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WAVELETS AND FILTER BANKS - NEW RESULTS AND APPLICATIONS

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Abstract

Wavelet transforms provide a new technique for time-scale analysis of non-stationary signals. Wavelet analysis uses orthonormal bases in which computations can be done efficiently with multirate systems known as filter banks. This thesis develops a comprehensive set of tools for (multidimensional) multirate signal analysis and uses them to investigate two multirate systems: filter banks and transmultiplexers. Several results in filter bank theory are obtained: a new parameterization of unitary filter banks, a theory of modulated filter banks, a theory of filter banks with symmetry restrictions, reduction of the multidimensional rational sampling rate filter bank problem to the uniform sampling rate filter bank problem, solution to the completion problem for filter banks (by reducing it to the (YJBK) parameterization problem in control theory) etc. Perfect reconstruction filter banks are shown to give structured decompostions of separable Hilbert spaces. Filter banks are used to construct several classes of wavelet bases: multiplicity M wavelet tight frames and frames, regular multiplicity M orthonormal bases, modulated wavelet tight frames etc. The thesis describes the design of optimal wavelets for signal representation and the wavelet sampling theorem. Application of wavelets in signal interpolation and in the approximation of linear-translation invariant operators is investigated.

To

Achan and Amma

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Chapter 1

Introduction

Fourier methods are inadequate for the analysis of non-stationary signals. A fundamental drawback of Fourier analysis is the inability to give *time-frequency* information of a given signal. A number of techniques, both linear and non-linear, signal independent and signal adaptive, have been proposed to solve this problem [20]: short-time Fourier transform, the Wigner-Ville distribution [94], the Choi-Williams distribution [13], the reduced interference distribution (RID) [47], the minimum cross-entropy (MCE) distribution [59], etc., to name a few. Wavelet analysis provides a novel technique for non-stationary signal analysis. A fundamental difference between wavelet methods and time-frequency methods is that the former introduces the concept of scale in place of frequency. Wavelet analysis, which is closely related to short-time Fourier analysis, gives a *time-scale* decomposition of signals, with an ability to study signals at various scales of resolution. Moreover, wavelet theory gives a rich family of *orthonormal* bases that can be tuned for specific applications.

Wavelet theory is related to multiscale analysis and pyramidal transforms used in computer vision through the concept of scale [8, 61]. From the intuitive idea that fine scale information requires higher rate of signal samples than coarse scale information, it is only natural to expect that time-scale analysis will be associated with multirate signal processing. This link between wavelet theory and multirate signal processing is through filter banks. Computations in wavelet analysis are usually associated with filter banks. This thesis argues that perfect reconstruction filter bank theory plays more than just a computational role in wavelet theory; filter bank theory is a starting point for some of the finer aspects of wavelet theory.

1.1 Outline of Thesis

The thesis solves a number of problems on a wide range of topics - the underlying common theme being multirate signal processing, filter banks and wavelets. Chapter 2 develops a general framework for multirate signals and systems. The basic framework for multirate signal analysis in one dimension is well known [70]. Recently, there has been an emphasis on extending one dimensional multirate results to multiple dimensions [54, 10, 50, 29, 34], motivated by image coding, video coding, etc. In multiple dimensions sampling rate conversion is accomplished using integer matrices. A fundamental problem with extending one dimensional results to multiple dimensions is that tools for handling integer matrices are not well-known in the digital signal processing community. The theory of integer matrices as relevant to multirate signal analysis is developed in Chapter 2. Several new results in the theory of integer matrices are derived; most notably the Representatives' Mapping Theorem and its interesting corollary that commuting matrices are left coprime iff they are right coprime. All results are consequences of one fundamental identity - the Aryabhatta/Bezout identity over integer matrices. A comprehensive set of tools for the analysis of multirate signals and systems is developed. The development makes transparent the important differences between one and multiple dimensions vis á vis the multirate signal analysis problem. Some of the new results in this Chapter 2 include

1. The Representatives' Mapping Theorem.
2. The Swapping and Commuting Theorem for upsamplers and downsamplers.
3. The Generalized Polyphase Representation.
4. Multirate identities for cascades of upsamplers, downsamplers, filters and delays.

While Chapter 2 is set in multiple dimensions, the rest of this thesis is mainly concerned with one dimensional multirate systems.

Chapter 3 discusses two important multirate problems, namely the filter bank problem and the transmultiplexer problem. Necessary and sufficient conditions for perfect reconstruction (PR) in filter banks and transmultiplexers are derived in a very general setting probably for the first time. Highlights of contributions in Chapter 3 include (all results are in 1-d unless otherwise specified)

1. Characterizations of PR for classes of filter banks and transmultiplexers.
2. A new theory of modulated filter banks that includes perfect reconstruction conditions for filter banks with FIR and IIR filters.
3. A classification of modulated filter banks.
4. Parameterization of unitary modulated FIR filter banks
5. Parameterization of unitary filter banks with various types of symmetry restrictions among the filters. This is particularly important in image processing applications where filters are sometimes required to be linear phase. Some of the results have been obtained earlier by other researchers [76].
6. Completion theory for filter banks and transmultiplexers, including new results on the completion of causal FIR and IIR filter banks.
7. Relationship between the famous Youla (YJBK) parameterization of compensators in control theory and the completion theory of PR filter banks.
8. Reduction of the multi-dimensional rational sampling rate filter bank problem to a uniform sampling rate filter bank problem.

The algebraic structure of perfect reconstruction filter banks gives a natural change of basis for separable Hilbert spaces. This establishes a connection between filter banks and certain specialized bases for $L^2(\mathbb{R})$. These connections are explored in detail in Chapter 4 leading to wavelet theory. We introduce the concept of *generalized frame pairs* in separable Hilbert spaces and show that PR filter banks provide a natural change of basis for generalized frame pairs. From this viewpoint we develop a theory of multiplicity M (or M-band), compactly supported wavelet frames and tight frames (WTFs), which gives added flexibility to the multiplicity 2 wavelet theory of Daubechies, Meyer, Mallat and others [21]. Similar to the multiplicity 2 case, K-regular, multiplicity M wavelet tight frames are explicitly constructed and parameterized. Necessary and sufficient conditions for a wavelet tight frame to be an orthonormal basis are given. Examples illustrating the relationship between the regularity of the scaling vector and smoothness of the wavelet basis are also given. A complete parameterization of compactly supported multiplicity M modulated wavelet tight frames is developed. Wavelet bases with symmetry restrictions are also constructed. The main contributions in Chapter 4 are:

1. Generalized frame pairs and Riesz basis pairs.
2. Parameterization of compactly supported multiplicity M WTFs.
3. Characterization of orthonormality for a multiplicity M WTF.
4. Design of K-regular WTFs.
5. State-space approach to WTFs.
6. Parameterization of compactly supported of modulated WTFs.
7. Construction of WTFs with "symmetric" wavelets.
8. Construction of Wavelet Frames.
9. Oversampling invariance results for Wavelet Frames.

In Chapter 5 computational aspects of wavelet analysis and filter bank analysis is studied. Interesting relationships between the moments of the scaling function of orthonormal wavelet bases are obtained. A theory of optimal and robust representation of band-limited signals in wavelet bases is developed and applied to various types of wavelet bases. One of the important consequences of this analysis is the *wavelet sampling theorem* which essentially states that the scaling expansion coefficients of a bandlimited function contain the same information as the Nyquist rate samples. Using this theory smooth modulated WTFs are constructed. The main contributions Chapter 5 are:

1. Efficient implementation of modulated filter banks.
2. Efficient computation of DWT in wavelet frames, general WTFs, and modulated WTFs.
3. Relationships between the moments of the scaling function of K-regular WTFs and its consequences in numerical analysis using wavelets.
4. Theory and algorithms for the optimal and robust representation of signals in compactly supported wavelet bases.
5. Construction of smooth WTFs and smooth modulated WTFs.
6. The Wavelet Sampling Theorem.
7. Wavelet-Galerkin approximation of linear-time-invariant operators (i.e., analog filters).
8. Wavelet-based lowpass and bandpass interpolation.

