(\kappa)}\mathbf{B}_{\psi(l)}\mathbf{T}_\psi^{-1}s, \mathbf{T}_\psi^{-1}s\rangle(k,\kappa)(l,\lambda)^* d\mu_\Gamma(\kappa)d\mu_G(l) \tag{49}$$

for some unitary operators, $\mathbf{A}_x^\diamond$ and $\mathbf{B}_x$, $x \in \mathbb{R}$, which are unitary representations of $(\mathbb{R},+)$ on $L^2(\mathbb{R},dx)$. Making the substitutions $l = \psi^{-1}(b)$ and $\kappa = \varphi^{-1}(\alpha)$ in (49) yields

$$\begin{aligned}(\widehat{P}(\widehat{\phi})s)(k,\lambda) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{\phi}(\varphi^{-1}(\alpha),\psi^{-1}(b))\langle \mathbf{A}^\diamond_\alpha \mathbf{B}_b \mathbf{T}_\psi^{-1}s, \mathbf{T}_\psi^{-1}s\rangle \\ &\quad (k,\varphi^{-1}(\alpha))(\psi^{-1}(b),\lambda)^* d\mu_\Gamma(\varphi^{-1}(\alpha))d\mu_G(\psi^{-1}(b)) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(\alpha,b)\langle e^{-j2\pi\alpha\mathcal{A}^\diamond}e^{j2\pi b\mathcal{B}}\mathbf{T}_\psi^{-1}s, \mathbf{T}_\psi^{-1}s\rangle(k,\varphi^{-1}(\alpha))(\psi^{-1}(b),\lambda)^* d\alpha db \end{aligned}\tag{50}$$

where in the last equality ϕ is defined as in (45), and we used (41) and the fact that $d\mu_G(\psi^{-1}(b)) = db$ and $d\mu_\Gamma(\varphi^{-1}(\alpha)) = d\alpha$, which follows from (32) and (34) (Theorem 1). Finally, using the functional equation (33) in Theorem 1, which relates the characters of $(G,\bullet)$ and $(\mathbb{R},+)$, (50) becomes

$$(\widehat{P}(\widehat{\phi})s)(k,\lambda) = \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(\alpha,b)\langle e^{-j2\pi\alpha\mathcal{A}^\diamond}e^{j2\pi b\mathcal{B}}\mathbf{T}_\psi^{-1}s, \mathbf{T}_\psi^{-1}s\rangle e^{j2\pi\psi(k)\alpha}e^{-j2\pi b\varphi(\lambda)} d\alpha db\ . \tag{51}$$

Comparing (51) with (22) and (20) we get the relation (43). Since we already know that $\mathbf{T}_\psi$ is an isometry, the only remaining thing to verify is that $\mathbf{V}_G$ is an isometry, which simply follows from (32) and (34). □

It is worth noting that the equivalence between the two approaches is simply based on *axis warping* transformations of the signal (via $\mathbf{U}_G = \mathbf{T}_\psi$) and the JSRs (via $\mathbf{V}_G$).[24] Moreover, the axis transformations are simply the group isomorphisms, ψ and φ. This also makes it almost trivial to infer, from Theorem 2, the equivalence relationships corresponding to other choices of marginals in Baraniuk's approach. First note that the signal warping transformation is always $\mathbf{T}_\psi^{-1}$, since the underlying signal space is $L^2(G,d\mu_G)$; only the axis warping of the JSRs changes. If a CED marginal is desired, the corresponding axis of the representation is warped by the group isomorphism ψ, whereas, if an IED marginal is desired, the corresponding axis transformation is the dual isomorphism φ. Thus, given any group and any type of JSR in Baraniuk's construction, we need only determine the group isomorphisms to find the equivalent in Cohen's method. Moreover, the proof of Theorem 2 also defines, albeit somewhat implicitly, the corresponding operators of the equivalent Cohen's representation. The next theorem explicitly characterizes the relationship between the operators of the two methods. Recall the definitions of the diagonal operators $\mathbf{\Lambda}^G$, $\mathbf{\Lambda}^\Gamma$, $\mathbf{\Lambda}_k^G$ and $\mathbf{\Lambda}_\kappa^\Gamma$ in Section 2.1.

