NEW DIMENSIONS IN WAVELET ANALYSIS

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ABSTRACT

In this paper we propose a new class of signal analysis tools that generalizes the popular wavelet and short-time Fourier transforms. The class allows skews and rotations of the analyzing wavelet in the time-frequency plane, in addition to the time and frequency translations and scalings employed by conventional transforms. In addition to providing a unifying framework for studying existing time-frequency representations, the general class provides a systematic method for designing new representations with properties useful for certain types of signals.

1. INTRODUCTION

The continuous wavelet transform (WT)

$$\phi(t,a) \;=\; |a|^{-\frac{1}{2}} \int s(\tau)\; g^*\left(\frac{\tau - t}{a}\right)\; d\tau \qquad (1)$$

and the short-time Fourier transform (STFT)

$$\psi(t,\omega) \;=\; \int s(\tau)\; g^*(\tau - t)\; e^{-j\omega\tau}\; d\tau \qquad (2)$$

are two-dimensional functions that indicate the joint time-frequency energy content of a one-dimensional signal [1]. They map the signal $s(\tau)$ to the time-frequency plane by projecting it onto an overdetermined basis of functions called *logons* that are transformed versions of the *analyzing wavelet* $g(\tau)$.

The logons can be interpreted as "tiling" the time-frequency plane in an overlapping fashion; the shapes and locations of the tiles are controlled by the transformation applied to the analyzing wavelet. We will refer to the representation of this transformation on the time-frequency plane as the *logon tiling function*. In the WT, the tiling function scales $g(\tau)$ by a factor a^{-1} (which scales its Fourier transform by a factor a) and then translates the result in time by t. Figure 1(a) illustrates the idealized time-frequency distribution of an analyzing wavelet and several logons obtained using this tiling function. Apparent from the figure is the proportional-bandwidth (constant-Q) nature of the WT analysis. In the STFT, $g(\tau)$ is time- and frequency-shifted. Figure 1(b) illustrates several logons obtained using this constant-bandwidth tiling function.

The WT and STFT work best for signals whose characteristics match the type of tiling function employed in the analysis. Thus, the WT performs best for signals having a constant-Q behavior (doppler-shifted transients and fractal signals are two examples), while the STFT performs best for signals having a constant-bandwidth behavior (frequency-shift keying signals, for example). While the signals encountered in many applications are often approximately constant-Q or approximately constant-bandwidth, there exists a large class of signals for which neither transform is well matched. Examples include dispersive or chirping signals that are well-matched only by slanted logons.

In this paper, we propose a new class of multidimensional signal representations that permits tiling functions more general than those employed in the WT and STFT. This class contains many new, interesting, and potentially useful representations, plus contains the WT and STFT as special cases.

2. A VERY GENERAL CLASS

Both the WT and STFT belong to a general class of linear signal transformations that can be represented (using the inner product notation $\langle d, e\rangle = \int d(\tau)\, e^*(\tau)\, d\tau$) as

$$\Gamma(x) \;=\; \langle s, f_x g\rangle, \qquad (3)$$

where s is the signal to be analyzed, g is the analyzing wavelet, and the operator f_x, which maps functions into functions, is called the *analysis map*. The analysis map is parameterized by the *analysis coordinate* x, which is assumed to come from some set X, called the *analysis space*. The interpretation of (3) is simple: each analysis coordinate x from X is associated with a (complex) number which is obtained by taking the inner product of the signal with a transformed version of the analyzing wavelet. The particular transformation used is determined by the value x.