Traditionally in signal processing vectors and matrices are represented in boldface. In this work sequences in $\mathbf{Z}^d$ will be denoted by $x(n)$, $y(n)$,..., where $n = (n_1, n_2, \ldots, n_d)$. We prefer *not* to use boldface notation so as to make transparent the relationship between the one dimensional and multidimensional results. For vectors x and y, x^y is the scalar defined by $x^y = \prod_{i=1}^{d} x_i^{y_i}$. For a vector x and a square matrix $M = \begin{bmatrix} m_1 & m_2 & \ldots & m_d \end{bmatrix}$, x^M is the vector (of the same type as x) defined by

$$x^M = (x^{m_1}, x_2^{m}, \ldots, x^{m_d}).$$

For a scalar x and a vector y, x^y is a vector (of the same type as y) given by

$$x^y = (x^{y_1}, x^{y_2}, \ldots, x^{y_d}).$$

The $\mathcal{Z}$-transform and Fourier transform of a sequence $x(n)$ are functions of d complex variables, $z = (z_1, z_2, \ldots, z_d)^T$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_d)^T$ defined by

$$X(z) = \sum_{n \in \mathbf{Z}^d} x(n) z^{-n} \quad \text{and} \quad X(\omega) = \sum_{n \in \mathbf{Z}^d} x(n) e^{-\imath \omega^T n}.$$

where $\imath = \sqrt{-1}$. For a matrix M, $|M|$ will denote the absolute value of the determinant of M.

Chapter 2

Fundamental Tools in Multirate Signal Analysis

There are three basic building blocks in linear, multirate signals and systems theory, namely, linear shift-invariant filters, upsamplers and downsamplers. Upsamplers and downsamplers provide the sampling rate conversion making the system *multirate*, while filters, besides accomplishing traditional filtering functions, are also *necessary* as *anti-aliasing* and *image removal* filters. Multirate signal analysis is mainly concerned with what happens when these operations occur in different orders and how to analyze them. For one dimensional signals, the necessary tools for multirate signal analysis are well known [70]. Recently many researchers have tried to extend these results to multi-dimensional signals, with varying degrees of success [34, 52, 10, 12, 29, 50]. This thesis obtains a comprehensive set of algebraic tools for the analysis of multidimensional multirate systems using the Aryabhatta/Bezout identity over integer matrices as a fundamental tool.

2.1 Lattices, Upsampling and Downsampling

In one dimension if $x(n)$ is the input of a 2-fold downsampler, the output $y(n)$ is $x(2n)$. Similarly, the output $y(n)$ of a 2-fold upsampler is obtained by interlacing the input sequence $x(n)$ with zeros. The "right" way to think about this is in terms of lattices. Both $x(n)$ and $y(n)$ are defined on a lattice of points, namely $\mathbf{Z}$. The output of a 2-fold downsampler is the input on a sublattice of $\mathbf{Z}$, namely $2\mathbf{Z}$. The upsampling operator chooses a bigger lattice and embeds the input sequence on a sublattice of this bigger lattice. This viewpoint readily generalizes to multi-dimensional downsampling and upsampling.

Let $\mathcal{L}$ denote integers in $\mathbb{R}^d$ (i.e., $\mathcal{L} = \mathbf{Z}^d$). For a non-singular matrix M over $\mathbb{R}$, the set $\mathcal{L}(M) = \{Mn \mid n \in \mathcal{L}\} = M\mathcal{L}$ is the *lattice generated* by M. For example,

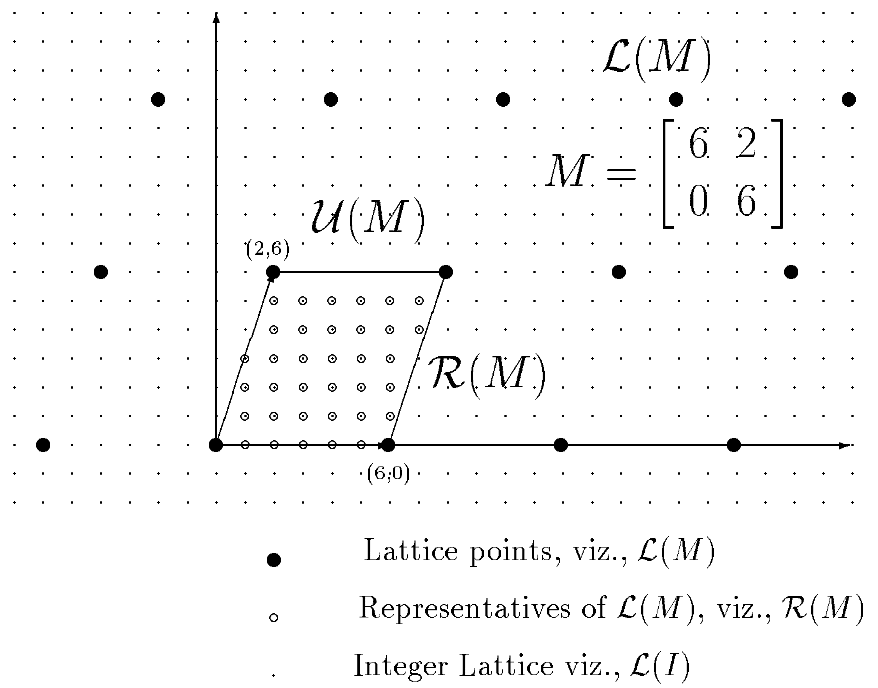

Figure 2.1: Lattice Generated by M

Fig. 2.1 shows the lattice generated by the matrix $M = \begin{bmatrix} 6 & 2 \\ 0 & 6 \end{bmatrix}$. Clearly, $\mathcal{L}$ is the lattice $\mathcal{L}(I)$, generated by the identity matrix I. A lattice $\mathcal{L}(M_1)$ is said to be a *sublattice* of the lattice $\mathcal{L}(M_2)$, if $\mathcal{L}(M_1) \subseteq \mathcal{L}(M_2)$. The geometric condition of one lattice being the sublattice of another has a neat algebraic characterization [9].

Fact 1 $\mathcal{L}(M_1) \subseteq \mathcal{L}(M_2)$ iff $M_1 = M_2K$ for some integer matrix K.

$\mathcal{L}(M) \subseteq \mathcal{L} = \mathcal{L}(I)$ iff M is integral. In Fig. 2.1 $\mathcal{L}(M)$ is a sublattice of $\mathcal{L}(I)$ and M is an integer matrix. The number of lattice points per unit volume of the lattice is $1/36 = 1/|M|$. More generally, for any lattice, $\mathcal{L}(M)$, $1/|M|$ is the average number of lattice points per unit volume [9, 84].

The *generator* for a given lattice is not *unique*. For example in Fig. 2.1

$$M = \begin{bmatrix} 8 & 2 \\ 6 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

is also a generator of the lattice. The generator is unique only up to *right* multiplication by a *unimodular* integer matrix.

Definition 1 An integer matrix M is *unimodular* if $\det M = \pm 1$.

Unimodular matrices are precisely those integer matrices that have integer inverse matrices. This follows from the fact that $M^{-1} = \frac{1}{\det M}\mathrm{adj}M$. If M is unimodular the number of lattice points per unit volume is $1/|M| = 1$. Since $\mathcal{L}(M)$ is a sublattice of $\mathcal{L}(I)$ which also has 1 lattice point per unit volume one would expect that $\mathcal{L}(M) = \mathcal{L}(I)$. More generally, for unimodular U, $\mathcal{L}(M) = \mathcal{L}(MU)$. To see this, define $MU = N$, and let $V = U^{-1}$. From Fact 1 it follows that $\mathcal{L}(N) \subseteq \mathcal{L}(M)$. Also since V is unimodular and $M = NV$, $\mathcal{L}(M) \subseteq \mathcal{L}(N)$ and hence the result.