Theorem 3. Let $\mathbf{K}_k$ and $\mathbf{L}_l$ be the two unitary operators in Baraniuk's approach whose JSRs, with $\mathbf{K}$-CED and $\mathbf{L}$-IED marginals, are desired. Then, in the equivalent Cohen's class of JSRs, the Hermitian operator $\mathcal{A}^\diamond$ corresponding to $\mathbf{K}_k = \mathbf{S}_\mathbf{K}^{-1}\mathbf{\Lambda}_k^G\mathbf{S}_\mathbf{K}$ is given by

$$\mathcal{A}^\diamond = \mathbf{S}_{\mathbf{A}^\diamond}^{-1}\mathbf{\Lambda}\mathbf{S}_{\mathbf{A}^\diamond} \quad \text{where} \tag{52}$$

$$(\mathbf{\Lambda}s)(c) = cs(c)\ , \quad c \in \mathbb{R}\ , \quad \text{and} \tag{53}$$

$$\mathbf{S}_{\mathbf{A}^\diamond} = \mathbf{T}_\psi^{-1}\mathbf{S}_{\mathbf{K}^\diamond}\mathbf{T}_\psi\ , \tag{54}$$

[24] We note that such warping transformations are employed in some specific classes of generalized JSRs [8, 10, 33]. For example, the hyperbolic class [8] is related to Cohen's class of TFRs [34], and the power class [10] is related to the affine class [5] via warping.

where $\mathbf{S}_{\mathbf{K}^\diamond} = \mathbb{F}_G^{-1}\mathbf{S}_{\mathbf{K}}$ is the transform determined by the eigenfunctions of $\overline{\mathbf{K}}_\kappa = \mathbf{K}^\diamond_\kappa = \mathbf{S}^{-1}_{\mathbf{K}^\diamond}\mathbf{\Lambda}^\Gamma_\kappa\mathbf{S}_{\mathbf{K}^\diamond}$, the dual operator of $\mathbf{K}_k$. Similarly, the Hermitian operator $\mathcal{B}$ corresponding to $\mathbf{L}_l = \mathbf{S}^{-1}_{\mathbf{L}}\mathbf{\Lambda}^G_l\mathbf{S}_{\mathbf{L}}$ is given by

$$\begin{aligned} \mathcal{B} &= \mathbf{S}^{-1}_{\mathbf{B}}\mathbf{\Lambda}\mathbf{S}_{\mathbf{B}} \quad \text{where} && (55)\\ \mathbf{S}_{\mathbf{B}} &= \mathbf{T}^{-1}_\varphi\mathbf{S}_{\mathbf{L}}\mathbf{T}_\psi\ . && (56)\end{aligned}$$

Proof: The relationships (52) and (55) are essentially inferred from (50) in the proof of Theorem 2 via the relationships

$$\begin{aligned} \mathbf{A}^\diamond_\alpha &= e^{-j2\pi\alpha\mathcal{A}^\diamond} && (57)\\ \mathbf{B}_b &= e^{j2\pi b\mathcal{B}}\ , && (58)\end{aligned}$$

with $\mathbf{A}^\diamond_\alpha = \mathbf{T}^{-1}_\psi\mathbf{K}^\diamond_{\varphi^{-1}(\alpha)}\mathbf{T}_\psi$, and $\mathbf{B}_b = \mathbf{T}^{-1}_\psi\mathbf{L}_{\psi^{-1}(b)}\mathbf{T}_\psi$. We first prove (52). We have

$$\begin{aligned} \mathbf{A}^\diamond_\alpha &= \mathbf{T}^{-1}_\psi\mathbf{K}^\diamond_{\varphi^{-1}(\alpha)}\mathbf{T}_\psi = \mathbf{T}^{-1}_\psi\mathbf{S}^{-1}_{\mathbf{K}^\diamond}\mathbf{\Lambda}^\Gamma_{\varphi^{-1}(\alpha)}\mathbf{S}_{\mathbf{K}^\diamond}\mathbf{T}_\psi\\ &= \mathbf{T}^{-1}_\psi\mathbf{S}^{-1}_{\mathbf{K}^\diamond}\mathbf{T}_\psi\mathbf{T}^{-1}_\psi\mathbf{\Lambda}^\Gamma_{\varphi^{-1}(\alpha)}\mathbf{T}_\psi\mathbf{T}^{-1}_\psi\mathbf{S}_{\mathbf{K}^\diamond}\mathbf{T}_\psi = \mathbf{S}^{-1}_{\mathbf{A}^\diamond}\mathbf{\Lambda}_\alpha\mathbf{S}_{\mathbf{A}^\diamond} && (59)\end{aligned}$$

where in the last equality $(\mathbf{\Lambda}_\alpha s)(c) = e^{-j2\pi c\alpha}s(c)$, $c \in \mathbb{R}$, and we used (54) and the fact that $\mathbf{T}^{-1}_\psi\mathbf{\Lambda}^\Gamma_{\varphi^{-1}(\alpha)}\mathbf{T}_\psi = \mathbf{\Lambda}_\alpha$, which readily follows from Theorem 1. (59) can be rewritten in the form (57) with $\mathcal{A}^\diamond$ exactly given by (52), and thus, by the uniqueness of the Hermitian operator defined by a group of unitary operators via Stone's theorem [21], $\mathcal{A}^\diamond$ is given by (52) and (54).

