MULTIPLE WINDOW TIME-VARYING SPECTRUM ESTIMATION

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ABSTRACT

We propose a new method for the time-varying spectrum estimation of non-stationary random processes. Our method extends Thomson's powerful multiple window spectrum estimation scheme to the time-frequency and time-scale planes. Unlike previous extensions of Thomsom's method, in this paper we identify and utilize optimally concentrated window and wavelet functions and develop a statistical test for extracting chirping line components. Examples on synthetic and real-world data illustrate the superior performance of the technique.

1. INTRODUCTION

Many methods exist for estimating the power spectrum of stationary signals. However, these methods are insufficient for the non-stationary signals that occur in important applications such as radar, sonar, acoustics, biology, and geophysics. These applications demand time-frequency representations that indicate how the power spectrum changes over time. To date research in time-frequency analysis has focused on deterministic signals. Only recently has attention turned to non-stationary random processes [1–5].

Unlike the power spectrum for stationary random processes, there is no unique definition for the time-varying spectrum of a non-stationary random process. Because it satisfies a number of desirable properties, we choose the *Wigner-Ville spectrum* (WVS) [1] as our definition of the time-varying spectrum in this paper. The WVS can be written as the expected value of the Wigner distribution [6] of one realization of the process $x(t)$

$$\begin{aligned}\mathbf{W}_x(t,f) &\equiv E\{W_x(t,f)\} \\ &= E\left\{\int x^*(t-\tau/2)\,x(t+\tau/2)\,e^{-j2\pi f\tau}d\tau\right\}. \quad (1)\end{aligned}$$

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The problem of time-varying spectrum estimation can be stated as the estimation of $\mathbf{W}_x(t,f)$ given only one realization of the non-stationary process $x(t)$.

A number of different WVS estimates have been proposed. The simplest is the empirical Wigner distribution $W_x(t,f)$ itself. However, while it is unbiased, it has infinite variance. Smoothing reduces the variance of the empirical Wigner distribution. Two-dimensional convolution of the Wigner distribution with a signal-independent smoothing kernel yields a distribution in the time and frequency shift-covariant Cohen's class [1, 6]. Two-dimensional affine convolution yields a distribution in the time and scale covariant affine class [7]. Sayeed and Jones [2] have developed a method for optimal kernel design for WVS estimation when the statistics of the process are known.

Unfortunately, the large amount of smoothing required to obtain a low variance WVS estimate can damage the resolution of line components in the data. Line components are deterministic chirping signals of the form $e^{j2\pi\gamma(t)}$, whose ideal time-frequency representations have the form $\delta(f-\gamma'(t))$.

Realizing that random and deterministic spectral components must be dealt with separately, Thomson introduced a powerful multiple window (MW) spectrum estimator for stationary signals in [8]. Because of its excellent performance, several groups have applied this technique, ad hoc, to non-stationary signals in a piecewise fashion [3–5, 9]. In this paper we refine these methods into an improved time-varying MW spectrum estimate for non-stationary signals. Our method preserves the resolution of line components, has low variance, and offers fine control over the bias-variance trade-off. We begin with a review of Thomson's MW method for stationary signals.

2. THOMSON'S MULTIPLE WINDOW METHOD

2.1. Summary of the Method

Thomson's MW method can be summarized in three steps [8]:

1. Detect and extract all significant sinusoids (stationary deterministic line components) in the data $x(t)$[1] using a statistical significance test (see Section 2.2) to obtain the part $y(t)$ of the data having a continuous spectrum

$$y(t) = x(t) - \{sinusoids\}. \quad (2)$$

2. Average K "orthogonal" periodogram estimates of $y(t)$ using prolate spheroidal data windows $\{v_k(t)\}$ [10]

$$P_T(f) = \sum_{k=0}^{K-1} d_k(f) \left| \int y(t)\, v_k(t)\, e^{-j2\pi ft}\, dt \right|^2. \quad (3)$$

The orthogonal prolate spheroidal windows are perfectly suited to stationary spectrum estimation, because they are simultaneously compactly supported in time and optimally concentrated in frequency. This concentration property results in a low bias estimate of the spectrum. The weights $d_k(f)$ can be chosen adaptively to further reduce the bias [8].

3. Reshape the spectrum $P_T(f)$ to account for the sinusoids excised in Step 1.

2.2. Thomson's F-test For Sinusoids

Before we can extract the significant sinusoids from the data $x(t)$ as in (2), we must detect their presence and estimate their parameters.

