DISCRETE FINITE VARIATION: A NEW MEASURE OF SMOOTHNESS FOR THE DESIGN OF WAVELET BASIS

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ABSTRACT

A new method for measuring and designing smooth wavelet basis which dispenses with the need for having a large number of zero moments of the wavelet is given. The method is based on minimizing the "discrete finite variation", and is a measure of the local "roughness" of a *sampled* version of the scaling function giving rise to "visually smooth" wavelet basis. Smooth wavelet basis are deemed to be important for several applications and in particularly for image compression where the goal is to limit spurious artifacts due to non-smooth basis functions in the presence of quantization of the individual subbands. The definition of smoothness introduced here gives rise to new algorithms for designing smooth wavelet basis with only one vanishing moment leaving free parameters, otherwise used for setting moments to zero, for optimization.

1. INTRODUCTION

Hölder and Sobolev exponents are fundamental mathematical measures of smoothness giving precise definition of differentiability. Several recent publications (in particular [6, 2]) has provided algorithms for estimating both the Hölder and the Sobolev exponent for a given wavelet basis given the associated scaling filter. However, to obtain good estimates it is necessary to provide the regularity or more specifically the number of zero moments of the wavelet function (or filter). Although this is a simple task if an estimate of the exponent for a given filter, designed to have a fixed number of zero moments, is desired, the task is significantly harder if one first have to estimate the number of zero moments. In fact, estimating the number of zero moments is exactly what has to be done if the Hölder or Sobolev exponents were to be used as cost functions in an optimization algorithm for finding the optimally smooth wavelet of a given support. It should be noticed that the problem of estimating the number of zero moments is equivalent to finding roots of a polynomial which is known to be a numerically ill conditioned problem. Due to the numerical problems neither the Hölder nor the Sobolev exponent are suited cost functions for designing smooth wavelets using nonlinear optimization. Recently a method for designing "more" differentiable wavelets (compared to the K-regular Daubechies solution) has been developed by Lang and Heller [4]. However, it does necessarily require that a large number of moments be set to zero.

First recall that the continuous (time) wavelet basis functions are obtained by iterating the associated filter bank an infinite number of times. Now realizing that any practical application will only involve a *finite* number of iterations, the associated functions are not continuous but discrete for which the strict definition of differentiability given by the Hölder and Sobolev exponents are not (necessarily) meaningful. As a result alternative descriptions of smoothness incorporating "visual" smoothness behavior (e.g., *local variation*) should be considered.

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Furthermore, one can show [5] that a good approximation to the discrete wavelet transform (DWT) can be implemented efficiently by approximate orthogonal rotations. However, by applying approximate orthogonal rotations one can only make the zeroth moment vanish exactly and only hope that higher order moments are small. Hence, with the goal of implementing the DWT efficiently, the fact that exact zero moments of higher order can not be achieved should be incorporate in to the design specifications. If in addition to only requiring that the first moment be set to zero it is imposed that the lattice angles be of the form

$$\beta_i = \sum_{j=0}^{W-1} b_{i,j} \arctan(1/2^j) \tag{1}$$

where W is the specified word-length and $b_{i,j} \in \{0,1\}$ one can in theory design (e.g., by integer programming) wavelet basis which can be implemented *exactly* by a small number of μ-rotations.

Based on these observations and the fact that prescribing only one zero moment (the zeroth moment of $\psi_1(t)$) the wavelet basis can at best be C^1 and hence measures of smoothness such as the Hölder and Sobolev exponents are insufficient for our purpose. The results presented in this paper is building on ideas first seen in [1] where total variation was introduced as a way to optimize the smoothness of the prototype filter for M-band cosine modulated orthonormal wavelets basis. The new measure, discrete finite variation, can be optimized independent of the number of vanishing moments giving rise to smooth wavelet basis. Furthermore, since the new measure does not depend on having a large number of vanishing moments for obtaining smooth basis it can in principle be used to solve the lattice angle quantization problem prescribed by (1).

