GENERALIZED DIGITAL BUTTERWORTH FILTER DESIGN *

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ABSTRACT

This paper presents a formula-based method for the design of IIR filters having more zeros than (nontrivial) poles. The filters are designed so that their square magnitude frequency responses are maximally-flat at $\omega = 0$ and at $\omega = \pi$ and are thereby generalizations of classical digital Butterworth filters. A main result of the paper is that, for a specified half-magnitude frequency and a specified number of zeros, there is only one valid way in which to split the zeros between $z = -1$ and the passband. IIR filters having more zeros than poles are of interest, because often, to obtain a good trade-off between performance and the expense of implementation, just a few poles are best.

1. INTRODUCTION

Probably the best known and most commonly used method for the design of IIR digital filters is the transformation of the classical analog filters (the Butterworth, Chebyshev I and II, and Elliptic filters) [5]. One advantage of this technique is the existence of formulas for these filters. Unfortunately, all such IIR filters have an equal number of poles and zeros. It is desirable to be able to design filters having more zeros than poles (away from the origin), for implementation purposes. This paper presents a method for the design of maximally-flat lowpass IIR filters having more zeros than poles and which possess a specified half-magnitude frequency. It is worth noting that not all the zeros are restricted to lie on $|z| = 1$. The method uses a formula, a transformation of variables, and a spectral factorization. Note that no phase approximation is done; the approximation is in the magnitude squared - as are the classical IIR filter types.

Another main result of the paper is that for a specified number of zeros and a specified half-magnitude frequency, there is only one valid way to divide the number of zeros between $z = -1$ and the passband. This is in contrast to the classical digital Butterworth filter, for which all the zeros lie at $z = -1$, regardless of the position of the half-magnitude frequency in $(0, \pi)$. The formulas given below provide a direct way to determine the number of zeros that must lie at $z = -1$ and the number of zeros that must contribute to the passband.

Given a half-magnitude frequency ω_o, the filters produced by the formulas described below are optimal in the sense that the maximum number of derivatives at $\omega = 0$ and $\omega = \pi$ are set to zero, under the constraint that the filter possesses the half-magnitude frequency ω_o. The IIR filters obtained by transforming the classical Butterworth filters, and the FIR filters obtained by Herrmann's formulas [1] are both special cases of the filters produced by the formulas given below.

Several authors have addressed the design and the advantages of IIR filters with unequal numerator and denominator degrees [2, 3, 4, 9, 10, 11]. In [8, 9], Saramäki finds that the classical Elliptic and Chebyshev filter types are seldom the best choice. In [2] Jackson improves the Martinez/Parks algorithm and notes that, for equi-ripple filters, the use of just 2 poles "is often the most attractive compromise between computational complexity and other performance measures of interest." However, to our knowledge, no formulas have been presented for the design of IIR filters in which zeros are not constrained to lie on the unit circle.

2. NOTATION

Let $B(z)/A(z)$ denote the transfer function of a digital filter. Its frequency response magnitude $M(\omega)$ is given by $|B(e^{\jmath\omega})/A(e^{\jmath\omega})|$. Throughout this paper, the degree of $B(z)$ will be denoted by $L + M$, where L is the number of zeros at $z = -1$ and M is the number of zeros that contribute to the passband. The degree of $A(z)$ will be denoted by N.

The zeros at $z = -1$ produce a flat behavior in the frequency response magnitude at $\omega = \pi$, while the remaining zeros, together with the poles, are used to produce a flat behavior at $\omega = 0$. The meanings of the parameters are shown in table 1. The half-magnitude frequency is that frequency at which the magnitude equals one half.

3. EXAMPLES

The classical digital Butterworth filter (defined by $L = N$, $M = 0$) is a special case of the filters discussed below. Figures 1 and 2 show a classical digital Butterworth filter of order 4 ($L = 4$, $M = 0$, $N = 4$).

The first generalization permits L to be greater than N: $L > N$ with $M = 0$. Figures 1 and 3 show an IIR filter with $L = 6$, $M = 0$, $N = 4$. It turns out that when $L > N$,

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Table 1. Notation.

Parameters	
$L+M$	total number of zeros
L	number of zeros at $z=-1$
N	total number of poles
ω_o	half-magnitude frequency
Flatness	
$L+M+N$	total degrees of flatness
$M+N$	degree of flatness at $\omega=0$
L	degree of flatness at $\omega=\pi$

Table 2. For L, M, and N shown, the interval of permissible half-magnitude frequencies ω_o is given by ω_{min} and ω_{max}. $L+M$ is the numerator degree and N is the denominator degree.