We assume the signal model

$$x(t) = y(t) + \sum_i \mu(f_i)\, e^{j2\pi f_i t} \quad (4)$$

with $y(t)$ zero mean and Gaussian. Define the k-th eigenspectrum $\chi_k(f)$ as the Fourier transform of the windowed data

$$\chi_k(f) = \int x(t)\, v_k(t)\, e^{-j2\pi ft}\, dt. \quad (5)$$

[1] In [8], Thomson sets up the spectrum estimation problem in discrete-time.

The expected value of $\chi_k(f)$ at f_i is given by

$$E[\chi_k(f_i)] = \mu(f_i)\, V_k(0) \quad (6)$$

where $V_k(f)$ is the Fourier transform of $v_k(t)$. Thus, using a simple linear regression, the complex amplitude $\mu(f_i)$ of each possible sinusoid can be estimated as

$$\widehat{\mu}(f_i) = \frac{\sum_{k=0}^{K-1} V_k(0)\, \chi_k(f_i)}{\sum_{k=0}^{K-1} V_k^2(0)}. \quad (7)$$

The eigenspectra yield a simple statistical test for whether sinusoids are really present in the data. Assuming that a sinusoid is present at frequency f, we subtract it from the data to obtain an estimate of the "background" continuous spectrum around f. Comparing this power in the background spectrum with the power in the assumed sinusoid results in an F variance-ratio test [8] with 2 and $2K-2$ degrees of freedom for the significance of the estimated line component

$$F(f) = \frac{(K-1)\, \widehat{\mu}(f)^2 \sum_{k=0}^{K-1} V_k(0)^2}{\sum_{k=0}^{K-1} |\chi_k(f) - \widehat{\mu}(f)\, V_k(0)|^2}. \quad (8)$$

If $F(f)$ exceeds a significance threshold, we say that a sinusoid exists at frequency f.

The probability of a miss increases with the threshold. On the other hand, the false alarm probability increases with decreasing threshold. When $F(f)$ exceeds the threshold when no sinusoid is present at frequency f, we say a *spurious peak* occurs.

Averaging orthogonal periodogram estimates reduces the variance of the MW power spectrum estimate by approximately K times compared to the variance of a single periodogram (in which $K = 1$). Furthermore, concentrated windows, adaptive weights, and sinusoid extraction keep resolution very high. These properties make Thomson's MW method the tool of choice for estimating the power spectrum of stationary random processes.

3. MULTIPLE WINDOW TIME-FREQUENCY ANALYSIS

The excellent performance of Thomson's MW method has led several groups to apply the method to time-varying spectrum estimation by simply sliding the estimate (3) along the signal and computing a MW spectrogram estimate about each time point [3–5,9]. While reasonably effective on certain classes of piecewise stationary signals, this approach suffers from two primary drawbacks. First, prolate spheroidal window functions have no inherent optimality properties in the joint time-frequency domain. Second, Thomson's F-test sinusoid extraction procedure fails on chirping line components of rapidly changing instantaneous frequency. In this section, we will properly extend Thomson's MW method to the time-frequency and time-scale planes by identifying sets of optimal windows/wavelets, and by developing a linear-chirp extraction algorithm that better matches non-stationary line components.

3.1. Hermite Windows

The foundation of the stationary MW method rests on the fact that the prolate spheroidal functions are optimal windows for estimating the spectrum of a time-limited signal. This optimality does not carry over into time-frequency, however, since the prolate spheroidal functions treat the time-frequency plane as two separate spaces rather than as one geometric whole [11–13].

For time-frequency signal analysis, it is natural to average over multiple orthogonal windows that are optimally concentrated in an appropriate time-frequency domain. To date, optimal orthogonal functions of this kind have been found only for a few very special domains. For instance, the Hermite functions are optimally concentrated in a circular time-frequency region and thus treat all time-varying spectral features in the same fashion [11–13]. The k-th order Hermite function, defined as

$$h_k(x) = \pi^{-1/4}(2^k k!)^{-1/2} \left(x - \frac{d}{dx}\right)^k e^{-x^2/2} \qquad (9)$$

for $k = 0, 1, 2, \cdots$, is concentrated in the circular region

$$R = \{(t,f): \quad t^2 + f^2 \leq C\} \qquad (10)$$

with C a constant. The first four Hermite windows and their concentration region are shown in Figure 1. Hermite functions concentrated in elliptical regions are easily obtained by compressions and dilations of the above functions.

