OPTIMAL PHASE KERNELS FOR TIME-FREQUENCY ANALYSIS

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Abstract— **We consider the design of kernels for time-frequency distributions through the phase, rather than amplitude, response. While phase kernels do not attenuate troublesome cross-components, they can translate them in the time-frequency plane. In contrast to previous work on phase kernels that concentrated on placing the cross-components on top of the auto-components, we set up a "don't care" region and place the cross-components there. The close connections between optimal allpass kernels and optimal lowpass kernels provide new insight into signal-dependent time-frequency analysis.**

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I Introduction

Time-frequency distributions (TFDs) are two-dimensional functions that indicate the joint time-frequency energy content of a signal. They have been utilized to study a wide range of signals in acoustics, biology, radar, sonar, geophysics, and speech processing. Most TFDs of current interest are members of Cohen's quadratic class [1] which can be generated by Fourier transformation of a weighted version of the ambiguity function (AF) of the signal to be analyzed. That is, if $P(t,f)$ is a bilinear TFD of the signal $s(t)$, then

$$P(t,f) \;=\; \iint A(\theta,\tau)\,\Phi(\theta,\tau)\,e^{-j2\pi(\theta t+\tau f)}\,d\theta\,d\tau \tag{1}$$

with $A(\theta,\tau)$ the AF of the signal

$$A(\theta,\tau) \;=\; \int s^*\!\left(t-\frac{\tau}{2}\right) s\!\left(t+\frac{\tau}{2}\right) e^{j2\pi\theta t}\,dt.$$

The weighting function $\Phi(\theta,\tau)$, called the *kernel* of the distribution, completely determines the properties of its corresponding TFD, via (1).

The AF is a quadratic function of the signal, and hence, it exhibits cross-components. If allowed to pass into the TFD, cross-components can reduce auto-component resolution, obscure the true signal features, and make interpretation of the distribution difficult. Therefore, the kernel is often selected to weight the AF such that the auto-components, which are centered at the origin of the (θ,τ) ambiguity plane, are passed, while the cross-components, which are located away from the origin, are suppressed [1–4]. Denoting the auto-component region in the (θ,τ) plane by $\mathcal{Q}$ (see Figure 1(a)), cross-component suppression requires the kernel to be lowpass, with $\Phi(\theta,\tau)\approx 1$ on $\mathcal{Q}$ and $\Phi(\theta,\tau)\approx 0$ on the complement $\mathcal{Q}^c$.

In Figure 1, we show three conventional TFDs of a signal composed of two parallel linear chirps. Since the Wigner distribution has kernel $\Phi_{\rm WD}=1$, it does not suppress cross-components. On the contrary, both the spectrogram of Figure 1(c) and cone-kernel distribution of Figure 1(d) employ lowpass kernels [1], and thus suppress cross-components at the expense of some smearing of the auto-components.

Unfortunately, TFDs derived from lowpass smoothing kernels cannot satisfy all of the desirable properties of a time-frequency energy density [1]. One property completely at odds with cross-

