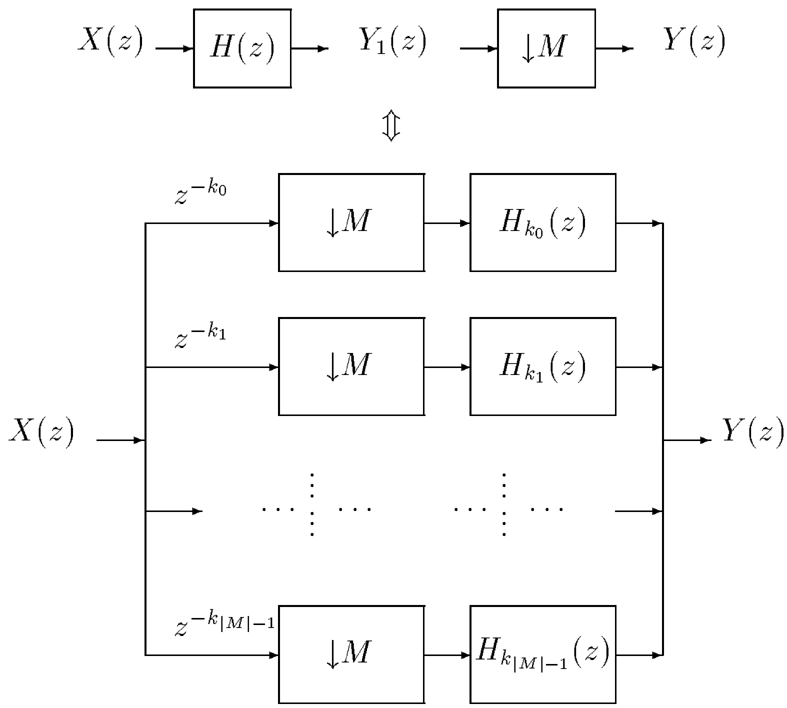

Figure 2.12: Filter-Downsampler (FD) Identity

would not have a neat (delay-free) formula for the output in terms of the polyphase components (for a comparison see [91]).

2.3.4 Upsampler-Filter Identity

This is the dual of the filter-downsampler situation, and hence once again the commutation is simplified by using polyphase components. To analyze this situation, it is simpler to use the same polyphase representation for both the signals and the filter. Let $X_k(z)$, $Y_k(z)$ and $H_k(z)$ denote the generalized polyphase components of $X(z)$, $Y(z)$ and $H(z)$ relative to a generalized set of representatives $\mathcal{S}(M)$ of $\mathcal{L}(M)$. Then $Y(z) = \sum_{k \in \mathcal{S}(M)} z^k Y_k(z^M)$ and

$$\begin{aligned} Y(z) &= H(z)\,[\uparrow M]\,X(z) \qquad (2.32)\\ &= H(z)X(z^M) \end{aligned}$$

$$= \sum_{k \in \mathcal{S}(M)} z^k H_k(z^M) X(z^M)$$

and therefore it follows that

$$Y_k(z) = H_k(z)X(z) \quad \text{for all } k \in \mathcal{S}(M). \tag{2.33}$$

Hence the mapping from $X(z)$ to $Y(z)$ may be represented as shown in Fig. 2.13.