Similarly, we have

$$\begin{aligned} \mathbf{B}_b &= \mathbf{T}^{-1}_\psi\mathbf{L}_{\psi^{-1}(b)}\mathbf{T}_\psi = \mathbf{T}^{-1}_\psi\mathbf{S}^{-1}_{\mathbf{L}}\mathbf{\Lambda}^G_{\psi^{-1}(b)}\mathbf{S}_{\mathbf{L}}\mathbf{T}_\psi\\ &= \mathbf{T}^{-1}_\psi\mathbf{S}^{-1}_{\mathbf{L}}\mathbf{T}_\varphi\mathbf{T}^{-1}_\varphi\mathbf{\Lambda}^G_{\psi^{-1}(b)}\mathbf{T}_\varphi\mathbf{T}^{-1}_\varphi\mathbf{S}_{\mathbf{L}}\mathbf{T}_\psi = \mathbf{S}^{-1}_{\mathbf{B}}\mathbf{\Lambda}_{-b}\mathbf{S}_{\mathbf{B}} && (60)\end{aligned}$$

where in the last equality we used (56) and the fact that $\mathbf{T}^{-1}_\varphi\mathbf{\Lambda}^G_{\psi^{-1}(b)}\mathbf{T}_\varphi = \mathbf{\Lambda}_{-b}$. Thus, by the same arguments as for $\mathcal{A}^\diamond$, we see that $\mathcal{B}$ is given by (55) and (56). □

It should be noted that whatever the underlying group, the Hermitian and unitary operators are completely characterized by their (generalized) eigenfunctions, since all the operators based on a particular group share the same diagonal operator (based on the eigenvalues) in their spectral representation. For Hermitian operators, the diagonal operator is of the form (14) or (15), and for unitary operators it is essentially multiplication with a character of the underlying group (see (4) and (6)). Thus, to relate different operators based on different groups we only need to relate their eigenfunctions, which is precisely what is done in Theorem 3. An alternative interpretation of (52) and (54), in terms of the operators of the Hermitian counterpart of Baraniuk's approach, is: $\mathcal{A}^\diamond = \mathbf{T}^{-1}_\psi\psi(\mathcal{K}^\diamond)\mathbf{T}_\psi$, where $\mathcal{K}^\diamond = \mathbf{S}^{-1}_{\mathbf{K}^\diamond}\mathbf{\Lambda}^G\mathbf{S}_{\mathbf{K}^\diamond}$. Similarly, $\mathcal{B} = \mathbf{T}^{-1}_\psi\varphi(\mathcal{L})\mathbf{T}_\psi$, where $\mathcal{L} = \mathbf{S}^{-1}_{\mathbf{L}}\mathbf{\Lambda}^\Gamma\mathbf{S}_{\mathbf{L}}$.

Example (continued). Recall that $(G, \bullet) = (\mathbb{R}_+, \times)$, and the dilation operator[25] $(\mathbf{D}_{k'}s)(k) \equiv s(k/k')$ is a unitary representation of G on $L^2(\mathbb{R}_+, dk/k)$ which gets mapped to the time-shift operator $(\mathbf{T}_{a'}s)(a) \equiv s(a - a')$ in $L^2(\mathbb{R}, dx)$ (Cohen's method). This is precisely the reason why, using the correspondence rule of (26), the joint time-scale representations based on the **T**-CED, **D**-CED marginals or the **T**-IED, **D**-IED marginals are trivial [16]: $\mathbf{T}_{a'}$ and $\mathbf{D}_{k'}$, represented in the same signal space, are identical! Working in the $L^2(\mathbb{R}_+, dk/k)$ space, the symmetric Wigner-like representation corresponding to **T**-CED, **D**-IED marginals yields exactly the Q-distribution of Altes [25]. The corresponding representation in Cohen's method, using Theorem 2, is the Wigner distribution

[25] Which is associated with scale in [16, 17].

(WD), W, corresponding to the variables time and frequency; that is, $(Qs)(e^t,f) = (W\mathbf{T}_\psi^{-1}s)(t,f)$, where $(\mathbf{T}_\psi^{-1}s)(t) = s(e^t)$, which is exactly the relationship derived in [25].