1.1. Wavelets

In the remainder of the paper the discussion will be restricted to 2-band orthonormal wavelet basis. However, most of the theory easily applies to the M-band solution as well as the biorthogonal solution. A 2-band compactly supported orthonormal wavelet bases are characterized by a scaling filter, $h_0(k)$, and wavelet filter, $h_1(k)$, both of finite length $N = 2K$, satisfying the linear condition

$$\sum_{n=0}^{N-1} h_0(n) = \sqrt{2} \tag{2}$$

and for all $k \in \mathbb{Z}$ the quadratic condition

$$\sum_{n=0}^{N-1} h_i(n) h_j(n+2k) = \delta(i-j)\delta(k) \quad i,j \in \{0,1\}\,. \tag{3}$$

The scaling function, $\psi_0(t)$, and the wavelet function, $\psi_1(t)$, is then defined for $i = 0, 1$ by the dyadic difference equation

$$\psi_i(t) = \sqrt{2} \sum_k h_i(k) \psi_0(2t - k). \tag{4}$$

Exact hardware implementations of a discrete wavelet transform (DWT) must satisfy both the orthogonality as well as the wavelet zero moment condition. In fact, for the maximally vanishing moment solution (Daubechies) there are $K = \frac{N}{2}$ vanishing moments. It is well known that implementing the DWT by lattice rotations orthogonality is structurally imposed through the basic building blocks (2×2 orthogonal rotations) [7]. It is also well know that by imposing that the sum of the lattice angles be $-\frac{\pi}{4}$ the zeroth moment is guaranteed to be zero (e.g., one can easily guarantee the implementation of an orthogonal wavelet basis). In fact, choosing any set of lattice angles as long as the sum is constrained to be $-\frac{\pi}{4}$ an orthogonal wavelet transformation is guaranteed independent of the properties (e.g., smoothness, regularity, stop band characteristics etc.) of the corresponding scaling and wavelet filters/functions.

If more than one vanishing moment (to high numerical accuracy) is required this can be achieved by implementing the lattice filter with floating point multipliers. However, if efficient implementations are desired [5] one has to give up the moment approximation property in favor of efficient arithmetic (replace floating point multiplies with a few shifts and adds).

The goal is to obtain an algorithm for designing "smooth" wavelet basis that can be implement efficiently without sacrificing properties obtained by the design. The first step in this direction is to obtain a new measure of smoothness coupling the constraints given by the lattice architecture and the design algorithm.

2. SMOOTHNESS

2.1. The Hölder and Sobolev measure

Definition 1. ***[Hölder continuity]***
Let $\varphi : \mathbb{R} \to \mathbb{C}$ and let $0 < \alpha \leq 1$. Then the function φ is Hölder continuous of order α if there exist a constant c such that

$$|\varphi(x) - \varphi(y)| \leq c\,|x - y|^{\alpha} \quad \textit{for all } x, y \in \mathbb{R} \tag{5}$$

Based on the above definition, φ has to be a constant if $\alpha > 1$. This is not very useful for determining continuity of order $\alpha > 1$. However, using the above definition Hölder continuity of any order $r > 0$ is defined as follows:

Definition 2. ***[Hölder exponent]***
A function $\varphi : \mathbb{R} \to \mathbb{C}$ is continuous of order $r = P + \alpha$ ($0 < \alpha \leq 1$) if $\varphi \in C^P$ and its Pth derivative is Hölder continuous of order α. Then r is said to be the Hölder exponent of φ

Similarly the Sobolev exponent is a measure of the decay of the Fourier transform and is given by the following definition.

Definition 3. ***[Sobolev exponent]***
Let $\varphi : \mathbb{R} \to \mathbb{C}$, then φ is said to belong to the Sobolev space of order s ($\varphi \in H^s$) if

$$\int_{\mathbb{R}} (1 + |\omega|^2)^s \, |\hat{\varphi}(\omega)|^2 \, d\omega < \infty \tag{6}$$

Then s is said to be the Sobolev exponent of φ

Notice, that although Sobolev exponent does not give explicit order of differentiability it does yield an lower and upper bound on r, the Hölder exponent, and hence the differentiability of φ. This can be seen from the following inclusions:

$$H^{s+1/2} \subset C^r \subset H^s \tag{7}$$

2.2. Bounded variation

Although bounded variation is defined for continuous functions it will be useful to assume that all functions are vector valued in this section. Let D_n be the nth order (generalized) discrete difference operator defined by

$$D_n f(x_i) = \sum_{j=0}^{n} \binom{n}{j} f(x_{i-j}) \quad \text{for } i \subset \mathbb{N}^+. \tag{8}$$

Then the generalized bounded variation of degree p (GBVp) is given by the following definition.