Let $\mathcal{U}$ denote the unit cube in $\mathbb{R}^d$. Then $\mathcal{U} = \left\{x \in \mathbb{R}^d \mid x_i \in [0,1)\right\} = [0,1)^d$. The *unit cell* $\mathcal{U}(M)$ of a lattice $\mathcal{L}(M)$ is defined to be the image of $\mathcal{U}$ under M: $\mathcal{U}(M) = \left\{Mx \in \mathbb{R}^d \mid x \in \mathcal{U}\right\} = M\mathcal{U} = M\mathcal{U}(I)$. Any point in $\mathbb{R}^d$ can be represented uniquely as a linear combination of points in $\mathcal{U}(M)$ and $\mathcal{L}(M)$.

Lemma 1 For every point $x \in \mathbb{R}^d$ there exists a *unique* decomposition of the form $x = x_l + x_u$ with $x_l \in \mathcal{L}(M)$ and $x_u \in \mathcal{U}(M)$.

Proof: By (geometrically) translating the unit cell to any integer in $\mathbb{R}^d$, it is clear that any point $y = M^{-1}x$ in $\mathbb{R}^d$ can be represented uniquely as $l + u$, $l \in \mathcal{L}$ and $u \in \mathcal{U}$. Take $x_l = Ml$ and $x_u = Mu$ to get the result. □

The decomposition will be called the $\mathcal{LU}$ decomposition of x with respect to $\mathcal{L}(M)$. In particular if $x = n$ is an integer and M is integer non-singular, then both x_l and x_u are integers. In this case the integer x_u is denoted by n mod M or $(n)_M$. No two points in the unit cell can differ by a lattice point (see Fig. 2.1).

Lemma 2 If $\alpha, \beta \in \mathcal{U}(M)$, then $\alpha - \beta \in \mathcal{L}(M)$ iff $\alpha = \beta$.

Proof: Firstly, note that in the scalar case if $\alpha, \beta \in [0, 1)$, then $\alpha - \beta \in (-1, 1)$ and therefore the only integer value it can take is 0. Similarly, if $\alpha, \beta \in [0, 1)^d$, then $\alpha - \beta \in (-1, 1)^d$, and hence the only integer value it can take is 0. This fact in conjunction with Lemma 1 gives the result. □

Any integer in the unit cell of a lattice is called a *representative* of the lattice. The set of all *representatives* of a lattice is denoted by $\mathcal{R}(M)$. We have the following relationship between $\mathcal{R}(M)$ and $\mathcal{U}(M)$: $\mathcal{R}(M) = \mathcal{L}(I) \bigcap \mathcal{U}(M)$.

There are precisely $\lceil |M| \rceil$ elements in $\mathcal{R}(M)$. Therefore for an integer matrix M there are $|M|$ representatives. While discussing filter bank theory it will be necessary to consider a more general notion than $\mathcal{R}(M)$ - *generalized* sets of representatives.

Definition 2 A set $\mathcal{S}(M) \subset \mathbb{R}^d$ (of $\lceil |M| \rceil$ points) is *a generalized set of representatives* of $\mathcal{L}(M)$ if $\mathcal{S}(M)(\bmod M) = \mathcal{R}(M)$.

For a given lattice $\mathcal{L}(M)$ there are infinitely many generalized sets of representatives. An important property of any generalized set of representatives $\mathcal{S}(M)$ is that for any two points in $\mathcal{S}(M)$, if the $\mathcal{U}$ parts of their $\mathcal{LU}$ decompositions are equal, the points are the same.

Lemma 3 If $k, l \in \mathcal{S}(M)$, then $k - l \in \mathcal{L}(M)$ iff $k = l$.

Proof: If $\mathcal{S}(M) = \mathcal{R}(M) \subseteq \mathcal{U}(M)$, then the result follows directly from Lemma 2. Else from the $\mathcal{LU}$ decomposition $k - l = (k_l - l_l) + (k_u - l_u)$ and hence $k - l \in \mathcal{L}(M)$, iff $k_u - l_u \in \mathcal{L}(M)$. But then from Lemma 2 $k_u = l_u$. Since $\mathcal{S}(M)$ is a generalized set of representatives, $k_l = l_l$ and hence $k = l$. □

Example 1 Consider the lattice of integers $\mathbf{Z}$ in $\mathbb{R}$. Any sublattice is given by $M\mathbf{Z}$ for integer M. Assume for simplicity that M is positive. Then unit cell of the lattice is the interval $[0, M)$. Any point $x \in \mathbb{R}$ can be represented uniquely as, $x_l + x_u$, where

$x_l = \lfloor \frac{x}{M} \rfloor$, $x_u = [x]$ and $[x]$ denotes the fractional part of x. No two points in $[0, M)$ differ by a lattice point (multiple of M). The representatives of this lattice are the integers $\{0, 1, 2, \ldots, M-1\}$, precisely M of them. The set $\{2, 3, \ldots, M+1\}$ forms one set of generalized representatives of $\mathcal{L}(M)$. It is obvious that two elements of this set cannot differ by a lattice point.

We now define the upsampling and downsampling operators corresponding to a non-singular integer matrix M (see Fig. 2.2).

Definition 3 Given a non-singular integer matrix M, the upsampling operator, $[\uparrow M]$, is defined by

$$y(n) = [\uparrow M]\, x(n) = \begin{cases} x(M^{-1}n) & \text{for } n \in \mathcal{L}(M) \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Definition 4 Given a non-singular integer matrix M, the downsampling operator, $[\downarrow M]$, is defined by

$$y(n) = [\downarrow M]\, x(n) = x(Mn). \tag{2.2}$$

Figure 2.2: Upsampling and Downsampling by M

In general, upsampling is a *reversible* process, while downsampling is an *irreversible* process. The input/output relationships of upsampler and downsampler operators in the frequency domain and $\mathcal{Z}$-transform domain are often very useful. The Fourier transform $X(\omega)$ of a signal $x(n)$ is periodic on the lattice $2\pi\mathcal{L}(I)$. Hence $X(\omega)$ is described completely by its values on $2\pi\mathcal{U}(I)$. Sometimes the periodicity lattice of $X(\omega)$ contains (i.e., is a super-set of) $2\pi\mathcal{L}$. This is precisely what happens after an upsampling operation:

$$Y(z) = [\uparrow M]\, X(z) = \sum_{n \in \mathbf{Z}^d} x(n) z^{-Mn} = X(z^M). \tag{2.3}$$

$$Y(\omega) = [\uparrow M]\, X(\omega) = \sum_{n \in \mathbf{Z}^d} x(n) e^{-\imath \omega^T M n} = \sum_{n \in \mathbf{Z}^d} x(n) e^{-\imath (M^T \omega)^T n} = X(M^T \omega). \tag{2.4}$$

$Y(\omega)$ being the Fourier transform of $y(n)$ is periodic on the lattice $2\pi\mathcal{L}(I)$. However, $Y(\omega)$ is also periodic on the lattice $2\pi\mathcal{L}(M^{-T})$:

$$Y(\omega + 2\pi M^{-T} k) = X(M^T \omega + 2\pi k) = X(M^T \omega) = Y(\omega).$$

The fundamental period *after upsampling* is $2\pi\mathcal{U}(M^{-T})$, a subset of $2\pi\mathcal{U}(I)$. There are $\left|M^T\right| = |M|$ copies of $Y(\omega)$ restricted to its fundamental period in $2\pi\mathcal{U}(I)$. The response in each fundamental period is the response of $X(\omega)$ (appropriately scaled) and therefore $Y(\omega)$ has $|M|$ *images* of $X(\omega)$ in $2\pi\mathcal{U}(I)$. Hence upsampling (when $|M| > 1$) leads to *imaging* of the spectrum. One can avoid imaging by using an *image removal* filter whose passband, for example, is $2\pi\mathcal{U}(M^{-T})$.