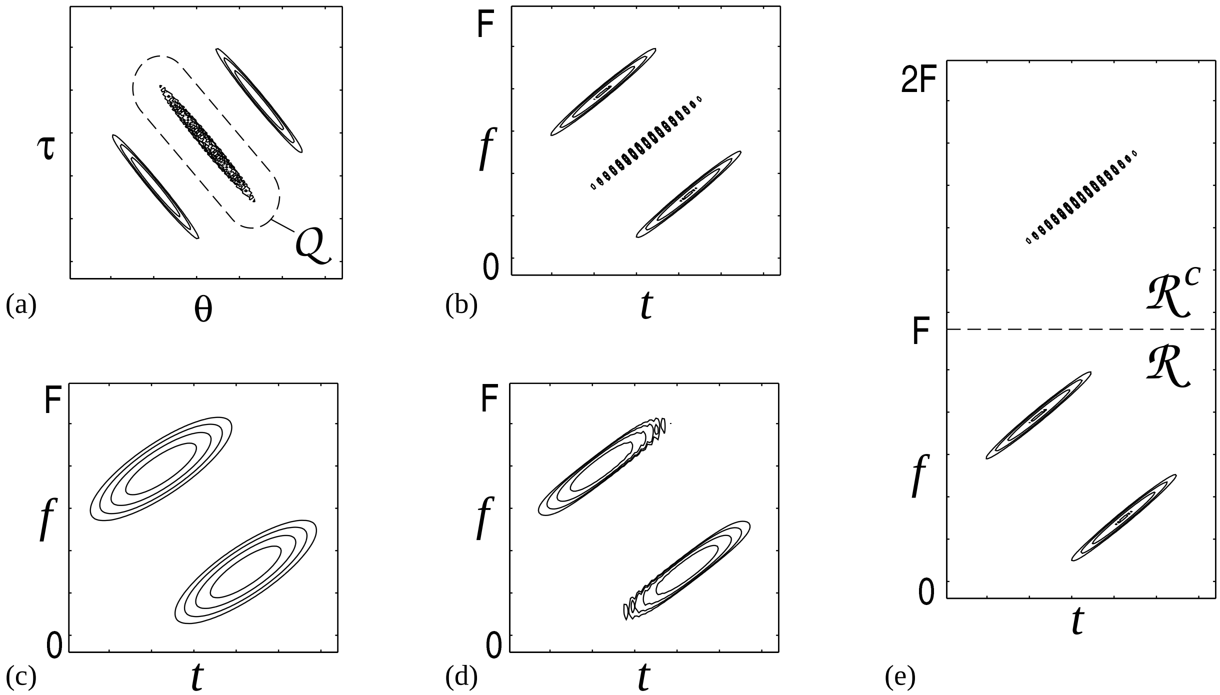


Figure 1: *Time-frequency distributions of a test signal consisting of two linear chirps. (a) Ambiguity function (AF), showing the auto-component region $\mathcal{Q}$ and cross-component region $\mathcal{Q}^c$. (b) Wigner distribution ($\Phi_{\rm WD} = 1$), which passes both auto- and cross-components into time-frequency. (c) Spectrogram, with lowpass kernel. (d) Cone-kernel distribution, with lowpass kernel. (e) Optimal phase kernel TFD, showing both the region of interest $\mathcal{R}$ and the don't care region $\mathcal{R}^c$.*

component suppression is *unitarity* (Moyal's formula), where we wish that

$$\iint P_1(t,f)\, P_2(t,f)\, dt\, df \;=\; \left| \int s_1(t)\, s_2^*(t)\, dt \right|^2 \tag{2}$$

with P_1 and P_2 the TFDs of the two signals s_1 and s_2. A quadratic TFD is unitary if and only if its kernel $|\Phi(\theta,\tau)| = 1$, meaning that both auto- and cross-components must be present in P. The desirability of unitarity (it is required for posing detection problems in time-frequency, for example [5–7]) motivates our study of *allpass kernels.*

II Phase Kernels

While an allpass kernel with $|\Phi(\theta,\tau)| = 1$ cannot suppress cross-components, it can *move* them. In particular, we can employ a phase factor in Φ to translate the cross-components away from the auto-components in the time-frequency plane, where they will not interfere with the interpretation of the distribution but where they will still be available for other purposes, such as calculations of the form (2).

Previous research has revealed the important role that phase can play in TFD kernel design [1,8–10]. However, to date researchers have concentrated on "strong support" properties and employed phase only to place the cross-components on top of the auto-components in the time-frequency plane. While this approach has lead to some new TFDs and interesting conclusions, TFDs derived in this way suffer from severe amplitude modulation artifacts.