3.2. Multiple Window WVS Estimate

Thomson's MW spectrum average (3) estimates the energy content of the signal at frequency f by projecting onto the windowed sinusoids $v_k(t)\, e^{j2\pi ft}$. By analogy, we estimate the energy content of a non-stationary signal at time t and frequency f by projecting onto the sliding windowed sinusoids $h_k(\tau - t)\, e^{j2\pi f\tau}$. The result

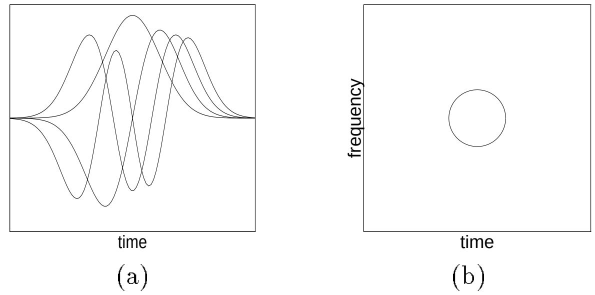

Figure 1: *(a) The first four Hermite functions in the time domain, and (b) their concentration region in time-frequency.*

can be written as the average of K Hermite-windowed spectrograms of the data

$$\widehat{\mathbf{W}}_x(t,f) = \sum_{k=0}^{K-1} d_k(t,f) \times \left| \int y(\tau)\, h_k(\tau - t)\, e^{-j2\pi f\tau} d\tau \right|^2. \qquad (11)$$

This WVS estimator has low variance thanks to the averaging but also minimized bias due to the optimal concentration of the Hermite windows. The bias-variance trade-off can easily be controlled and optimized by changing the number of windows K and by tuning the adaptive weighting functions $d_k(t,f)$ as in [8].

3.3. Cohen's Class Interpretation

The MW WVS estimate (11) belongs to Cohen's class of time-frequency distributions, each of which can be written as

$$W_x(t,f) \ ** \ \phi(t,f) \qquad (12)$$

with $\phi(t,f)$ a kernel function. The kernel generating the spectrogram is precisely the Wigner distribution of the window function. Furthermore, the Wigner distribution of the k-th order Hermite function is the k-th order Laguerre function [14]

$$L_k(t,f) = e^{-\frac{\pi}{2}\left|t^2+f^2\right|} \times \sum_{m=0}^{k} \frac{k!}{(k-m)!m!} \frac{\left(-\left(\pi\left|t^2+f^2\right|\right)\right)^m}{m!}. \qquad (13)$$

Therefore, we have a closed form expression for the kernel corresponding to the MW WVS estimate (11) as a weighted sum of K Laguerre functions; in this interpretation, the MW WVS estimate reads

$$\widehat{\mathbf{W}}_x(t,f) = W_x(t,f) \ ** \ \sum_{k=0}^{K-1} d_k(t,f)\, L_k(t,f). \qquad (14)$$

Since the weight functions $d_k(t, f)$ are tuned for each signal, we see that the MW WVS estimate employs a signal-dependent kernel.

3.4. Extraction Of Line Components

As in Thomson's method for stationary signals, the averaging inherent in (11) will degrade the resolution of line components. Following Thomson's programme, we will first detect and extract all line components in the data before performing (11), and then reshape the estimate. We assume the signal model

$$x(t) = y(t) + \sum_i \mu_i(t)\, e^{j\,2\pi\gamma_i(t)} \qquad (15)$$

with $y(t)$ zero mean and Gaussian.

A straightforward application of Thomson's sinusoid extraction algorithm to $x(t)$ as in [4, 5] relies on an assumption that the chirp functions $e^{j\,2\pi\gamma_i(t)}$ can be closely approximated locally as sinusoids. Unfortunately, this is not the case for most chirping components; in these cases, the approach fails. In order to detect and extract highly non-stationary chirps, we have developed a statistical significance test for linear chirps of the form $e^{j\,2\pi(ft+ct^2)}$. Linear chirps can closely approximate all but the most rapidly changing chirp functions.

The test for linear chirp components flows as in Section 2.2, except that a test like (8) must be performed at each time t, frequency f, and chirp rate c. This results in a three dimensional F-test statistic $F(t, f, c)$.

Due to the repeated application of the test, the number of spurious peaks in F increases far beyond that seen in Thomson's method for stationary signals. These peaks must be suppressed in order to create a readable time-frequency image.

To suppress spurious peaks that peek above the significance threshold, we employ the following nonlinear cleaning algorithm:

1. Threshold the data volume $F(t, f, c)$ and slice it along the chirp-rate dimension c.
2. For each fixed c_i, apply a nonlinear filter to $F(t, f, c_i)$ to remove peaks that have not coalesced into a region larger than the Heisenberg uncertainty principle mandates. (Intuition: spurious peaks are isolated in $F(t, f, c_i)$; true peaks lie along curves in $F(t, f, c_i)$.)
3. Combine the results from each c_i to obtain the final test statistic.