Downsampling leads to *aliasing distortion*. Consider the sum $\sum_{k \in \mathcal{R}(M^T)} e^{\imath 2\pi k^T M^{-1} n}$. If $n \in \mathcal{L}(M)$, $M^{-1}n$ is an integer vector and hence $k^T M^{-1} n$ is an integer, and the sum becomes $\sum_{k \in \mathcal{R}(M^T)} 1 = |M|$. If $n \notin \mathcal{L}(M)$, the sum is zero. Hence the output of the downsampler is given by

$$\begin{aligned} Y(z) = [\downarrow M]\, X(z) &= \sum_{n \in \mathbf{Z}^d} x(Mn) z^{-n} \\ &= \sum_{n \in \mathbf{Z}^d} \left\{ \frac{1}{|M|} \sum_{k \in \mathcal{R}(M^T)} x(n) e^{-\imath 2\pi k^T M^{-1} n} \right\} z^{-M^{-1} n} \\ &= \frac{1}{|M|} \sum_{k \in \mathcal{R}(M^T)} X(z^{M^{-1}} e^{-\imath 2\pi k^T M^{-1}}). \end{aligned} \tag{2.5}$$

Substituting $z = e^{\imath \omega^T}$ we get

$$Y(\omega) = [\downarrow M]\, X(\omega) = \frac{1}{|M|} \sum_{k \in \mathcal{R}(M^T)} X(M^{-T}(\omega - 2\pi k)). \tag{2.6}$$

After downsampling $Y(\omega)$ is periodic on $2\pi\mathcal{L}(I)$. At a given ω, there are $|M|$ *alias* components of $X(\omega)$ appropriately shifted. It is impossible to reconstruct $X(\omega)$ from $Y(\omega)$ unless all but one of the components of $X(\omega)$ in Eqn. 2.6 is zero. This can be accomplished by using *anti-aliasing* filters before downsampling. Fig. 2.3 and Fig. 2.4 illustrate *imaging* and *aliasing* respectively in one dimension with $M = 2$.

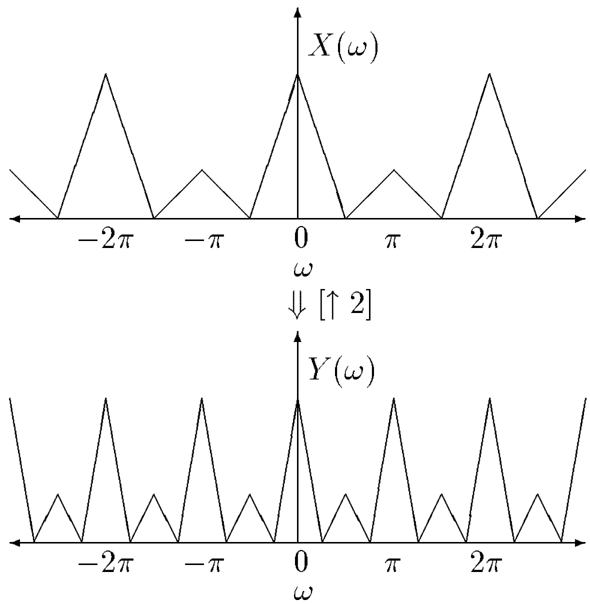

Figure 2.3: Upsampling and Spectral Imaging

2.2 Cascades of Upsamplers and Downsamplers

In order to study multidimensional multirate filter banks it is often necessary to know what happens when we take a product (cascade) of upsampling and downsampling operators. The main difficulty stems from the non-commutativity of matrix multiplication. Analysis of general cascades of upsamplers and downsamplers can be studied by only considering the four possibilities for the cascade of two upsamplers/downsamplers. For this we require the notions of *greatest common right/left divisors (gcr(l)d's) and least common right/left multiples (lcr(l)m's)* of integer ma-

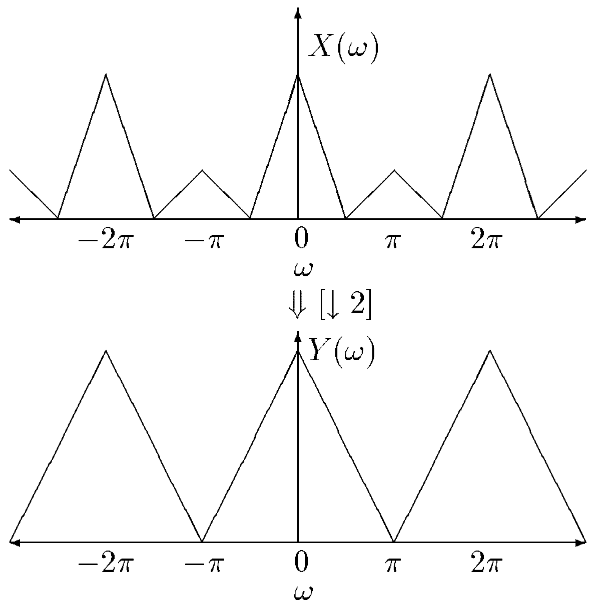

Figure 2.4: Downsampling and Aliasing

trices (see Appendix A). Using the Aryabhatta/Bezout over integer matrices several new results for matrices relevant to the multirate signal analysis problem are obtained. A fundamental mathematical result is the Representatives' Mapping Theorem.

Example 2 (Construction of gcrd/gcld and lcrm/lclm) Let

$$M = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & -1 \\ -1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{bmatrix}.$$

Both matrices are non-singular with $\det M = 6$, and $\det N = 8$.

A gcrd of M and N: It can be shown that

$$D_r = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix}$$

is a *common right divisor* since

$$M = \begin{bmatrix} -2 & 6 & -3 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} D_r \text{ and } N = \begin{bmatrix} -2 & 4 & -2 \\ 2 & -3 & 1 \\ 0 & -2 & 0 \end{bmatrix} D_r.$$

Moreover, in this case D_r turns out to be a greatest common right divisor. Since $\det D_r = 2$, it is not unimodular and hence M and N are *not* right coprime.

A gcld of M and N: The matrix

$$D_l = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 3 & -1 \end{bmatrix}$$

is a *common left divisor* since

$$M = D_l \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{bmatrix} \text{ and } N = D_l \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ -7 & 5 & 1 \end{bmatrix}.$$

Again D_l is a gcld of M and N and since $\det D_l = -2$, M and N are not left coprime.

An lclm of M and N:

$$M_l = \begin{bmatrix} 2 & -2 & -2 \\ 6 & -3 & -3 \\ 0 & 2 & -2 \end{bmatrix}$$

is a *common left multiple* since

$$M_l = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 3 & -4 \\ 0 & 2 & 0 \end{bmatrix} M \text{ and } M_l = \begin{bmatrix} -1 & -2 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} N.$$

M_l is also an lclm.

An lcrm of M and N:

$$M_r = \begin{bmatrix} -2 & 0 & 0 \\ -5 & 4 & -3 \\ -14 & 12 & -6 \end{bmatrix}$$

is a *common right multiple* since

$$M_r = M \begin{bmatrix} 4 & -4 & 2 \\ -5 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } M_r = N \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -7 & 5 & -3 \end{bmatrix}.$$

M_r is an lcrm of M and N.

Fact 2 *(Aryabhatta/Bezout Identity)* M and N are right coprime iff there exist right coprime matrices X and Y, left coprime matrices $\tilde{M}$ and $\tilde{N}$, and left coprime matrices $\tilde{X}$ and $\tilde{Y}$ such that

$$\begin{bmatrix} \tilde{Y} & \tilde{X} \\ \tilde{N} & -\tilde{M} \end{bmatrix} \begin{bmatrix} M & X \\ N & -Y \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \qquad (2.7)$$

An example of the Aryabhatta/Bezout identity follows:

Example 3 Consider the matrices

$$M = \begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix} \text{ and } N = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}.$$

M and N are right coprime although $\det M = -4$ and $\det N = -2$ are not relatively prime integers. Since

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & -1 & 4 & 0 \\ 0 & 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

one can identify X, Y, $\tilde{X}$, $\tilde{Y}$, $\tilde{M}$ and $\tilde{N}$ by partitioning the matrices as follows:

$$\begin{bmatrix} M & X \\ N & -Y \end{bmatrix} = \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ \hline -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right]$$

and

$$\begin{bmatrix} \tilde{Y} & \tilde{X} \\ \tilde{N} & -\tilde{M} \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ \hline 1 & -1 & 4 & 0 \\ 0 & 2 & -4 & 1 \end{array}\right].$$