Definition 4. ***[Generalized bounded variation of degree p]***
Let $\varphi(x)$ be a vector valued function such that $\varphi : [a, b] \to \mathbb{R}$ and consider any m points partitioning $[a, b]$ such that

$$a = x_0 < x_1 < \cdots < x_{m-1} = b$$

then for $n > 1$

$$\mathcal{BV}_n^p = \sum_{i=0}^{n-1} |D_n \varphi(x_i)|^p \tag{9}$$

is defined to be the generalized bounded variation of degree p of the function $\varphi(x)$ on $[a, b]$.

Given the definition of GBVp, the generalized total variation of degree p (GTVp) is defined as the least upper bound of the GBVp over all possible partitions of $[a, b]$. Using the above definition of GBVp a formal definition of GTVp follows.

Definition 5. ***[Generalized total variation of degree p]***
Let $\mathcal{X}$ be the set of m points partitioning $[a, b]$ such that

$$\mathcal{X} = \{x_i : a = x_0 < x_1 < \cdots < x_{m-1} = b\}$$

then

$$\mathcal{TV}_n^p = \sup_{\mathcal{X}} \mathcal{BV}_n^p = \sup_{\mathcal{X}} \sum_{i=0}^{n-1} |D_n \varphi(x_i)|^p \tag{10}$$

is defined to be the generalized total variation of degree p of $\varphi(x)$ on $[a, b]$. Furthermore, in case $\mathcal{TV}_n^p < \infty$ then we say the the function $\varphi(x)$ is of bounded variation.

Functions of bounded variation has a number of interesting properties [3, pp. 530-543], however, for the purpose of applying ideas from the theory of bounded variation to the design of smooth wavelets these properties are not relevant and in fact "total variation" is not the exact quantity that should be minimized. This can intuitively be seen by observing the functions given in Fig. 1. Notice that both Fig. 1a and Fig. 1b has the same total variation, namely $\mathcal{TV}_1^1 = c$, and yet the functions are very different with respect to smoothness. Also, noticing that (10) involves taking the $\sup$ over all possible partitions on $[a, b]$ this is computationally prohibitive.

Hence, a measure of smoothness should take all this in to account and most importantly it should objectively quantify the differences observed in Fig. 1. In fact, the measure should favor Fig. 1b since it is smoother than Fig. 1a. In the next section a measure related to both bounded variation as well as Hölder is introduced that will quantify the desired distinction. The new measure will be referred to as the discrete finite variation (DFV) indicating that it is based on the (local) variation of discrete samples of the wavelet generated by a finitely iterated filter bank.

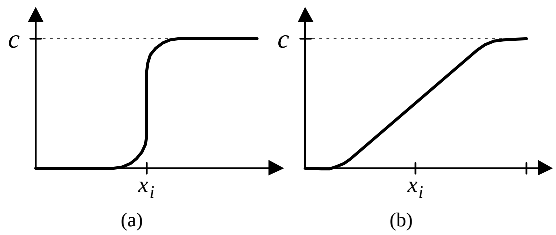

Figure 1: Bounded variation and locally smooth functions.

2.3. Discrete Finite Variation

Based on the above description the desired property of the new measure can be stated as follows: *locally the "discrete" wavelet should not change to fast*. This is exactly why Fig. 1b is prefered over Fig. 1a where at x_i the rate of change is approaching infinity. Also, notice the use of the word "discrete" in the above description. This comes from the fact that typically only a few stages (scales) of the wavelet transform is computed in any real application and hence the associated wavelet is not continuous (at that scale) but discrete.