Note that, in Baraniuk's method, the above example can also be construed, quite misleadingly though, as the joint representation based on a *single* (!) variable corresponding to the unitary operator $\mathbf{D}_{k'}$: just choose both the operators as $\mathbf{D}_{k'}$ and design the representation to yield CED marginal in one case and the IED marginal in the other case. The reason for the apparent confusion is that when a CED marginal is desired in Baraniuk's method, the underlying operator is effectively the *dual* operator which, as we argue in the next section using Stone's theorem, corresponds to a different variable (the dual variable to be precise). Thus, we are actually dealing with two operators, duals of each other, and hence two dual variables. This is also manifested in the fact that the corresponding Cohen's representation is the WD which is based on the dual variables of time and frequency [11]. This example illustrates the importance of precisely defining the operator correspondence principles and the relationship between Hermitian and unitary operator correspondences. Such issues are briefly discussed in the next section.

Example (continued). We can also illustrate the extended Baraniuk's method in this example. In Cohen's method, let $\mathcal{B} = \mathcal{F}$, the frequency Hermitian operator, and let $\mathcal{A} = \mathcal{C} \equiv \frac{j}{4\pi}\mathbf{I} - \mathcal{T}\mathcal{F}$, the "Mellin" Hermitian operator [11, 32]. The characteristic function operator for Weyl correspondence (23) is given by $\mathbf{M}^W_{(\alpha,\beta)} = e^{j2\pi(\alpha\mathcal{C}+\beta\mathcal{F})} = e^{j2\pi(e^\alpha-1)\beta\mathcal{F}/\alpha}e^{j2\pi\alpha\mathcal{C}}$ [32, 7], which yields the affine Wigner distribution [32, 7, 4] via (20)

$$(Ps)(a,b) = \int_{-\infty}^{\infty} S(\mu(x)e^x b)S^*(\mu(x)b)e^{x/2}e^{-j2\pi ax}\mu(x)dx \ , \tag{61}$$

where $\mu(x) \equiv \frac{x}{1-e^x}$, and $S \equiv \mathbb{F}s$ denotes the Fourier transform of s. Using (38), the corresponding unitary operators in Baraniuk's method, for both IED marginals, are given by (see foonote 22) $\mathbf{K}_{k'} = \mathbf{T}_\psi e^{j2\pi\ln(k')\mathcal{C}}\mathbf{T}_\psi^{-1}$ and $\mathbf{L}_{l'} = \mathbf{T}_\psi e^{j2\pi\ln(l')\mathcal{F}}\mathbf{T}_\psi^{-1} \equiv \mathbf{D}_{\frac{1}{l'}}$, where $\mathbf{K}_{k'}$ can be expressed more explicitly as $(\mathbf{K}_k s)(l) = k^{-1/2}s(l^{\frac{1}{k}})$, $s \in L^2(\mathbb{R}_+, dk/k)$.[26] Using (42), the corresponding characteristic function operator (Weyl's rule) in the extended Baraniuk's method is given by $\widehat{\mathbf{M}}^W_{(k,l)} = \mathbf{T}_\psi \mathbf{M}^W_{(\ln(k),\ln(l))}\mathbf{T}_\psi^{-1} = \mathbf{D}_{l^{\nu(k)}}\mathbf{K}_k$ where $\nu(k) \equiv \frac{1-k}{\ln(k)}$. The resulting form of the affine Wigner distribution in Baraniuk's method can be directly computed via

$$(\widehat{P}s)(\kappa,\lambda) = \int_G\int_G \left\langle \widehat{\mathbf{M}}^W_{(k,l)}s, s\right\rangle e^{-j2\pi\kappa\ln(k)}e^{-j2\pi\lambda\ln(l)}dk/k\ dl/l \ , \tag{62}$$

or, alternatively, we can use (43) in Theorem 2 to obtain it from (61) by using the change of variables $a \mapsto \kappa$, $b \mapsto \lambda$, and replacing $S = \mathbb{F}s$ with $\hat{S} = \mathbb{F}\mathbf{T}_\psi^{-1}s \equiv \mathbb{F}_{\mathbb{R}_+}s$, the Mellin transform, defined in (16).[27] Due to this warping, the resulting analogue of the affine WD for $L^2(\mathbb{R}_+, dk/k)$ offers different signal analysis characteristics than (61).

5 Some Implications

Characterization of the relationship between the two types of operator correspondences is essential for correct interpretation of JSRs, as demonstrated by the discussion of the example in the last section. In the next subsection, using the notion of *shift* and *dual* operators, we explicitly characterize the relationship between Hermitian and unitary operator correspondences, which yields the appropriate interpretation for unitary operators associated with variables. Moreover, although

[26] The operator $\mathbf{K}_k$ is similar to the parameterized warping operator used in [10] to relate the power class to the affine class.