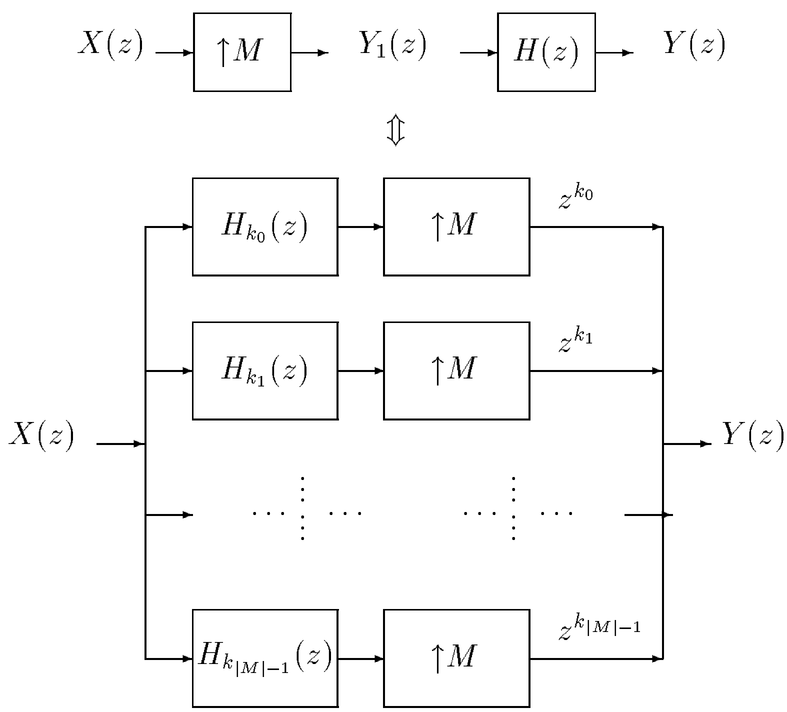

Figure 2.13: Upsampler-Filter (UF) Identity

Remark: The Filter-Downsampler and Upsampler-Filter identities are true for *any* generalized polyphase representation, and in particular for the first-orthant polyphase representation.

2.3.5 Upsampler-Delay-Downsampler Identity

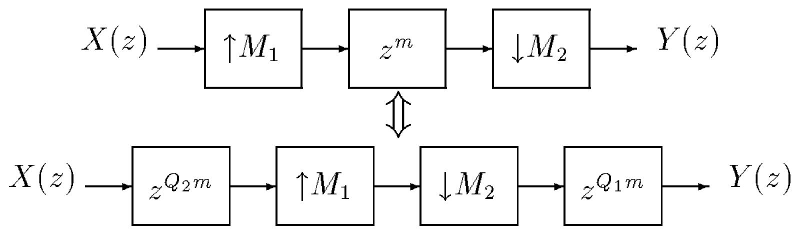

Figure 2.14: Upsampler-Delay-Downsampler (UΔD) Identity

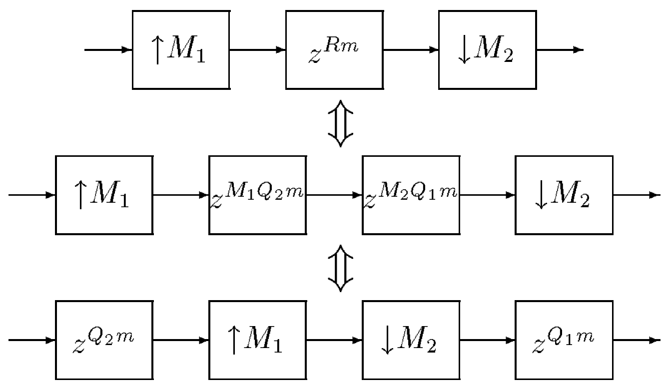

Figure 2.15: Upsampler-Delay-Downsampler Reduction Steps

In this situation the delay could be considered as a filter and one could use the UF or FD identities (both involving polyphase representations) to shift the filter out from between the up/downsamplers. However, a look at Fig. 2.13 and Fig. 2.12 should make it clear that this *will* introduce new delays (coming from the polyphase representation) between the upsampler and downsampler. Therefore delays *squeezed* between upsamplers and downsamplers cannot be *released* (by using the UF and FD identities). This problem is quite different from the the one where a delay being squeezed between a downsampler and an upsampler. In that case the DF identity or the FU identity can be used to *release* the delay onto the upsampler side or downsampler side. This section shows the conditions under which squeezed delays between upsamplers and downsamplers can be released.

Lemma 5 A delay *squeezed* between an M_1-fold upsampler and M_2-fold downsampler can be released iff M_1 and M_2 are left coprime.

Proof: Let z^m be squeezed between $[\uparrow M_1]$ and $[\downarrow M_2]$. We have to show that there exist integer matrices Q_1 and Q_2 such that the delay can be released as shown in Fig. 2.14. Let $R = \text{gcrd}(M_1, M_2)$. Then from the Aryabhatta/Bezout identity, it follows that there exist matrices P_2 and Q_1 such that

$$M_1 Q_2 + M_2 Q_1 = R \tag{2.34}$$

and therefore any integer of the form Rm can be expressed in the form $M_1 Q_2 m + M_2 Q_1 m$. This composite delay may be thought of as two filters in cascade and by using the FU and DF identities on these filters we can release the delay. In particular, when M_1 and M_2 are left coprime, $R = I$ and we can release any delay. Conversely, given that we can release the delay we can show that the delay has to be of the form Rm. Fig. 2.15 outlines the steps involved in the reduction. □