Let J be the number of stages in the iterated filter bank used to generate the wavelet, and let $\psi_0^J(x_i)$ supported on $[0, N-1]$, be the length L sequence of samples of the scaling function such that $0 = x_0 < x_1 < \cdots < x_{L-1} = N-1$ with $x_i - x_{i-1} = \Delta x$ for all $i = 1 \ldots, L-1$. Then, with D_n defined by (8) define the vector, $\mathcal{V}_n$, of nth order differences by

$$\mathcal{V}_n = D_n \psi_0^J(x_i) \tag{11}$$

and a function is said to be of DFV if $\|\mathcal{V}_n\|_p < \infty$. Hence, the distinction between bounded variation and DFV is that DFV only takes into account the variation of adjacent points while bounded variation cares about all possible partitions of the support.

An algorithm for designing smoothly varying wavelets can then easily be constructed by minimizing the ℓ^p norm of $\mathcal{V}_n$ over h_0 subject to the constraints in (2) and (3). For shorthand notation we will refer to the solution of minimizing the ℓ^p norm of $\mathcal{V}_n$ by $\mathcal{DFV}_n^p$. Although, the above procedure has a lot of resembles with obtaining smooth derivatives, which is exactly what Hölder and Sobolev achieves, notice that nowhere is it required that $\psi_0(x)$ or any of its derivatives are continuous and hence higher order moments does not have to be zero. Furthermore, the definition of smoothness given by DFV does take into account that for practical applications the wavelet functions are not continuous.

3. EXAMPLES

This section gives results of the new design and compare the results with the K-regular solutions due to Daubechies. Table 1 compares DFV and Hölder exponent for the K-regular solution and the $\mathcal{DFV}_1^2$ solution. The solution of the optimal DFV was obtained by performing a constrained minimization using the MATLAB constrained optimization algorithm: `constr`. In Fig. 2 the corresponding scaling function for $N = 12$ is plotted. Notice, that although the Hölder regularity is significantly different (2.1553 versus 0.8964) the new design is "visually" smoother. In Fig. 3 the corresponding first and second order numerical derivatives are plotted. Notice that higher Hölder regularity does indeed correspond to smoother first and second order derivatives and yet one could not have guessed that by observing the functions in Fig. 2 only. In Table 2 scaling filter coefficients for the optimal $N = 12$ solution is given and Table 3 show the first 6 moments of both filters.

Table 1: Comparison of DFV and Hölder regularity for a length $N = 2K$ maximally vanishing moments (K-regular) wavelet basis and the wavelet basis obtained by optimizing DFV ($\min \mathcal{DFV}_2^1$).

N	DFV		Hölder	
	K-regular	$\min \mathcal{DFV}_1^2$	K-regular	$\min \mathcal{DFV}_1^2$
6	0.2024	0.1962	1.0816	0.8498
8	0.1804	0.1735	1.6008	0.9132
10	0.1731	0.1680	1.9336	0.8976
12	0.1697	0.1655	2.1553	0.9124
14	0.1678	0.1649	2.4261	0.8955

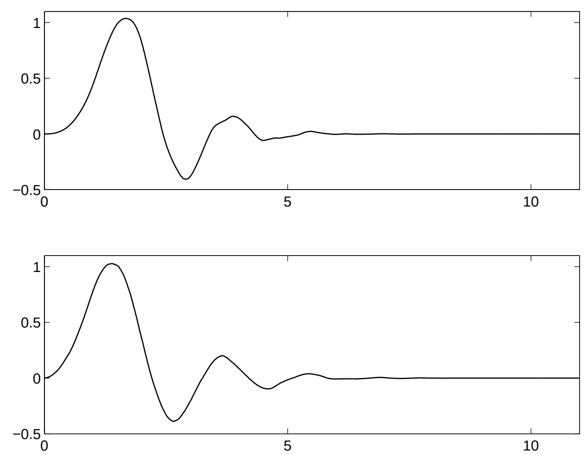

Figure 2: Scaling function $\psi_0(t)$ for length $N = 12$. Top figure shows the 6-regular (Daubechies) solution and the bottom figure shows the solution obtained by minimizing $\mathcal{DFV}_1^2$.