[27] We note that the form of the affine WD in Baraniuk's method can also be interpreted, due to the warping operations involved, in terms of a different parameterization of the affine group.

Baraniuk's method explicitly allows the flexibility of IED or CED marginals, it is not obvious in Cohen's approach [16]. Using the notions of *shift* and *dual* operators we show in Section 5.2 that the Hermitian operator counterpart of Baraniuk's approach can generate CED marginals; it is just a matter of using the correct operator correspondence. The equivalence results of the previous section then yield the same conclusion for Cohen's method.

5.1 Relationship Between Hermitian and Unitary Operator Correspondences

Time and frequency variables help to provide the perspective for relating the Hermitian and unitary operator correspondences. The Hermitian operators associated with time and frequency are [14, 34]

$$(\mathcal{T}s)(t) \equiv ts(t) \quad \text{and} \quad (\mathcal{F}s)(t) \equiv \frac{-j}{2\pi}\frac{d}{dt}s(t) \ . \tag{63}$$

By Stone's theorem, $\mathcal{T}$ and $\mathcal{F}$ define two unitary operators given by

$$(\mathbf{F}_\theta s)(t) \equiv (e^{j2\pi\theta\mathcal{T}}s)(t) = e^{j2\pi\theta t}s(t) \quad \text{and} \quad (\mathbf{T}_\tau s)(t) \equiv (e^{-j2\pi\tau\mathcal{F}}s)(t) = s(t-\tau) \ . \tag{64}$$

Thus, the unitary operator associated with $\mathcal{T}$ via Stone's theorem is the frequency-shift operator, and the unitary operator associated with $\mathcal{F}$ is the time-shift operator. Thus, mathematically, it is tempting to associate time with $\mathbf{F}_\theta$ and frequency with $\mathbf{T}_\tau$. However, intuition and physical interpretation dictate the more natural association of $\mathbf{T}_\tau$ with time and $\mathbf{F}_\theta$ with frequency. That is, the unitary time operator produces a translation in the natural time-representation (with respect to the eigenfunctions of $\mathcal{T}$) of s, whereas the unitary frequency operator produces a translation in the frequency-representation of the signal. Note that the time and frequency variables are based on $(\mathbb{R},+)$, with the group Fourier transform being the usual Fourier transform; that is $\mathbb{F}_G = \mathbb{F}$ where $(\mathbb{F}s)(f) = \int s(t)e^{-j2\pi ft}dt$.[28] Thus, the unitary operators for time and frequency can be interpreted as *shift* operators with shift being the group operation: simple translation, in this case. We now make the notion of shift precise for arbitrary groups.

Definition: Shift Operator. Let $k \in G$ be a variable associated with the Hermitian operator $\mathcal{K}^\diamond = \mathbf{S}_{\mathcal{K}^\diamond}^{-1}\mathbf{\Lambda}^G\mathbf{S}_{\mathcal{K}^\diamond}$, and let $\lambda \in \Gamma$ be another variable associated with $\mathcal{L} = \mathbf{S}_{\mathcal{L}}^{-1}\mathbf{\Lambda}^\Gamma\mathbf{S}_{\mathcal{L}}$.[29] Then, a (unitary) shift operator, $\mathbf{K}_{k'}$, $k' \in G$, for the variable k, is one which produces a group translation in the natural k-representation of the signal; that is, $(\mathbf{S}_{\mathcal{K}^\diamond}\mathbf{K}_{k'}s)(l) = (\mathbf{S}_{\mathcal{K}^\diamond}s)(l \bullet k'^{-1})$. Similarly, a shift operator, $\mathbf{L}^\diamond_{\lambda'}$, $\lambda' \in \Gamma$, for the variable λ, is one which satisfies $(\mathbf{S}_{\mathcal{L}}\mathbf{L}^\diamond_{\lambda'}s)(\kappa) = (\mathbf{S}_{\mathcal{L}}s)(\kappa \circ \lambda'^{-1})$.