By using Moyal's formula [2], the energetic version of the transform (3) can be represented as the two-dimensional inner product of two Wigner time-frequency distributions

$$\begin{aligned} |\Gamma(x)|^2 &= |\langle s, f_x g\rangle|^2 \\ &= 2\pi\, \langle W_s, F_x W_g\rangle \\ &= 2\pi \iint W_s(\tau,\Omega)\; F_x W_g(\tau,\Omega)\; d\tau d\Omega. \qquad (4) \end{aligned}$$

Here $W_s(\tau,\Omega)$ and $W_g(\tau,\Omega)$ are the Wigner distributions

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of the signal and analyzing wavelet, respectively, τ and Ω represent time and frequency, and F_x is the operator defined by $F_x W_g = \mathcal{W} f_x \mathcal{W}^{-1} W_g$. ($\mathcal{W} f$ represents the mapping from a signal f to its Wigner distribution W_f.) F_x is a mapping from the subset $\mathcal{V}$ of two-dimensional functions that are valid Wigner distributions into $\mathcal{V}$. Given F_x, the operator f_x is easily obtained as

$$f_x g = \mathcal{W}^{-1} F_x \mathcal{W} g. \qquad (5)$$

Because of the close relationship between F_x and f_x, we will also refer to F_x as the analysis map.

The Wigner distribution representation of $\Gamma(x)$ is very useful for conceptual purposes, because both the time-frequency plane coordinates, τ and Ω, appear explicitly. This representation is particularly illuminating when F_x is a transformation of the coordinates (τ, Ω) of W_g. Then F_x is precisely the coordinate transformation that, when applied to W_g, deforms and translates it as prescribed by the logon tiling function discussed in the Introduction.

The WT and STFT are easily shown to belong to the general class. For the WT, we have [3]

$$\begin{aligned} f_x g(\tau) &= |a|^{-\frac{1}{2}}\, g\left(\tfrac{\tau - t}{a}\right) \\ F_x W_g(\tau, \Omega) &= W_g\left(\tfrac{\tau - t}{a}, a\Omega\right) \\ x = (t, a), &\quad X = \mathbb{R}^2. \end{aligned} \qquad (6)$$

For the STFT, we have

$$\begin{aligned} f_x g(\tau) &= g(\tau - t)\, e^{j\omega\tau} \\ F_x W_g(\tau, \Omega) &= W_g(\tau - t, \Omega - \omega) \\ x = (t, \omega), &\quad X = \mathbb{R}^2. \end{aligned} \qquad (7)$$

In both cases F_x is a coordinate transformation that generates the appropriate logon tiling function.

3. THE TWO-DIMENSIONAL AFFINE CLASS

The class defined in (3) is very general – unfortunately too general to be practical. Therefore, rather than consider this class in its totality, we will now derive an important subclass that more directly extends the WT and STFT.

Note from (6) and (7) that the analysis map F_x for both the WT and STFT takes the form of a two-dimensional, area-preserving, affine coordinate transformation; that is,

$$F_x W_g(\tau, \Omega) = W_g(\tau', \Omega'), \qquad (8)$$

where

$$\begin{bmatrix} \tau' \\ \Omega' \end{bmatrix} = A\left(\begin{bmatrix} \tau \\ \Omega \end{bmatrix} - b\right) = A\begin{bmatrix} \tau \\ \Omega \end{bmatrix} + b'. \qquad (9)$$

Here A is a 2×2 real matrix with $|A| = 1$, and b and b' are 2×1 real vectors. We will study the subclass of (3)–(4) obtained from such (two-dimensional) affine analysis maps[1].

3.1. Analysis Map F_x

Affine analysis maps can generate a broad range of affine logon tiling functions. In addition to the translations and scalings employed by the WT and STFT, logons may be skewed and rotated in the time-frequency plane.

When the diagonal elements of A are nonzero, an analysis map of the form (8)–(9) can be decomposed into a composition of five distinct one-parameter transformations (recall that $|A| = 1$) on the time-frequency plane

$$A\left(\begin{bmatrix} \tau \\ \Omega \end{bmatrix} - b\right) = \begin{bmatrix} 1 & 0 \\ q & 1 \end{bmatrix}\begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix} \bullet \left(\begin{bmatrix} \tau \\ \Omega \end{bmatrix} - \begin{bmatrix} t \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \omega \end{bmatrix}\right). \qquad (10)$$

The component transformations correspond to (from left to right) skewing in the frequency direction, skewing in the time direction, axis scaling, translation in time, and translation in frequency. Axis scaling and time and frequency translation are familiar from the WT and STFT; the two skew transformations are new. Note that if $|A| \neq 1$, then $F_x W_g$ will not, in general, be a valid Wigner distribution [6]. In this case, f_x and the representation (3) are undefined.