$L+M$	L	M	N	ω_{min}/π	ω_{max}/π
4	4	0	4	0	1
5	5	0	4	0	0.5349
	4	1	4	0.5349	1
6	6	0	4	0	0.4620
	5	1	4	0.4620	0.6017
	4	2	4	0.6017	1
7	7	0	4	0	0.4140
	6	1	4	0.4140	0.5299
	5	2	4	0.5299	0.6446
	4	3	4	0.6446	1

the restriction that M equal zero limits the range of achievable half-magnitude frequencies. This motivates the second generalization.

In addition to permitting L to be greater than N, the second generalization permits M to be greater than zero: $L \geq N$ and $M > 0$. Figures 1 and 4 show an IIR filter with $L = 16$, $M = 7$, $N = 4$.

As mentioned above, for a specified half-magnitude frequency ω_o and a specified number of zeros ($L+M$), there is only one correct way to split the zeros between $z=-1$ and the passband. To illustrate this property, it is helpful to construct a table that indicates the appropriate values for L, M and N. When $N=4$ and $L+M$ equals $4,\dots,7$, table 2 gives the appropriate choice for L and M to achieve a desired half-magnitude frequency. As can be seen from the table, the intervals cover the interval $(0,\pi)$ and do not overlap. As explained below, these intervals can be unambiguously computed by inspecting the roots of an appropriate set of polynomials.

4. DERIVATIONS

The approach described below provides formulas for two nonnegative polynomials $P(x)$ and $Q(x)$. Then, by (*i*) using a suitable transformation ($x = \frac{1}{2}(1-\cos\omega)$ as in [1]) and (*ii*) taking a spectral factor, a stable IIR filter $B(z)/A(z)$ is obtained having a magnitude squared frequency response $|M(\omega)|^2$ given by

$$|M(\omega)|^2 = \frac{P(\frac{1}{2}-\frac{1}{2}\cos\omega)}{Q(\frac{1}{2}-\frac{1}{2}\cos\omega)}.$$

Accordingly, $P(x)/Q(x)$ is designed to approximate a low-pass response over $x \in [0,1]$. This results in a formula-based method. No iterations are required for finding $P(x)$ and $Q(x)$.

To begin, we derive the classical digital Butterworth filter. This establishes notation and makes clear the way in which the generalization uses the same ideas in its derivation.

4.1. Classical Digital Butterworth Filter

Let the degree of $P(x)$ be L, the degree of $Q(x)$ be L, and define $F(x) = P(x)/Q(x)$. In order to obtain L degrees of flatness at $x=1$, $F(x)$ must have the following form:

$$F(x) = \frac{P(x)}{Q(x)} = \frac{(1-x)^L}{Q(x)}. \tag{1}$$

In order that $F(x)-1$ have an L degree of flatness at $x=0$, $F(x)$ must satisfy

$$F(x) - 1 = \frac{P(x)-Q(x)}{Q(x)} = -\frac{cx^L}{Q(x)} \tag{2}$$

where c is an appropriately chosen constant. Solving eqs (1) and (2) for $Q(x)$ gives

$$Q(x) = (1-x)^L + cx^L. \tag{3}$$

To choose c to achieve a specified half-magnitude frequency ω_o is straight-forward. The eq $|M(\omega_o)| = \frac{1}{2}$ becomes $F(x_o) = \frac{1}{4}$ where $x_o = \frac{1}{2}(1-\cos\omega_o)$. Solving for c, one obtains $c = 3\frac{(1-x_o)^L}{x_o^L}$.

4.2. First Generalization

Let L denote the number of zeros at $x=1$ and let N denote the number of poles with $L \geq N$. Then, as above,

$$F(x) = \frac{P(x)}{Q(x)} = \frac{(1-x)^L}{Q(x)} \tag{4}$$

where $Q(x)$ has degree N. But

$$F(x) - 1 = \frac{P(x)-Q(x)}{Q(x)} = -\frac{x^N U(x)}{Q(x)} \tag{5}$$

where $U(x)$ is a polynomial of degree at most $L-N$. Solving eqs (4) and (5) for $Q(x)$ gives

$$Q(x) = (1-x)^L + x^N U(x). \tag{6}$$

Table 3. Permissible ranges for c for the first generalization.