Thus, based on above arguments, the unitary operator associated with a variable should be the corresponding shift operator as defined above. We now characterize the shift operator for a given variable. In terms of the group translation operators $\Upsilon^G_{k'}$ and $\Upsilon^\Gamma_{\lambda'}$, defined in (3) and (5), the shift operators, defined above, can be expressed as

$$\mathbf{K}_{k'} = \mathbf{S}_{\mathcal{K}^\diamond}^{-1}\Upsilon^G_{k'^{-1}}\mathbf{S}_{\mathcal{K}^\diamond} \quad \text{and} \quad \mathbf{L}^\diamond_{\lambda'} = \mathbf{S}_{\mathcal{L}}^{-1}\Upsilon^\Gamma_{\lambda'^{-1}}\mathbf{S}_{\mathcal{L}} \ . \tag{65}$$

Substituting the fundamental relationships (7) in the above equations yields the following characterization of the shift operators in terms of their spectral representations:

$$\mathbf{K}_{k'} = \mathbf{S}_{\mathcal{K}^\diamond}^{-1}\mathbb{F}_G^{-1}\mathbf{\Lambda}^G_{k'}\mathbb{F}_G\mathbf{S}_{\mathcal{K}^\diamond} = \mathbf{S}_{\mathbf{K}}^{-1}\mathbf{\Lambda}^G_{k'}\mathbf{S}_{\mathbf{K}} \quad , \quad \mathbf{S}_{\mathbf{K}} = \mathbb{F}_G\mathbf{S}_{\mathcal{K}^\diamond} \tag{66}$$

$$\mathbf{L}^\diamond_{\lambda'} = \mathbf{S}_{\mathcal{L}}^{-1}\mathbb{F}_G\mathbf{\Lambda}^\Gamma_{\lambda'}\mathbb{F}_G^{-1}\mathbf{S}_{\mathcal{L}} = \mathbf{S}_{\mathbf{L}^\diamond}^{-1}\mathbf{\Lambda}^\Gamma_{\lambda'}\mathbf{S}_{\mathbf{L}^\diamond} \quad , \quad \mathbf{S}_{\mathbf{L}^\diamond} = \mathbb{F}_G^{-1}\mathbf{S}_{\mathcal{L}} \ . \tag{67}$$

The uniqueness of shift operators follows from the uniqueness of the spectral representation of groups of unitary operators [21].

[28] Moreover, time and frequency variables are dual in the sense of the definition of Section 2.1 since the eigenfunctions of $\mathcal{T}$ and $\mathcal{F}$ are related as $\mathbf{S}_{\mathcal{T}} = \mathbb{F}_G^{-1}\mathbf{S}_{\mathcal{F}}$.

[29] See footnote 14.

Discussion. By Stone's theorem, the most direct choice of unitary operators to be associated with the variables k and λ is the *character* operators $\mathbf{K}^{\diamond}_{\kappa} \equiv (\mathcal{K}^{\diamond}, \kappa)^*$ and $\mathbf{L}_l \equiv (l, \mathcal{L})$, respectively[30] (which corresponds to associating time with $\mathbf{F}_\theta$ and frequency with $\mathbf{T}_\tau$ in the case $G = \mathbb{R}$). However, we have argued that if we start with a variable and a corresponding Hermitian operator, the appropriate unitary operator to be associated with the variable is the *shift* operator which is precisely the *dual* of the unitary (*character*) operator defined by the Hermitian operator via Stone's theorem. This choice for unitary operator correspondence is further justified by the fact that given a variable belonging to a group, the parameter of the corresponding *shift* operator belongs to the group itself, whereas the parameter of the *character* operator belongs to the dual group.

5.2 CED and IED Marginals

Consider two variables k and λ with the corresponding Hermitian operators $\mathcal{K}^{\diamond}$ and $\mathcal{L}$ as defined in the previous section. Suppose we generate joint k-λ distributions using the Hermitian operator counterpart of Baraniuk's method outlined in Section 3. If we choose the *shift* operators, $\mathbf{K}_{k'}$ and $\mathbf{L}^{\diamond}_{\lambda'}$, as the unitary operators corresponding to k and λ, respectively, then, by definition of shift operators, we note from (30) and (31) that the k-marginal is $\mathbf{K}$-covariant and λ-marginal is $\mathbf{L}^{\diamond}$-covariant. On the other hand, if we choose the *character* operators $\mathbf{K}^{\diamond}_{\kappa}$ and $\mathbf{L}_l$ as the corresponding unitary operators, then the marginals are *invariant* to the unitary operators. Thus, we see that the Hermitian operator-based approach can generate both CED and IED marginals; it is just a matter of associating the *appropriate* unitary operators with the variables. In fact, in light of our argument for associating shift operators with variables, the Hermitian operator approach always generates CED marginals.

6 Conclusions

Recently, in an attempt to tailor JSRs to a wide variety of signal characteristics, there has been significant research on joint distributions of variables other than time and frequency. Time-scale representations constituted the first such generalizations spurred by the interest in the wavelet transform. More recently, research in such generalized JSRs has culminated in two main classes of general theories for JSRs of arbitrary variables. On one hand are the distributional approaches of Cohen [1, 6] and Baraniuk [16, 17], which emphasize the marginal properties of JSRs, and on the other hand are the covariance-based formulations proposed by Hlawatsch and Bölcskei [18, 19] and Sayeed and Jones [20], which are based on the covariance of JSRs to certain unitary operators.