Using the Aryabhatta/Bezout identity we obtain the *Representatives' Mapping Theorem*, the one dimensional version of which is a standard result in number theory. If $M_1\mathbf{Z}$ and $M_2\mathbf{Z}$ are sublattices of $\mathbf{Z}$ (generated by positive integers M_1 and M_2) their unit cells are $[0, M_1)$ and $[0, M_2)$ respectively and their representatives are the integers modulo M_1 and the integers modulo M_2 respectively; $\mathcal{L}(M_i) = \{\ldots, -2M_i, -M_i, 0, M_i, 2M_i, \ldots\}, \mathcal{U}(M_i) = [0, M_i)$, and $\mathcal{R}(M_i) = \{0, 1, 2, \ldots, M_i - 1\}$. If M_1 and M_2 are relatively prime then [77]

$$M_2\mathcal{R}(M_1) \bmod M_1 = \mathcal{R}(M_1) \quad \text{and} \quad M_1\mathcal{R}(M_2) \bmod M_2 = \mathcal{R}(M_2). \tag{2.8}$$

Eqn. 2.8 states that the representatives of a lattice are mapped back onto itself under multiplicative mapping provided the multiplication is by an integer coprime to the generator of the lattice. The extension of this result to the case of general lattices in $\mathbf{Z}^d$ is the *Representatives' Mapping Theorem.*

Theorem 1 *(Representatives' Mapping Theorem)* If M_1 and M_2 are left coprime, there exist right coprime matrices N_1 and N_2, such that

$$M_2\mathcal{R}(N_1) \bmod M_1 = \mathcal{R}(M_1). \tag{2.9a}$$

$$M_1\mathcal{R}(N_2) \bmod M_2 = \mathcal{R}(M_2). \tag{2.9b}$$

$$N_1^T\mathcal{R}(M_2^T) \bmod N_2^T = \mathcal{R}(N_2^T). \tag{2.9c}$$

$$N_2^T\mathcal{R}(M_1^T) \bmod N_1^T = \mathcal{R}(N_1^T). \tag{2.9d}$$

Conversely if N_1 and N_2 are right coprime, there exist M_1 and M_2 left coprime, such that Eqns. 2.9a-2.9d are true.

Proof: Since M_1 and M_2 are left coprime, by the Aryabhatta/Bezout identity there exist integer matrices P_1, P_2, Q_1, Q_2, N_1, and N_2 such that

$$\begin{bmatrix} P_1 & P_2 \\ M_1 & -M_2 \end{bmatrix} \begin{bmatrix} N_2 & Q_2 \\ N_1 & -Q_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

In particular we have the equations:

$$M_1 N_2 = M_2 N_1. \tag{2.10a}$$

$$|M_1|\,|N_2| = |M_2|\,|N_1|\,. \tag{2.10b}$$

$$P_1 N_2 + P_2 N_1 = I. \tag{2.10c}$$

First consider consider Eqn. 2.9a. It states that under the action of M_2, there exists a set that maps into the set of representatives of the lattice M_1. Moreover, this set is the set of representatives of a certain lattice N_1. We first show that $M_2\mathcal{R}(N_1)\text{mod}M_1 \subseteq \mathcal{R}(M_1)$ and then show the converse.

$\boxed{M_2\mathcal{R}(N_1)\text{mod}M_1 \subseteq \mathcal{R}(M_1)}$ For any $n \in \mathbf{Z}^d$, n mod $M_1 \in \mathcal{R}(M_1)$. Therefore, in particular, for any fixed $k \in \mathcal{R}(N_1)$, $M_2 k$ mod $M_1 \in \mathcal{R}(M_1)$.

$\boxed{M_2\mathcal{R}(N_1)\text{mod}M_1 \supseteq \mathcal{R}(M_1)}$ In order to prove this we show that the mapping induced by M_2 is a one-to-one onto mapping of the representatives of the lattice $\mathcal{L}(N_1)$ onto the representatives of the lattice $\mathcal{L}(M_1)$.

one-to-one We need to show that for any two integers k and l in the unit cell of the lattice generated by N_1, if

$$M_2 k \text{ mod } M_1 = M_2 l \text{ mod } M_1, \tag{2.11}$$

then $k = l$. From Lemma 1 the $\mathcal{LU}$ decomposition of k and l gives us

$$k = N_1\alpha \quad \text{and} \quad l = N_1\beta \tag{2.12}$$

with $\alpha, \beta \in \mathcal{U}$. Hence from Eqn. 2.11 there exists an $n \in \mathbf{Z}^d$ such that

$$\begin{array}{lll} & M_2 N_1(\alpha - \beta) = M_1 n & \text{Eqn. 2.12.} \\ \Rightarrow & M_1 N_2(\alpha - \beta) = M_1 n. & \text{Eqn. 2.10a} \\ \Rightarrow & N_2(\alpha - \beta) = n. & M_1 \text{ is invertible} \\ \Rightarrow & P_1 N_2(\alpha - \beta) = P_1 n. & \\ \Rightarrow & (I - P_2 N_1)(\alpha - \beta) = P_1 n. & \text{Eqn. 2.10c} \\ \Rightarrow & (\alpha - \beta) = P_1 n + P_2(l - k). & \text{Eqn. 2.12} \\ \Rightarrow & (\alpha - \beta) = 0. & \text{Lemma 2} \\ \Rightarrow & k - l = 0. & \end{array}$$

The last step follows from the fact that if $\alpha, \beta \in \mathcal{U}$ then the only integer of the form $\alpha - \beta$ is 0. Hence the mapping is one-to-one. This implies that

$$|N_1| \leq |M_1|. \tag{2.13}$$

onto It suffices to show that there are at most $|N_1|$ representatives of the lattice $\mathcal{L}(M_1)$. In that case every representative of the lattice $\mathcal{L}(N_1)$ maps into a unique representative of $\mathcal{L}(M_1)$, and conversely. Using the above arguments with the transposed form of the Aryabhatta/Bezout identity we can show that under multiplication by N_2^T, $\mathcal{R}(M_1^T)$ is mapped into $\mathcal{R}(N_1^T)$ in a one-to-one fashion and hence

$$\left|M_1^T\right| \leq \left|N_1^T\right|. \tag{2.14}$$

From Eqn. 2.13 and Eqn. 2.14 it follows that $|M_1| = |N_1|$ and hence the mapping is onto.

We have thus proved Eqn. 2.9a. Eqn. 2.9b follows using the same arguments by replacing M_1 with M_2, N_1 and N_2 etc. Eqn. 2.9c and Eqn. 2.9d also can be shown by considering the Aryabhatta/Bezout identity in its transposed (dual) form:

$$\begin{bmatrix} N_2^T & N_1^T \\ Q_2^T & -Q_1^T \end{bmatrix} \begin{bmatrix} P_1^T & M_1^T \\ P_2^T & -M_2^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

□

Remark: The crucial fact used in the proof is that M_1, M_2, N_1 and N_2 are non-singular. It follows that if Q_1 and P_1 (or equivalently Q_2 and P_2) are non-singular the following equations are also true.

$$P_2\mathcal{R}(Q_1) \bmod P_1 = \mathcal{R}(P_1) \tag{2.15a}$$

$$P_1\mathcal{R}(Q_2) \bmod P_2 = \mathcal{R}(P_2) \tag{2.15b}$$

$$Q_1^T\mathcal{R}(P_2^T) \bmod Q_2^T = \mathcal{R}(Q_2^T) \tag{2.15c}$$

$$Q_2^T\mathcal{R}(P_1^T) \bmod Q_1^T = \mathcal{R}(Q_1^T) \tag{2.15d}$$

The Representatives' Mapping Theorem has several consequences some of which illuminate the similarities and differences between the scalar and matrix cases.