The parameters associated with the component transformations define an analysis coordinate $x = (t, \omega, a, p, q)$ in a five-dimensional analysis space X. Figure 2 shows W_g for a Gaussian window analyzing wavelet g and $F_x W_g$ corresponding to the point $x = (10, 1.5, 0.5, 0, -0.2)$.

3.2. Components of the Analysis Maps F_x and f_x

Each matrix in (10) is associated with an operator on W_g. Application of (5) to the components of F_x yields the corresponding component operators of f_x. The component operators are summarized below [6].

Time shift:	$\mathcal{T}_t W_g(\tau, \Omega) = W_g(\tau - t, \Omega)$
	$\overline{\mathcal{T}}_t g(\tau) = g(\tau - t)$
Frequency shift:	$\mathcal{F}_\omega W_g(\tau, \Omega) = W_g(\tau, \Omega - \omega)$
	$\overline{\mathcal{F}}_\omega g(\tau) = e^{j\omega\tau} g(\tau)$
Axis Scaling:	$\mathcal{A}_a W_g(\tau, \Omega) = W_g(a^{-1}\tau, a\Omega)$
	$\overline{\mathcal{A}}_a g(\tau) = \lvert a\rvert^{-\frac{1}{2}} g(a^{-1}\tau)$
Skew in time:	$\mathcal{P}_p W_g(\tau, \Omega) = W_g(\tau + p\Omega, \Omega)$
	$\overline{\mathcal{P}}_p g(\tau) = (jp)^{-\frac{1}{2}}\, e^{-j\tau^2/2p} \star g(\tau)$
Skew in frequency:	$\mathcal{Q}_q W_g(\tau, \Omega) = W_g(\tau, \Omega + q\tau)$
	$\overline{\mathcal{Q}}_q g(\tau) = e^{-j\tau^2 q/2} g(\tau)$.

Just as F_x is the composition of $\mathcal{T}_t, \mathcal{F}_\omega, \mathcal{A}_a, \mathcal{P}_p$, and $\mathcal{Q}_q$, so f_x is the composition of $\overline{\mathcal{T}}_t, \overline{\mathcal{F}}_\omega, \overline{\mathcal{A}}_a, \overline{\mathcal{P}}_p$, and $\overline{\mathcal{Q}}_q$. Skews in the time and frequency directions are produced by modulating either the wavelet or its spectrum by a linear-FM ("chirp") function. (For the operator $\overline{\mathcal{P}}_p$, $\star$ denotes convolution. When $p = 0$, this operator has no effect, since $(jp)^{-\frac{1}{2}}\, e^{-j\tau^2/2p}$ becomes a Dirac delta function.)

Using these operators, a transform from the two-dimensional affine class can be written as

$$\Gamma(t, \omega, a, p, q) = \langle s,\ \overline{\mathcal{F}}_\omega \overline{\mathcal{T}}_t \overline{\mathcal{A}}_a \overline{\mathcal{P}}_p \overline{\mathcal{Q}}_q\, g\rangle \qquad (11)$$

$$|\Gamma(t, \omega, a, p, q)|^2 = 2\pi\, \langle W_s,\ \mathcal{F}_\omega \mathcal{T}_t \mathcal{A}_a \mathcal{P}_p \mathcal{Q}_q\, W_g\rangle. \qquad (12)$$

3.3. Analysis Space X

As pointed out earlier, the analysis space X is five-dimensional. We emphasize that even though the two-

[1] We recently discovered that this class of transformations has been suggested independently by Berthon [4] and Mann and Haykin [5].

dimensional Wigner time-frequency distribution was used as a conceptual tool in the development of the affine class, $\Gamma(x)$ is more than just a distribution of *time* and *frequency*. For example, if the values t_0, ω_0, a_0, and p_0 are fixed, then $\Gamma(t_0, \omega_0, a_0, p_0, q)$ measures the similarity of the signal to wavelets having varying amounts of linear-FM modulation.