In this paper, we focussed on the general methods proposed by Cohen and Baraniuk, and one of the main results of the paper was that the two approaches, despite being apparently quite different, are completely equivalent. In addition, we explicitly characterize the relationship between the operators associated with the variables of interest in the two methods. From a theoretical viewpoint, by unifying the two main distributional approaches, the results of this paper facilitate a better understanding of the theory of JSRs of arbitrary variables.

Quite remarkably, the two types of JSRs generated by the two methods are *unitarily equivalent*, and the unitary transformations relating them are simply input- and output-*axis warping* transformations. This fact has important implications for properties of arbitrary JSRs. In particular, by appropriately prewarping the signal axis and postwarping the axes of JSRs from Cohen's method (which possess *translational* covariance properties), we can generate JSRs with radically different covariance properties. These warped representations (the JSRs in Baraniuk's method), in view of our equivalence results, can be interpreted as JSRs corresponding to the *same* variables, albeit with respect to different bases or signal spaces. Such flexibility in choosing covariance properties can potentially be very useful for detecting or estimating the effects of certain unitary signal

[30] The *character* and *shift* operators are *dual* (see Section 2.1) because $\mathbf{S}_{\mathbf{K}^{\diamond}} = \mathbf{S}_{\mathcal{K}^{\diamond}} = \mathbb{F}_G^{-1}\mathbf{S}_{\mathbf{K}}$ and $\mathbf{S}_{\mathbf{L}} = \mathbf{S}_{\mathcal{L}} = \mathbb{F}_G \mathbf{S}_{\mathbf{L}^{\diamond}}$, where we have used the fact that a Hermitian operator and the corresponding character operator share the same eigenfunctions (e.g. $\mathbf{S}_{\mathbf{K}^{\diamond}} = \mathbf{S}_{\mathcal{K}^{\diamond}}$).

transformations (which may model a channel or system, for example) [35, 36]; from a practical perspective, certain changes are easier than others to detect or estimate.

Fundamental to both Cohen's and Baraniuk's approach is the idea of associating variables with operators; Cohen uses Hermitian operators, whereas Baraniuk uses groups of unitary operators. An important issue, which is fundamental to the understanding of generalized JSRs and which has been left unaddressed in existing treatments, is the relationship between the two types of operator *correspondences*. By interpreting Baraniuk's approach in terms of Hermitian operators and using the concept of *shift* and *dual* operators, we have precisely characterized the relationship between the Hermitian and unitary operators associated with a variable. In particular, given a variable and a Hermitian operator associated with it, the corresponding unitary operator should be the *shift* operator associated with the variable.

Finally, we note that Baraniuk's method may conceptually be the preferred method in certain situations, but Cohen's approach, being based on the Fourier transform, has a computational advantage. In fact, an important practical implication of the equivalence results is that we can replace the group transforms in Baraniuk's approach, which are computationally *inefficient* in general, with the Fourier transform and simple pre- and post-processing. Moreover, depending on the variable in question, either the corresponding Hermitian or the unitary operator may have a simpler description. Thus, the results of this paper, in addition to providing a better understanding of the theory and application of generalized JSRs, also allow the adoption of the most convenient approach in any given situation.

Appendix

Proof of Theorem 1. We first need to verify that φ defined in (33) is one-to-one and onto Γ_2 and satisfies

$$\varphi(\kappa_1 \circ \kappa_2) = \varphi(\kappa_1) \cdot \varphi(\kappa_2) \quad \text{for all } \kappa_1, \kappa_2 \in \Gamma_1 \; . \tag{68}$$

To show φ is one-to-one, suppose that $\varphi(\kappa_1) = \varphi(\kappa_2)$. Then from (33) we have

$$(\psi^{-1}(l), \kappa_1)_{G_1} = (l, \varphi(\kappa_1))_{G_2} = (l, \varphi(\kappa_2))_{G_2} = (\psi^{-1}(l), \kappa_2)_{G_1} \quad \text{for all } l \in G_2 \tag{69}$$

which implies, using the fact that ψ is onto, that $(k, \kappa_1 \circ \kappa_2^{-1}) = 1$ for all $k \in G_1$, from which it follows that $\kappa_1 = \kappa_2$ [22], which proves that φ is one-to-one.