Many different phase shifting schemes are possible; we will use only the simplest scheme here to explain the basic principles of our approach. For a signal bandlimited to the region of interest $\mathcal{R} = [0, F]$ Hz, we set up the *don't care region* $\mathcal{R}^c = [F, 2F]$ Hz.[1] The simple phase kernel [11]

$$\Phi_{\mathcal{Q}}(\theta,\tau) \;=\; \begin{cases} 1, & (\theta,\tau) \in \mathcal{Q} \\ e^{j\pi\tau F}, & (\theta,\tau) \in \mathcal{Q}^c \end{cases} \tag{3}$$

partitions the components, sending AF (auto) components from $\mathcal{Q}$ to the region of interest $\mathcal{R}$ in the time-frequency plane and sending AF (cross) components from $\mathcal{Q}^c$ to the don't care region $\mathcal{R}^c$. (See Figure 1(e).)

III Optimal Phase Kernels

While phase kernels appear an interesting adjunct to more conventional real-valued kernels, we are still left with the question of how to choose the zero-phase region $\mathcal{Q}$ for a given signal. Since the locations of the auto- and cross-components depend on the signal to be analyzed, we expect to obtain good performance for a broad class of signals only by using a *signal-dependent* kernel.

We propose a novel procedure for selecting a signal-dependent phase kernel. Given a signal, the method automatically designs a kernel that is optimal with respect to a set of performance criteria that attempts to capture, mathematically, the kernel properties that lead to good performance.

Our formulation singles out the phase kernel that optimally translates cross-components while passing auto-components. For the performance index, we choose the L^2 norm of the TFD in the region $\mathcal{R}$ of interest, which has been shown to be an effective measure of TFD concentration [3,4].

[1] Implementation of this simple scheme in discrete-time clearly would require that the signal be double over-sampled. Shifting in the time direction would remove this constraint, but would also make on-line implementation more complicated.

Since unconstrained maximization of this measure would result in the maximally concentrated yet cross-component-dominated Wigner distribution, we constrain the kernel to be an allpass filter of the form (3). Further, to ensure that the zero-phase region $\mathcal{Q}$ remains connected to the origin in the (θ, τ) plane (where the auto-components live), we constrain $\mathcal{Q}$ to be a *radial region* of finite area, in which the ray from each point in $\mathcal{Q}$ to the origin remains within $\mathcal{Q}$. In other words, in polar coordinates $r^2 = \theta^2 + \tau^2$ and $\psi = \arctan(\tau/\theta)$, we require that

$$(r_2, \psi) \in \mathcal{Q} \quad \Rightarrow \quad (r_1, \psi) \in \mathcal{Q} \qquad \forall\ r_1 \leq r_2, \ \ \forall\ \psi. \tag{4}$$

Such a constraint is in the same spirit as the "radially nonincreasing" constraint of [3] (see (11) in the Appendix).

We define the optimal phase kernel as the solution to the following optimization problem:

$$\max_{\Phi_{\mathcal{Q}}} \iint_{\mathcal{R}} |P(t,f)|^2 \, dt \, df \tag{5}$$

$$\text{subject to (3), (4), and area}(\mathcal{Q}) \leq \alpha. \tag{6}$$

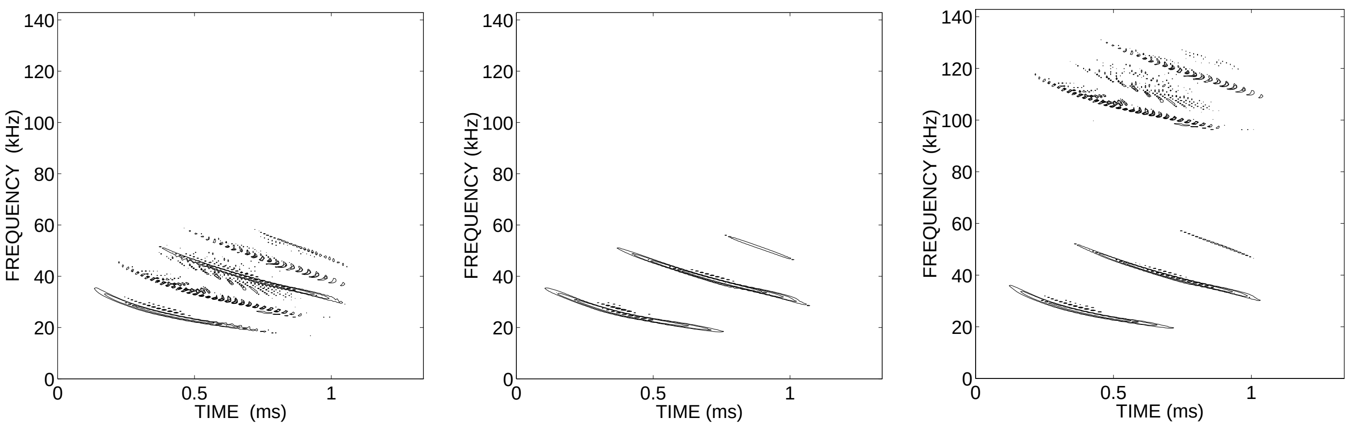
The parameter α limits the size of the zero-phase region of the optimal kernel. Note that the constraints do not dictate the exact shape of the zero-phase region of the kernel; the shape is determined by maximizing the performance measure.