Corollary 1 Let M_1 and M_2 be non-singular, left coprime and let N_1 and N_2 be non-singular, right coprime. If $M_1N_2 = M_2N_1$ then

$$|M_1| = |N_1| \quad \text{and} \quad |M_2| = |N_2|\,. \tag{2.16}$$

Proof: First note that we can always obtain an Aryabhatta/Bezout identity involving the four matrices M_1, M_2, N_1, and N_2. This can be done by taking an Aryabhatta/Bezout corresponding to the N's and tweaking the unimodular matrix U in order to makes the lower block becomes $[M_1 \;\; -M_2]$. Now the proof of Theorem 1 gives the result. □

The result implies that commuting matrices behave like integers as far as coprimeness is concerned.

Corollary 2 Non-singular commuting matrices are left coprime iff they are right coprime.

Proof: Let M_1 and M_2 be left coprime. By hypothesis, we have an Aryabhatta/Bezout identity with N_1 and N_2 right coprime and

$$M_1 N_2 = M_2 N_1 = M. \tag{2.17}$$

Since M_1 and M_2 commute, their product $\hat{M}$ is a right common multiple of both M_1 and M_2. Hence there exists a matrix R such that $M_2 = N_2 R$ and $M_1 = N_1 R$. Therefore $\hat{M} = MR$. From Corollary 1 $|M| = |M_1||N_2| = |M_1||M_2| = \left|\hat{M}\right|$, and therefore R is unimodular. Hence the result. □

Remark: Corollary 2 has been obtained recently by other researchers [10]. Their technique does not use the Representatives' Mapping Theorem.

Another result that we require later is the following:

Lemma 4 If $k, l \in \mathcal{R}(M)$, (or more generally if $k, l \in \mathcal{S}(M)$, a generalized set of representatives of $\mathcal{L}(M)$), then

$$[\downarrow M]\, z^{k-l} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

Proof: Let $x(n) = \delta(n - k + l)$. Then $y(n) = [\downarrow M]\, x(n) = \delta(Mn - k + l)$ and hence is non-zero only when $k - l = Mn \in \mathcal{L}(M)$. But from Lemma 3, $k - l \in \mathcal{L}(M)$ iff $k = l$, and therefore, $y(n) = 0$ if $k \neq l$ and $y(n) = \delta(Mn) = \delta(n)$ when $k = l$. Taking the $\mathcal{Z}$-transform on both sides we get the result. □

We now analyze cascades of upsamplers and downsamplers.

2.2.1 Upsampler-Upsampler Identity

Consider the cascade of two upsamplers as shown in Fig. 2.5. In the Fourier transform domain using Eqn. 2.4 we have $Y(\omega) = Y_1(M_2^T \omega) = X(M_1^T(M_2^T \omega)) = X((M_2 M_1)^T \omega)$. Therefore $[\uparrow M_2]\,[\uparrow M_1] = [\uparrow M_2 M_1]$ as shown in Fig. 2.5.

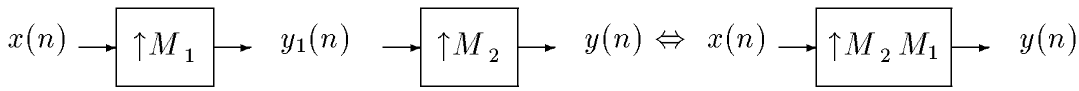

Figure 2.5: Upsampler-Upsampler (UU) Identity

2.2.2 Downsampler-Downsampler Identity

For a cascade of downsamplers as shown in Fig. 2.6, using Eqn. 2.1 we have $y(n) = y_1(M_2 n) = x(M_1 M_2 n)$ and therefore $[\downarrow M_2]\,[\downarrow M_1] = [\downarrow M_1 M_2]$.

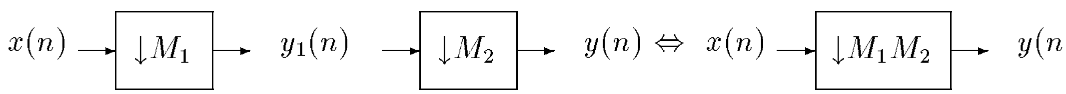

Figure 2.6: Downsampler-Downsampler (DD) Identity

2.2.3 Upsampler-Downsampler Identity

In Fig. 2.7, for simplicity let $\mathcal{L}(M_2) \subseteq \mathcal{L}(M_1)$. Since upsampling onto $\mathcal{L}(M_1)$ is followed by by downsampling onto a sublattice of $\mathcal{L}(M_1)$, intuitively one expects this to be equivalent to upsampling on a *reduced* lattice. Indeed even more is true. Since $y(n) = [\downarrow M_2]\, y_1(n) = y_1(M_2 n)$, and $y_1(n) = [\uparrow M_1]\, x(n)$, Eqn. 2.2 implies

$$y(n) = \begin{cases} x(M_1^{-1} M_2 n) & \text{for } M_2 n \in \mathcal{L}(M_1) \\ 0 & \text{otherwise.} \end{cases} \tag{2.19}$$

If M is a gcld of M_1 and M_2 (i.e., $\mathcal{L}(M_1)$ and $\mathcal{L}(M_2)$ are sublattices of $\mathcal{L}(M)$), then $M_1 = MK_1$ and $M_2 = MK_2$. Therefore $M_1^{-1} M_2 = K_1^{-1} K_2$ and $M_2 n \in \mathcal{L}(M_1) \Leftrightarrow K_2 n \in \mathcal{L}(K_1)$ (from the nonsingularity of M). Moreover,

$$y(n) = \begin{cases} x(K_1^{-1} K_2 n) & \text{for } K_2 n \in \mathcal{L}(K_1) \\ 0 & \text{otherwise.} \end{cases} \tag{2.20}$$

From Eqn. 2.19 and Eqn. 2.20 we get $[\downarrow M_2]\,[\uparrow M_1] = [\downarrow K_2]\,[\uparrow K_1]$ implying the Upsampler-Downsampler identity in Fig. 2.7. In an upsampler-downsampler cascade

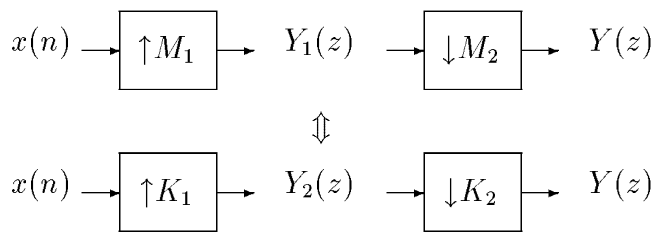

Figure 2.7: Upsampler-Downsampler (UD) Identity

one may always assume that M_1 and M_2 are left coprime. In particular, when $M_2 = M_1 K$, M_1 is indeed a gcld and therefore one can collapse it into upsampling by K as our intuition suggests!

2.2.4 Downsampler-Upsampler Identity

In a downsampler-upsampler cascade there is no simplification. Assume downsampling by M_1 followed by upsampling by M_2. Then

$$y(n) = \begin{cases} x(M_1 M_2^{-1} n) & \text{for } n \in \mathcal{L}(M_2) \\ 0 & \text{otherwise.} \end{cases} .$$

If $M_1 = K_1 M$ and $M_2 = K_2 M$ (notice that there are no sublattice conditions), then $M_1 M_2^{-1} = K_1 K_2^{-1}$. However, $n \in \mathcal{L}(M_2)$ is *different* from $n \in \mathcal{L}(K_2)$ implying reduction is not possible *unless* M_2 and K_2 are related by a unimodular matrix (i.e., M_1 and M_2 are right coprime). This impossibility of of reduction is even seen in the one dimensional case. It has to do the inherent irreversibility of the downsampling.

In summary, pure cascades of upsamplers/downsamplers can be reduced to a single upsampler/downsampler that is the product of the constituents. Upsamplers followed by downsamplers can always be reduced to the case of a single upsampler followed by a downsampler with left coprime matrices. No simplifications are possible when we have a downsampler followed by an upsampler.

2.2.5 Swapping Upsamplers and Downsamplers

Swapping of upsamplers and downsamplers is useful in many applications like the *rational sampling rate filter bank* problem (Section 3.8). The following result characterizes conditions under which one can swap upsamplers and downsamplers. It explicitly constructs the swapped upsamplers and downsamplers.