N even	$c \geq 0$
N odd	$c \geq \binom{L-1}{N}$

Since $Q(x)$ has degree N, $Q(x)$ must equal the polynomial obtained by taking only the first $N+1$ coefficients of $(1-x)^L + x^N U(x)$. $U(x)$ can always be chosen so that the remaining coefficients are zero. Introducing the notation $\mathcal{T}_N$ for polynomial truncation (discarding all terms beyond the N^{th} power), $Q(x)$ can be written as

$$\begin{aligned} Q(x) &= \mathcal{T}_N\{(1-x)^L\} + cx^N && (7) \\ &= \sum_{i=0}^{N} \binom{L}{i} (-x)^i + cx^N. && (8) \end{aligned}$$

The free parameter, c, must lie within the ranges shown in table 3. (When c is chosen to lie in the ranges shown in the table, then $0 < F(x) < 1$ for $x \in (0,1)$.) To choose c to achieve a specified half-magnitude frequency ω_o, let $x_o = \frac{1}{2}(1 - \cos\omega_o)$. Solving $F(x_o) = \frac{1}{4}$ for c yields

$$c = \frac{4(1-x_o)^L - \mathcal{T}_N\{(1-x)^L\}(x_o)}{x_o^N}. \qquad (9)$$

If c does not lie in the range shown in table 3 then the specified ω_o is too high for the current choice of L and N. The greatest ω_o achievable for a fixed L and N can be found by setting c equal to the appropriate value shown in table 3 and solving eq (9) for x_o. It is seen that x_o is a root of an L degree polynomial,

$$\mathcal{T}_N\{(1-x)^L\} + cx^N - 4(1-x)^L = 0, \qquad (10)$$

having exactly one real root in (0,1).

To obtain filters having wider passbands with the same number of zeros and (nontrivial) poles, it is necessary to move at least one zero from $x = 1$ ($z = -1$) to the passband. This leads us to the next subsection.

4.3. Second Generalization

The second generalization possesses zeros lying away from $z = -1$. These zeros are used to obtain a higher degree of flatness at $\omega = 0$ (see figure 4). The filters possess a degree of flatness of $M + N$ at $\omega = 0$, and a degree of flatness of L at $\omega = \pi$.

Following the same reasoning as above, a closed form solution was found to be given by:

$$P(x) = (1-x)^L(R(x) + cT(x)). \qquad (11)$$

$$Q(x) = \mathcal{T}_N\{P(x)\} \qquad (12)$$

where $R(x)$ and $T(x)$ are given by

$$R(x) = \sum_{k=0}^{M-1} \binom{M+N-k-1}{N} \binom{L-N+k-1}{k} x^k \qquad (13)$$

Table 4. Permissible ranges for c for the second generalization.

N even	$-1 \leq c \leq \dfrac{L-N}{M+N}$
N odd	$\dfrac{L-N}{N} \leq c$

and

$$T(x) = x \sum_{k=0}^{M-1} \binom{M+N-k-2}{N-1} \binom{L-N+k}{k} x^k. \qquad (14)$$

$\binom{n}{k}$ is a binomial coefficient. To evaluate $\binom{n}{k}$ for negative values of n we use the convention [7]: $\binom{n+k-1}{k} = (-1)^k \binom{-n}{k}$ for $k \geq 0$. Also, $\binom{n}{k} = 0$ for $n \geq 0$, $k < 0$; and $\binom{n}{k} = 0$ for $n \geq 0$, $k > n$.

The free parameter c can be chosen to position the half-magnitude frequency ω_o. c must lie in the ranges shown in table 4. To choose c to achieve a specified ω_o, let $x_o = \frac{1}{2}(1 - \cos\omega_o)$. Solving $F(x_o) = \frac{1}{4}$ for c yields

$$c = \frac{4(1-x_o)^L R(x_o) - \mathcal{T}_N\{(1-x)^L R(x)\}(x_o)}{\mathcal{T}_N\{(1-x)^L T(x)\}(x_o) - 4(1-x_o)^L T(x_o)}. \qquad (15)$$

If c does not lie in the range given in table 4, then the specified ω_o is either too high or too low for the current choice of L, M and N – it is necessary to alter the distribution of zeros between $x = 1$ ($z = -1$) and the passband.