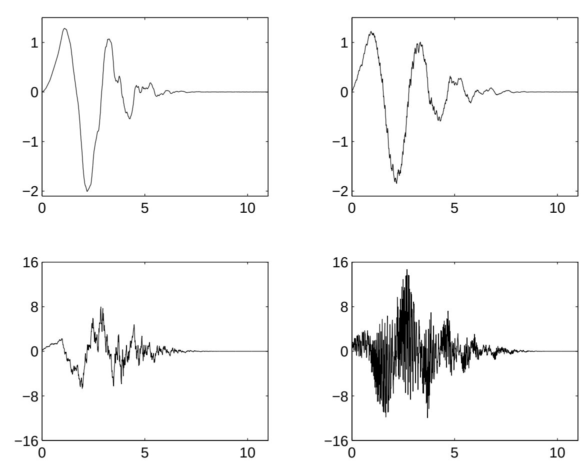

Figure 3: For $N = 12$ the top row shows the numerical first derivative for the 6-regular solution (left) and the optimal DFV solution (right). The bottom row shows the corresponding numerical second derivative.

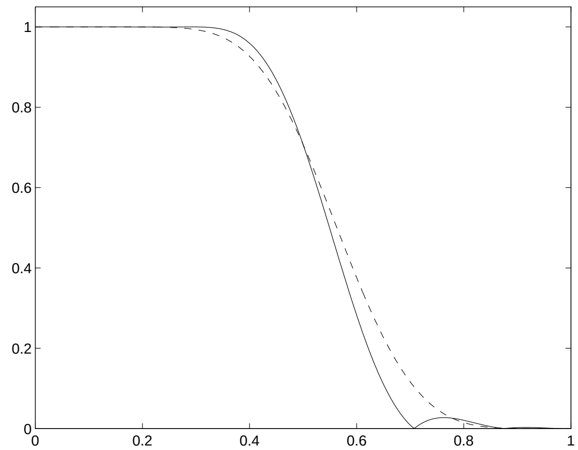

Figure 4: Frequency response of the length $N = 12$ scaling filters. Solid line is the optimal DFV solution and the dashed line corresponds to the 6-regular solution of Daubechies.

Table 2: Coefficients for the 6-regular Daubechies and optimal DFV of length $N = 12$ scaling filter, $h_0(i)$

i	6-regular	min $\mathcal{DFV}_1^2$
0	0.11154074335011	0.19245354387245
1	0.49462389039847	0.61548666606234
2	0.75113390802111	0.68371407583713
3	0.31525035170920	0.15741065475483
4	-0.22626469396546	-0.25034258010171
5	-0.12976686756727	-0.09471248100885
6	0.09750160558732	0.12424589905097
7	0.02752286553031	0.03213128171802
8	-0.03158203931749	-0.05910916266348
9	0.00055384220116	0.00183896311853
10	0.00477725751095	0.01614500519119
11	-0.00107730108531	-0.00504830345833

Table 3: Discrete wavelet moments m_i of the 6-regular Daubechies and optimal DFV of length $N = 12$ wavelets

i	6-regular	min $\mathcal{DFV}_1^2$
0	8.8818e-16	1.6653e-16
1	6.6613e-15	-1.5449e-05
2	5.3291e-14	2.5649e-01
3	3.4106e-13	4.9725e+00
4	2.7285e-12	9.0241e+01
5	2.1828e-11	1.5307e+03

4. SUMMARY

This paper introduces a new definition of smoothness which is more meaningful and less restrictive than optimization of Hölder and Sobolev exponents. The method dispenses with the traditional measures of smoothness which requires that a large number of moments be set to zero in favor of a measure that describes the "visual smoothness" of the finitely iterated filters. Results show that minimization of discrete finite variation (a measure of local roughness) results in wavelet basis which are visually smoother than the K-regular solutions due to Daubechies without having to impose a large number of zero moments of $\psi_1(x)$.

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