Although the linear chirp detection and extraction algorithm is computationally expensive, it is readily parallelizable.

4. MULTIPLE WINDOW TIME-SCALE ANALYSIS

For random processes containing high frequency components of short duration and low frequency components of long duration, time-frequency techniques are not appropriate. These types of processes are better matched by time-scale representations from the affine class [7]. The smoothing kernels in the affine class change with frequency to accommodate high frequency components of short duration and low frequency components of long duration. The regions of smoothing at different parts in the time-frequency plane for Cohen's class and the affine class are shown in Figure 2.

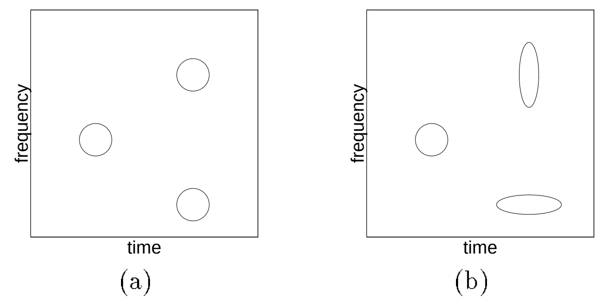

Figure 2: *Smoothing regions in the time-frequency plane in (a) Cohen's class, (b) Affine class.*

The Morse wavelets [15–17] play a rôle in time-scale analogous to that of the Hermite windows in time-frequency. They are defined in the frequency domain as the generalized Laguerre functions

$$\Psi_k(f) = f^{\beta/2}\, e^{-f/2}\, \frac{d^\beta}{df^\beta}\left[e^f \, \frac{d^{\beta+k}}{df^{\beta+k}} \left(f^{\beta+k}\, e^{-f}\right)\right], \quad (16)$$

with k the order of the Morse wavelet, and β the degree of flatness at $f = 0$. The Morse functions are mutually orthogonal and maximally concentrated in the tear-drop shaped time-frequency region [15–17]

$$R = \left\{ (t, f): \quad t^2 + \frac{C_1}{f^2} + 1 \le \frac{C_2}{|f|} \right\} \qquad (17)$$

with C_1 and C_2 constants. The first four Morse wavelets and their concentration region R are shown in Figure 3.

We form a time-scale MW WVS estimate of the data $x(t)$ as the weighted average of the squares of K wavelet transforms using the Morse wavelets

$$\widehat{\mathbf{W}}_x(t, a) = \sum_{k=0}^{K-1} d_k(t, a) \times \left| a^{-1/2} \int x(\tau)\, \psi_k\left(\frac{\tau - t}{a}\right) d\tau \right|^2 . \qquad (18)$$

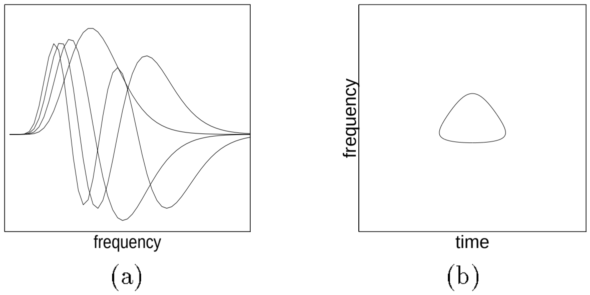

Figure 3: *(a) The first four Morse wavelets in the frequency domain, and (b) their concentration region in time-frequency.*

where ψ_k is the k-th order Morse wavelet expressed in the time domain. The resulting estimate belongs to the affine class of time-scale covariant distributions. Its kernel is the weighted sum of the Wigner distributions of the K Morse wavelets.

As in the time-frequency case, averaging degrades the resolution of chirping line components. Using the algorithm of Section 3.4, we can detect and extract the line components from the data before computing the estimate (18).

Lilly and Park have also considered multi-wavelet spectrum estimation [18]. In their work, they employed different wavelets and did not consider line component extraction.

5. EXAMPLES

In Figure 4, we illustrate the performance of the MW WVS estimate using a test signal composed of a chirp with sinusoidal instantaneous frequency in an additive bandpass Gaussian noise of linearly rising center frequency. It is not possible to identify the components of the test signal from the empirical Wigner distribution due to its large variance. The spectrogram smoothes the Wigner distribution. Unfortunately the amount of smoothing needed to reduce the variance smears the line components excessively. A sliding version of Thomson's method as proposed in [3–5] does not perform well for this non-stationary data, since a local sine approximation to the chirping line component is inadequate. The time-frequency MW estimate of Figure 4(d) on the other hand has both high resolution and low variance simultaneously.