Although, in general, $\Gamma(x)$ is a five-dimensional distribution, analyses can be carried out in lower-dimensional subsets of X. The WT and STFT have already been discussed; they lie on the two-dimensional planes parameterized by $x = (t, 0, a, 0, 0)$ and $x = (t, \omega, 1, 0, 0)$, respectively.

The subset of analysis does not have to be planar; it could even be curved. An example of a tilted surface of analysis is given in Fig. 3. The logons in the figure are associated with points lying on the two-dimensional surface $x = (t, \omega, 1, 0.08t, 0)$. Several new transforms that correspond to other lower-dimensional subsets of X are presented in the next section.

4. EXAMPLES

4.1. The Scale-Skew Transform

A new two-dimensional transform that shares many of the desirable properties of the WT and STFT is the *scale-skew wavelet transform* corresponding to the two-dimensional analysis plane $x = (0, 0, a, p, 0)$. The following result is easily proved.

Theorem 1 *The scale-skew transform is isometric and continuously-invertible, provided the analyzing wavelet $g(\tau)$ satisfies the following admissibility condition, stated here in terms of its Fourier transform $G(\Omega)$:*

$$\int \left| \frac{G(\Omega)}{\Omega} \right|^2 d\Omega \ < \ \infty. \tag{13}$$

An interesting property of this transform is that it can completely represent a signal without using time or frequency translations of the wavelet. Similar results apply for many other analysis surfaces.

4.2. The Dispersion Transform

Dispersion artifacts occur in signals acquired from media in which the wave propagation velocity varies with the frequency of the signal. The components of a dispersed signal are tilted in time-frequency; see [7] for a good example. A fundamental limitation of both the WT and STFT is illustrated in Figs. 4(a) and (b) for a simulated dispersed signal: since the WT and STFT logons remain at a fixed angle in the time-frequency plane, they smear out dispersed signal components and are poorly concentrated.

The *dispersion transform* (DT) evaluates $\Gamma(x)$ at points on the two-dimensional surface $x = (t, \omega, 1, p(t, \omega), 0)$. Logons corresponding to several points on such a surface were shown in Fig. 3 for $p(t, \omega) = 0.08t$. The DT of the example dispersed signal for points on this surface is shown in Fig. 4(c). Generally, in situations where the dispersive characteristics of the medium are known or can be estimated, the DT will yield a more concentrated time-frequency representation than either the WT or STFT. Dispersion transforms based on time-scale-skew ($x = (t, 0, a, p(t, a), 0)$) are also possible.

4.3. The Bowtie Transform

Mann and Haykin's *bowtie chirplet transform* [5] is a three-dimensional transform which has proven useful for studying the acceleration signature of Doppler shifted signals. It is based on time and frequency translations and the frequency skew. See [5] for more details.

4.4. Nonuniform Filterbanks

The two-dimensional surface in X parameterized by $x = (t, \omega, a(\omega), 0, 0)$ yields a time-frequency transform in which the analysis frequency vs. analysis bandwidth relationship is unconstrained (contrary to the WT and STFT). This allows for better matching of the transform to signals whose behavior is neither constant-Q nor constant-bandwidth.

5. CONCLUSIONS

In addition to providing a unifying framework for studying existing linear signal representations like the WT and STFT, the general class (3) and its five-parameter subclass (11) provide a systematic method for designing new representations with properties useful for certain classes of signals. The example transforms presented in Section 4 indicate that there is much to be gained by stepping out of the strict time-frequency paradigm.

Currently, we are using group representation theory to determine a set of admissibility conditions that an analysis subset must satisfy to generate an isometric, invertible transform. We are also studying the discretization of the general class to form orthonormal bases and frames, as well as the bilinear class of representations obtained from (4) by replacing W_g with an arbitrary two-dimensional function.

Finally, note that the generalization of current time-frequency and time-scale representations does not have to stop at the two-dimensional affine class. In addition to skews and rotations of logons, transforms based on higher-dimensional analysis spaces (using quadratic-FM transformation functions, $e^{jc\tau^3}$, for example) will manipulate the *curvature* of logons during analysis.

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