This optimization problem has an elegant and efficient solution in terms of the "1/0" lowpass kernel optimization of [3, 12] (see the Appendix for details):

> *The zero-phase region $\mathcal{Q}_{\rm opt}$ of the optimal phase kernel coincides with the region of support of the optimal 1/0 kernel.*

This result is intuitively reasonable, since the support of the optimal 1/0 kernel corresponds closely to the support of the AF auto-components.

Time-frequency analysis with the optimal phase kernel distribution therefore follows a four-step procedure: (1) compute the AF of the signal; (2) solve for the optimal 1/0 kernel using the fast algorithm given in [12]; (3) set the phase kernel zero-phase region $\mathcal{Q}_{\rm opt}$ equal to the region of support of the optimal 1/0 kernel; and (4) Fourier transform the AF $\times$ phase kernel product.



(a) (b) (c)

Figure 2: *Time-frequency distributions of the echolocation chirp of a large brown bat,* Eptesicus fuscus. *Antialiasing filtering and a Hilbert transform has bandlimited the signal to the range* $\mathcal{R} = [0, 70]$ *kHz. (a) Wigner distribution, largely obscured by cross-components. (b) Optimal 1/0 kernel TFD, which compromises unitarity by suppressing cross-components. (c) Optimal phase kernel TFD, which preserves unitarity by translating cross-components. The frequency band* $\mathcal{R}^c = (70, 140]$ *kHz serves as the don't care region. Data provided by Curtis Condon, Ken White, and Al Feng of the University of Illinois.*

IV Examples

Figure 1(e) illustrates the optimal phase kernel TFD for the two-chirp test signal. The auto-components lie concentrated in the bandlimited region $[0, F]$ Hz of interest, while the cross-component is relegated to the don't care region $[F, 2F]$ Hz.

Figure 2 compares the Wigner distribution with the optimal 1/0 and optimal phase kernel TFDs on the echolocation chirp of a large brown bat, *Eptesicus fuscus*.

V Conclusions

In this paper, we have proposed a new approach to TFD kernel design that departs from conventional amplitude-only approaches. Optimal phase kernels represent the complement the optimal lowpass kernels developed in [3, 4] and share many of their useful features. The signal-dependent, optimal phase kernel TFD provides a good time-frequency representation by adjusting the shape of its kernel to optimally pass auto-components and move cross-components, regardless of their location and orientation in the time-frequency plane. Our approach is based on quantitative optimality criteria, is automatic, and is computationally efficient (a fast algorithm yields the optimal phase kernel in $O(N^2 \log N)$ computations, with N the number of signal samples to analyze).

In addition to unitarity, the simple phase kernels of (3) retain the outer time support and time marginal properties of the Wigner distribution. To secure the frequency marginal, a supplementary constraint can be appended to the optimization formulation, as in [3, 4]. Further constraints will result in other desirable properties; however, not all constraints are compatible with the allpass nature of the phase kernel.