For ontoness, we need to show that for each $\lambda \in \Gamma_2$, there exists a $\kappa \in \Gamma_1$ such that $\varphi(\kappa) = \lambda$. Given $\lambda \in \Gamma_2$, define $\kappa \in \Gamma_1$ as

$$(k, \kappa)_{G_1} \equiv (\psi(k), \lambda)_{G_2} \; , \quad \text{for all } k \in G_1. \tag{70}$$

Then, by (33) we have, for all $l \in G_2$, $(l, \varphi(\kappa))_{G_2} = (\psi^{-1}(l), \kappa)_{G_1} = (\psi\psi^{-1}(l), \lambda)_{G_2} = (l, \lambda)_{G_2}$, where the second to last equality follows from (70). Thus, $\lambda = \varphi(\kappa)$, and hence φ is onto. Continuity of φ and φ^{-1} follows from the fact that the mapping (k, κ) is continuous on $G \times \Gamma$ [22, p. 10].

To show (68), note that for all $l \in G_2$ and for all $\kappa_1, \kappa_2 \in \Gamma_1$,

$$\begin{aligned}
(l, \varphi(\kappa_1 \circ \kappa_2))_{G_2} &= (\psi^{-1}(l), \kappa_1 \circ \kappa_2)_{G_1} \qquad \text{using (33)} && (71)\\
&= (\psi^{-1}(l), \kappa_1)_{G_1} (\psi^{-1}(l), \kappa_2)_{G_1} \text{ by definition of characters} \in \Gamma_1 && (72)\\
&= (l, \varphi(\kappa_1))_{G_2} (l, \varphi(\kappa_2))_{G_2} \qquad \text{by (33)} && (73)\\
&= (l, \varphi(\kappa_1) \cdot \varphi(\kappa_2))_{G_2} \text{ by definition of characters} \in \Gamma_2 && (74)
\end{aligned}$$

from which we conclude that (68) holds.

We now prove (34). We first note that the set function $\tilde{\mu}_{\Gamma_1} : \Gamma_1 \to [0, \infty)$ defined by $\tilde{\mu}_{\Gamma_1}(F) = \mu_{\Gamma_2}(\varphi(F))$, for all measurable sets $F \subset \Gamma_1$, is also a Haar measure on Γ_1, and thus, by the uniqueness of Haar measure [22], we must have $\mu_{\Gamma_1} = \lambda \tilde{\mu}_{\Gamma_1}$ for some $\lambda > 0$. Thus, we only need to show that $\lambda = 1$, which can be shown by proving (34) for any one particular measurable set $F \subset \Gamma_1$. Let $F \subset \Gamma_1$ be a measurable set with $\mu_{\Gamma_1}(F) < \infty$. Define $H \in L^2(\Gamma_1, d\mu_{\Gamma_1})$ by $H(\kappa) \equiv I_F(\kappa)$, the indicator function of F. Then, $h = \mathbb{F}_G^{-1} H$ exists in $L^2(G_1, d\mu_{G_1})$ and using (1) we have

$$H(\kappa) = \int_{G_1} h(k)(k,\kappa)^*_{G_1} d\mu_{G_1}(k) = \int_{G_1} h(k)(\psi(k),\varphi(\kappa))^*_{G_2} d\mu_{G_1}(k) \quad \text{using (33)}, \tag{75}$$

$$H(\varphi^{-1}(\lambda)) = \int_{G_2} h(\psi^{-1}(l))(l,\lambda)^*_{G_2} d\mu_{G_1}(\psi^{-1}(l)) \quad , \quad l = \psi(k), \lambda = \varphi(\kappa) \tag{76}$$

$$= \int_{G_2} h(\psi^{-1}(l))(l,\lambda)^*_{G_2} d\mu_{G_2}(l) \qquad \text{using (32)} . \tag{77}$$

By inverting (77) we get

$$h(\psi^{-1}(l)) = \int_{\Gamma_2} H(\varphi^{-1}(\lambda))(l,\lambda)_{G_2} d\mu_{\Gamma_2}(\lambda) . \tag{78}$$

Note that $\psi^{-1}(0) = 0$, and that the relation $h = \mathbb{F}_G^{-1} H$ implies that $h(0) = \int_{\Gamma_1} H d\mu_{\Gamma_1} = \mu_{\Gamma_1}(F)$. Thus, from (78) we have

$$\mu_{\Gamma_1}(F) = h(0) = h(\psi^{-1}(0)) = \int_{\Gamma_2} I_F(\varphi^{-1}(\lambda)) d\mu_{\Gamma_2}(\lambda) = \int_{\Gamma_2} I_{\varphi(F)}(\lambda) d\mu_{\Gamma_2}(\lambda) = \mu_{\Gamma_2}(\varphi(F)) , \tag{79}$$

which proves that $\lambda = 1$ and hence (34). This completes the proof.

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