For fixed L, M, and N, the minimum and maximum permissible values of the half-magnitude frequency ω_o can be computed by (*i*) setting c to the values in table 4, (*ii*) solving eq (15) for x and (*iii*) using $\omega = \arccos(1-2x)$. When c is finite, it is seen that x_o is a root of the $L + M$ degree polynomial:

$$\begin{aligned} &\mathcal{T}_N\{(1-x)^L(R(x) + cT(x))\} - \\ &\qquad 4(1-x)^L(R(x) + cT(x)) = 0. \end{aligned} \qquad (16)$$

When N is odd, c can be arbitrarily large, in which case the appropriate polynomial eq becomes:

$$\mathcal{T}_N\{(1-x)^L T(x)\} - 4(1-x)^L T(x) = 0. \qquad (17)$$

Each polynomial has exactly one real root in the interval (0,1). Thus the appropriate L and M can be found systematically by finding the roots of the appropriate polynomials and constructing a table like table 2.

5. CONCLUSION

By using appropriate formulas and a transformation, and by taking a spectral factor, maximally flat IIR filters having more zeros than (nontrivial) poles can be easily designed - and without the restriction that all zeros lie on the unit circle. In addition, for a fixed number of zeros and a fixed number of (nontrivial) poles, the formulas above give a direct way of finding the correct way to split the number of zeros between $z = -1$ and the passband.

More information can be found on the World Wide Web at URL `http://www-dsp.rice.edu`.

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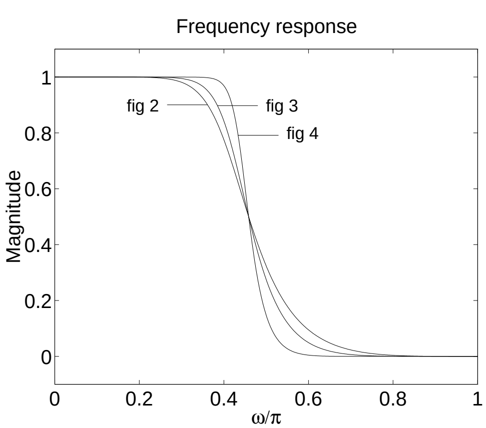

Figure 1. Magnitudes

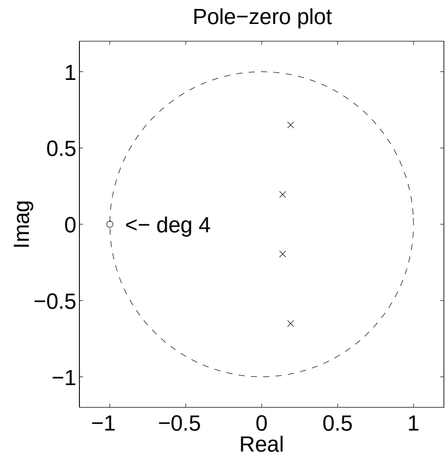

Figure 2. $L = 4$, $M = 0$, $N = 4$.

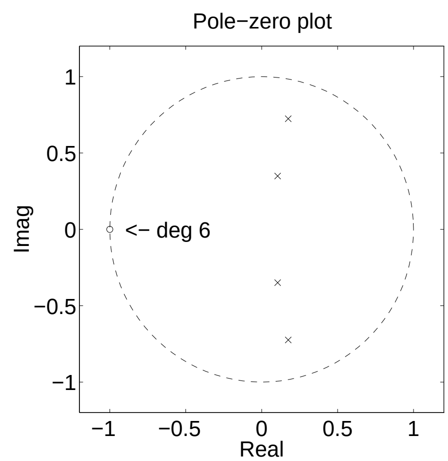

Figure 3. $L = 6$, $M = 0$, $N = 4$.

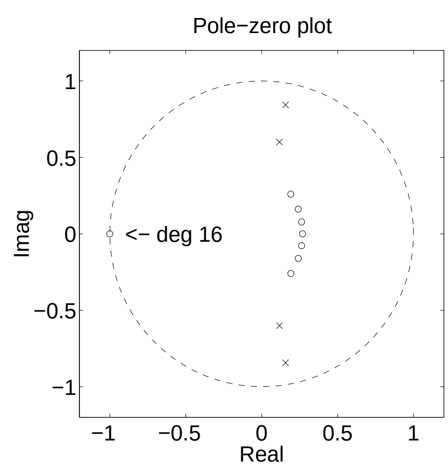

Figure 4. $L = 16$, $M = 7$, $N = 4$.