Appendix: 1/0 Optimal Kernels

The optimal phase kernel is closely tied to the optimal 1/0 lowpass kernel of [3, 12]. Given a signal and its AF, the optimal 1/0 kernel is defined as the real, non-negative function $\Phi_{1/0}$ that solves the following optimization problem:

$$\max_{\Phi} \iint |A(\theta,\tau)\,\Phi(\theta,\tau)|^2 \, d\theta \, d\tau \tag{7}$$

subject to

$$\Phi(0,0) \;=\; 1 \tag{8}$$

$$\Phi(\theta,\tau) \text{ is radially nonincreasing} \tag{9}$$

$$\iint |\Phi(\theta,\tau)|^2 \, d\theta \, d\tau \;\leq\; \alpha, \qquad \alpha \geq 0. \tag{10}$$

The radially nonincreasing constraint (9) can be expressed explicitly in polar coordinates as

$$\Phi(r_1,\psi) \;\geq\; \Phi(r_2,\psi) \qquad \forall\; r_1 \leq r_2, \;\; \forall\; \psi. \tag{11}$$

Note the similarity to (4).

The constraints (8)–(10) and performance measure (7) are formulated so that the optimal 1/0 kernel passes auto-components and attenuates cross-components. The constraints force the optimal kernel to be a lowpass filter of fixed volume α; maximizing the performance measure encourages the passband of the kernel to lie over the auto-components. Both the performance measure and the constraints are insensitive to the orientation angle and aspect ratio (scaling) of the signal components in the (θ, τ) plane.

By controlling the volume under the optimal kernel, the parameter α controls the tradeoff between cross-component suppression and auto-component smearing. Reasonable bounds are $1 \leq \alpha \leq 5$. At the lower bound, the optimal kernel shares the same volume as a spectrogram kernel, whereas at the upper bound, the optimal kernel smooths only slightly. In fact, as $\alpha \rightarrow \infty$, the 1/0 optimal-kernel distribution converges to the Wigner distribution of the signal.

A distinctive feature of the optimal 1/0 kernel is that $\Phi_{1/0} = 0$ everywhere except on a radial region of area α, where $\Phi_{1/0} = 1$ [12]. We now use this fact to demonstrate that the zero-phase region $\mathcal{Q}_{\text{opt}}$ of the optimal phase kernel corresponds to the region of support of the optimal 1/0 kernel.

First, note that since phase kernels of the form (3) map all components in $\mathcal{Q}^c$ to the don't care region $\mathcal{R}^c$, we can use Parseval's theorem to rewrite the performance measure (5) in the AF domain

$$\iint_{\mathcal{R}} |P(t,f)|^2 \, dt \, df \;=\; \iint_{\mathcal{Q}} |A(\theta,\tau)|^2 \, d\theta \, d\tau.$$

With this transformation, we can translate the phase kernel optimization formulation (5), (6) into an equivalent optimization over 1/0 lowpass kernels:

$$\max_{\Psi_{\mathcal{Q}}} \iint |A(\theta,\tau) \, \Psi_{\mathcal{Q}}(\theta,\tau)|^2 \, d\theta \, d\tau \tag{12}$$

subject to

$$\Psi_{\mathcal{Q}} \;=\; \begin{cases} 1 \text{ on } \mathcal{Q} \\ 0 \text{ on } \mathcal{Q}^c \end{cases} \tag{13}$$

$$\mathcal{Q} \text{ a radial region of area}(\mathcal{Q}) \leq \alpha. \tag{14}$$

The region of support of the solution to this optimization coincides with the zero-phase region $\mathcal{Q}_{\text{opt}}$ of the optimal phase kernel.

To seal the proof, note that all kernels feasible under (13), (14) are also feasible under (8)–(10); therefore the optimal 1/0 kernel $\Phi_{1/0}$ solves *both* (7)–(10) and (12)–(14). Thus, the region of support of $\Phi_{1/0}$ coincides with the zero-phase region $\mathcal{Q}_{\text{opt}}$ of the optimal phase kernel.

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