In Figure 5, we demonstrate the ability of the linear chirp detection algorithm to detect four hyperbolic chirps simultaneously. The data is a digitized 2.5 microsecond echo-location pulse emitted by the Large Brown Bat, Eptesicus Fuscus. There are 400 samples and the sampling period is 7 microseconds. Comparing

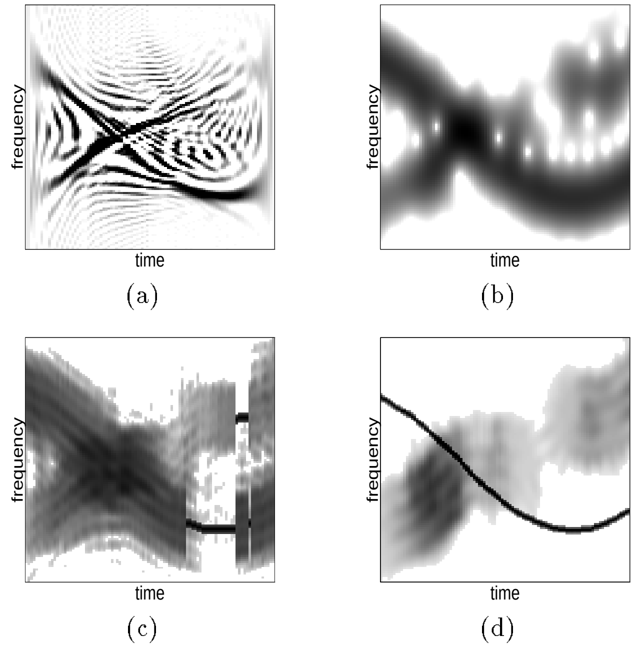

Figure 4: *Four time-varying spectrum estimates of a test signal. (a) Empirical Wigner distribution. (b) Spectrogram. (c) Sliding window Thomson's method [3–5]. (d) Multiple window method.*

the multiple window method with the other two plots, we see that the detection algorithm is able to pull out even the weakest high frequency line component successfully. The method even reveals the aliasing of the hyperbolic chirp due to under-sampling.

6. CONCLUSIONS

In this paper, we have motivated and developed multiple-window time-frequency and time-scale analysis for time-varying signals by fully extending Thomson's work [8] on multiple-window spectrum estimation for stationary signals.

Our contribution differs from the previous work done in multiple-window method spectrum estimation of time-varying signals in two majors ways:

1. We have identified the optimal windows to use for time-varying spectrum estimation. They are the Hermite functions [11–13] for time-frequency analysis, and the Morse wavelets [15–17] for time-scale analysis. These windows are optimal in the sense that they are the most concentrated in the time-frequency and time-scale planes resulting in low bias spectral estimates.

2. We have developed an algorithm to detect and extract non-stationary line components from the

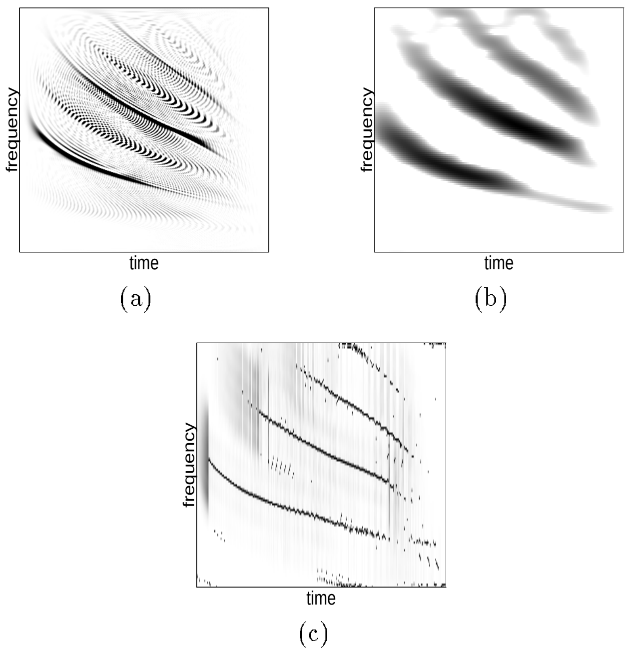

Figure 5: *Three spectrum estimates of the echolocation pulse emitted by the Large Brown Bat, Eptesicus Fuscus: (a) Empirical Wigner distribution. (b) Spectrogram. (c) Multiple window method.*

data by approximating them as piece-wise linear chirps. We then form the MW WVS estimate of the chirp-free data and reshape the spectrum to account for the excised line components. This preserves the resolution of the line components.

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