> **Theorem 2** *(The Swapping Theorem)* If M_1 and M_2 are left coprime there exist right coprime matrices N_1 and N_2 such that
>
> $$[\downarrow M_2]\,[\uparrow M_1] = [\uparrow N_1]\,[\downarrow N_2]\,. \tag{2.21}$$
>
> Conversely, given N_1 and N_2 right coprime, there exist left coprime matrices M_1 and M_2, such that Eqn. 2.21 holds.

Proof: From the hypothesis, and the Aryabhatta/Bezout identity, we have right coprime matrices N_1 and N_2 such that,

$$\begin{bmatrix} P_1 & P_2 \\ M_1 & -M_2 \end{bmatrix} \begin{bmatrix} N_2 & Q_2 \\ N_1 & -Q_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{2.22}$$

We now show that N_1 and N_2 can be used for swapping. From the Aryabhatta/Bezout identity, notice in particular that the following equations are true:

$$N_1^T M_2^T = N_2^T M_1^T. \tag{2.23a}$$

$$N_1^T P_2^T + N_2^T P_1^T = I \tag{2.23b}$$

First consider upsampling by M_1 followed by downsampling by M_2. In the Fourier domain we have (from Eqn. 2.6),

$$\begin{aligned}
Y(\omega) &= \frac{1}{|M_2^T|} \sum_{k \in \mathcal{R}(M_2^T)} Y_1(M_2^{-T}(\omega - 2\pi k)) \\
&= \frac{1}{|M_2^T|} \sum_{k \in \mathcal{R}(M^T)} X(M_1^T M_2^{-T}(\omega - 2\pi k)) \\
&= \frac{1}{|M_2^T|} \sum_{k \in \mathcal{R}(M_2^T)} X(N_2^{-T} N_1^T \omega - 2\pi N_2^{-T} N_1^T k) \quad \text{from Eqn. 2.23a} \\
&= \frac{1}{|N_2^T|} \sum_{k \in \mathcal{R}(M_2^T)} X(N_2^{-T} N_1^T \omega - 2\pi N_2^{-T} N_1^T k) \quad \text{from Corollary 1} \\
&= \frac{1}{|N_2^T|} \sum_{k \in \mathcal{R}(N_2^T)} X(N_2^{-T} N_1^T \omega - 2\pi N_2^{-T} k) \qquad \text{from Lemma 1.}
\end{aligned}$$

The last expression is precisely the Fourier domain equation for the process of downsampling by N_2 followed by upsampling by N_1. □

If M_1 and M_2 are commuting coprime matrices one can *commute* upsampling and downsampling.

Theorem 3 *(The Upsampler/Downsampler Commuting Theorem)* If M_1 and M_2 commute and are coprime $[\uparrow M_1][\downarrow M_2] = [\downarrow M_2][\uparrow M_1]$ and $[\uparrow M_2][\downarrow M_1] = [\downarrow M_1][\uparrow M_2]$.

Proof: Since M_1 and M_2 are (left) coprime, the Swapping Theorem applies. It suffices to show that one can choose $N_1 = M_1$ and $N_2 = M_2$. First note that $M_2 N_1 = M_1 N_2$ and $M_2 M_1 = M_1 M_2$. By Corollary 2 M_1 and M_2 are also right coprime. Hence both $M_2 M_1$ and $M_2 N_1$ are lcrms of M_2. Therefore there exists a unimodular U such that $M_2 N_1 U = M_2 M_1 = M_1 M_2 = M_1 N_2 U$. Multiplying the Aryabhatta/Bezout identity in Eqn. 2.22 on the left and right respectively by the unimodular matrices $\begin{bmatrix} U^{-1} & 0 \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}$ we get

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} U^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ M_1 & -M_2 \end{bmatrix} \begin{bmatrix} N_2 & Q_2 \\ N_1 & -Q_1 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} U^{-1}P_1 & U^{-1}P_2 \\ M_1 & -M_2 \end{bmatrix} \begin{bmatrix} N_2U & Q_2 \\ N_1U & -Q_1 \end{bmatrix}. \quad (2.24)$$

Relabeling $U^{-1}P_1$ and $U^{-1}P_2$ by P_1 and P_2 and using the fact that $N_1U = M_1$ and $N_2U = M_2$ one has

$$\begin{bmatrix} P_1 & P_2 \\ M_1 & -M_2 \end{bmatrix} \begin{bmatrix} M_2 & Q_2 \\ M_1 & -Q_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

From the above Aryabhatta/Bezout identity and Theorem 2 one gets Theorem 3. □

2.3 Commuting Filters and Upsamplers/Downsamplers

Once we know how to handle filters and delays with upsamplers/downsamplers we will have all the basic tools for the analysis of arbitrary multirate linear-shift invariant systems. The situation in which a filter is followed by an upsampler (Filter-Upsampler) and the situation in which the filter follows a downsampler (Downsampler-Filter) are duals and easy to analyze [84, 70]. The reverse situations of Filter-Downsampler and Upsampler-Filter combinations have important identities associated with them and play an important role in filter bank theory. The identities involve polyphase representations. We introduce the new concept of a *generalized* polyphase representation giving more general results than available in the literature. A novel identity, Upsampler-Delay-Downsampler identity, is also obtained.

2.3.1 Filter-Upsampler Identity

Let $y_1(n) = h(n) * x(n)$ and $y(n) = [\uparrow M]\, y_1(n)$ as shown in Fig. 2.8. In the Fourier and $\mathcal{Z}$ domain we have $Y_1(\omega) = H(\omega)X(\omega)$ and $Y_1(z) = H(z)X(z)$ and therefore $Y(\omega) = H(M^T\omega)X(M^T\omega)$ and $Y(z) = H(z^M)X(z^M)$. Equivalently $Y(\omega) = H(M^T\omega)\,\{[\uparrow M]\, X(\omega)\}$ and $Y(z) = H(z^M)\,\{[\uparrow M]\, X(z)\}$. This establishes the equivalence in Fig. 2.8.

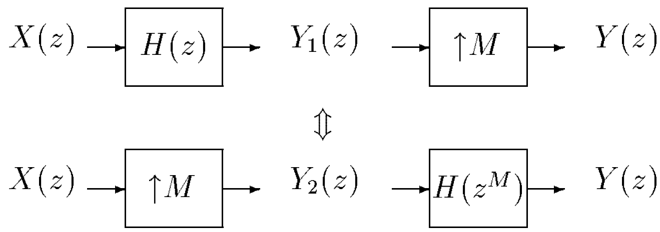

Figure 2.8: Filter-Upsampler (FU) Identity

2.3.2 Downsampler-Filter Identity

Here $y_1(n) = x(Mn)$, and $y(n)(= y_1(n) * h(n))$ is given by the convolution

$$\begin{aligned} y(n) &= \sum_{k \in \mathbf{Z}^d} h(k) y_1(n-k) = \sum_{k \in \mathbf{Z}^d} h(k) x(Mn - Mk) \\ &= \sum_{k} h(M^{-1}Mk) x(Mn - Mk) \\ &= \sum_{k' \in \mathcal{L}(M)} h(M^{-1}k') x(Mn - k') \\ &= [\downarrow M] \{ [\uparrow M] \, h(n) * x(n) \}. \end{aligned} \tag{2.25}$$

The equivalence is shown in Fig. 2.9.

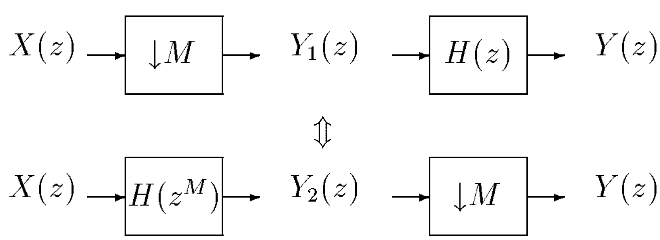

Figure 2.9: Downsampler-Filter (DF) Identity

2.3.3 Filter-Downsampler Identity

The Filter-Downsampler identity is useful from both theoretical and computational points of view. Since downsampling throws away samples it is more efficient to downsample a signal *before* filtering and the Filter-Downsampler identity does precisely that. One requires the notion of a *polyphase* representation [4, 70] - also known as *lifting* in control theory and mathematics community [1]. The essential idea is to *"lift"* a scalar valued sequence/function into a vector-valued sequence/function by blocking. The components of this *lifted* signal are referred to as *polyphase* components of the original sequence. We also introduce the new concept of a *generalized polyphase representation.*

The polyphase representation of $x(n)$ is the vector valued signal $x_p(n)$ (whose components are labeled for convenience by the representatives of $\mathcal{L}(M)$ and) given by

$$x_k(n) = x(Mn - k) = [\downarrow M]\{x(n-k)\} \quad \text{for } k \in \mathcal{R}(M). \tag{2.26}$$

There are $|M|$ polyphase components. If $X_k(z)$ is the $\mathcal{Z}$ transform of $x_k(n)$, then

$$X(z) = \sum_{k \in \mathcal{R}(M)} z^k X_k(z^M). \tag{2.27}$$

The above polyphase representation will be called the *first-orthant* polyphase. (also referred to as the *synthesis* polyphase [91] or Type 1 polyphase [85]). Another polyphase representation, the *dual first-orthant* polyphase, (also called the *analysis* polyphase and Type 2 polyphase) is defined by

$$x_k(n) = x(Mn + k) = [\downarrow M]\{x(n+k)\} \quad \text{for } k \in \mathcal{R}(M). \tag{2.28}$$

There are infinitely many choices for the polyphase representation corresponding to different choices of lifting the signal $x(n)$ to $x_p(n)$, the only restriction being that the components of the polyphase *must* be labeled from a set of generalized representatives of the lattice $\mathcal{L}(M)$. From Eqn. 2.27 we can get the identities shown in Fig. 2.10 and Fig. 2.11, called the polyphase-inverse-polyphase (PIP) and inverse-polyphase-polyphase (IPP) identities respectively. Given any set $\mathcal{S}(M)$ of *generalized representatives* of $\mathcal{L}(M)$, the *generalized* polyphase representation of a signal $x(n)$ relative to $\mathcal{S}(M)$, is the vector signal $x_p(n)$, whose components (precisely $|M|$ of them

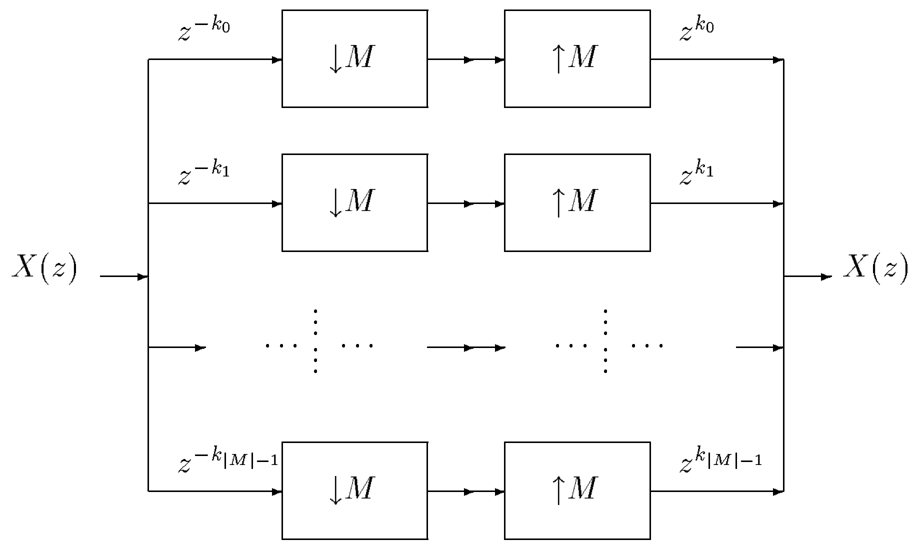

Figure 2.10: The Polyphase-Inverse-Polyphase (PIP) Identity

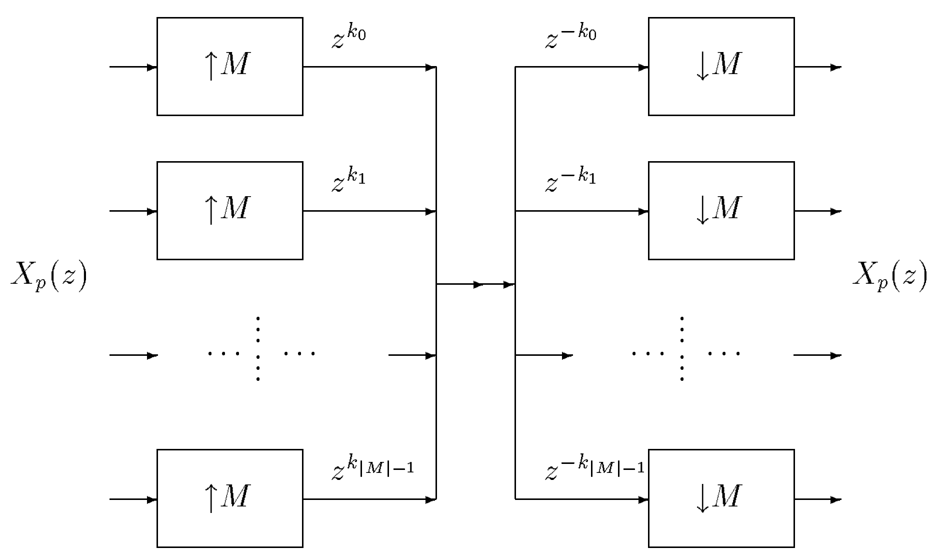

Figure 2.11: The Inverse-Polyphase-Polyphase (IPP) Identity

labeled from $\mathcal{S}(M)$) are given by

$$x_k(n) = x(Mn - k) \quad \text{for } k \in \mathcal{S}(M). \tag{2.29}$$

The dual of the generalized polyphase representation relative to $\mathcal{S}(M)$ is obtained by replacing k by $-k$ in Eqn. 2.29. Corresponding to this representation we have the generalized PIP (GPIP) and generalized IPP (GIPP) identities respectively, which look exactly like the PIP and IPP identities in Fig. 2.10 and Fig. 2.11, except that $k_i \in \mathcal{S}(M)$ instead of $\mathcal{R}(M)$.

We are now ready to obtain the Filter-Downsampler identity. Let $X_p(z)$ denote a *generalized* polyphase representation of $X(z)$ relative to $\mathcal{S}(M)$, and let $H_p(z)$ denote the *dual* polyphase representation of $H(z)$. If $Y(z)$ is the output as shown in Fig. 2.12, then

$$Y(z) = H_p^T(z)X_p(z) = \sum_{k\in\mathcal{S}(M)} H_k(z)X_k(z), \tag{2.30}$$

because

$$\begin{aligned}
Y(z) &= [\downarrow M]\, H(z)X(z) \\
&= [\downarrow M] \left\{ \sum_{k\in\mathcal{S}(M)} z^{-k}H_k(z^M) \right\} \left\{ \sum_{l\in\mathcal{S}(M)} z^{l}X_l(z^M) \right\} \\
&= [\downarrow M] \left\{ \sum_{k,l\in\mathcal{S}(M)} z^{l-k}H_k(z^M)X_l(z^M) \right\} \\
&= \left\{ [\downarrow M] \sum_{k,l\in\mathcal{S}(M)} z^{l-k} \right\} H_k(z)X_l(z) \\
&= \sum_{k,l\in\mathcal{S}(M)} H_k(z)X_l(z)\delta(l-k) \quad \text{from Lemma 3} \\
&= \sum_{k\in\mathcal{S}(M)} H_k(z)X_k(z).
\end{aligned} \tag{2.31}$$

Remark: The analysis of the filter-downsampler situation is simplified by the careful choice of notation. If both the filter and signals had been represented by the *same* polyphase representation, then the analysis would have become cumbersome, and we