2.4 Generalization of Kovacevic's Theorem

This section gives another application of *The Swapping Theorem* (Theorem 2). In the two dimensional case a set of necessary and sufficient conditions for commuting upsamplers and downsamplers has been reported [52]. Using the tools developed earlier we now show that the result extends readily to higher dimensions. The proof in the two dimensional case involved the detailed analysis of scalar lcms in a number of special situations and used a special representation of integer matrices generating a given lattice [52]. The power of the methods developed in this chapter should be evident from the proof of this theorem.

Theorem 4 *(Generalized Kovacevic's Theorem)* Given commuting matrices M_1 and M_2,

$$[\downarrow M_2]\,[\uparrow M_1] = [\uparrow M_1]\,[\downarrow M_2] \qquad (2.35)$$

iff the determinant of generator of the the greatest common sublattice (gcs) of $\mathcal{L}(M_1)$, and $\mathcal{L}(M_2)$, viz., $\mathcal{L}(M_1)\cap\mathcal{L}(M_2)$, is equal to $\pm\,|M_1|\,|M_2|$.

Proof: Let M be the generator of the gcs of $\mathcal{L}(M_1)$ and $\mathcal{L}(M_2)$. Then from Fact 1, there exist integer matrices K_1 and K_2 with K_1 and K_2 right coprime, such that $M = M_1K_1 = M_2K_2$. Therefore M is an lcrm of M_1 and M_2. By assumption $\widehat{M} = M_1M_2 = M_2M_1$ and hence $\widehat{M}$ is a *common right multiple* of M_1 and M_2. There exists a matrix R (which is a gcrd of M_1 and M_2) such that $\widehat{M} = MR$. Now R is unimodular iff $|M| = \left|\widehat{M}\right| = |M_1|\,|M_2|$. In this case M_1 and M_2 are right coprime. Hence by Corollary 3 we can interchange upsampling and downsampling. □

The various multirate identities developed in this chapter is tabulated in Table. 2.1.

Table 2.1: Summary of Multirate Identities

Filter-Filter (FF)	$H_1(z)$	$H_2(z)$	$H_1(z)$	$H_2(z)$
Downsampler-Downsampler (DD)	$[\downarrow M_1]$	$[\downarrow M_2]$	$[\downarrow M_1M_2]$	
Upsampler-Upsampler (UU)	$[\uparrow M_1]$	$[\uparrow M_2]$	$[\uparrow M_2M_1]$	
Upsampler-Downsampler (UD)	$[\uparrow M_1]$	$[\downarrow M_2]$	$[\uparrow K_1]$	$[\downarrow K_2]$
Downsampler-Upsampler (DU)	$[\downarrow M_1]$	$[\uparrow M_2]$	–	
Filter-Upsampler (FU)	$H(z)$	$[\uparrow M]$	$[\uparrow M]$	$H(z^M)$
Downsampler-Filter (DF)	$[\downarrow M]$	$H(z)$	$H(z^M)$	$[\downarrow M]$
Filter-Downsampler (FD)	$H(z)$	$[\downarrow M]$	$[\downarrow M]$	$H_k(z)$
Upsampler-Filter (UF)	$[\uparrow M]$	$H(z)$	$H_k(z)$	$[\uparrow M]$

Chapter 3

Filter Banks and Transmultiplexers

3.1 Introduction

Filter Banks and transmultiplexers have been studied for several years in Digital Signal Processing [75, 82, 83, 85, 91, 88, 62, 84, 70]. The introduction of the Smith-Barnwell example of a two-channel perfect reconstruction (PR) filter bank sparked a flurry of new results in filter bank theory that have recently found applications in wavelet theory. This chapter describes the filter bank and transmultiplexer problems and gives several solutions depending on restrictions on the classes of filters used.

3.1.1 The Filter Bank Problem

The structure of the classical one-dimensional filter bank problem is depicted in Fig. 3.1. The input signal $x(n)$ is filtered by a set of M filters $\{h_i\}, i \in \mathcal{R}(M)$. The desired filter responses are shown in Fig. 3.2. The response of the i^{th} filter occupies only a subband of $[-\pi, \pi]$. The filter outputs are called *subband* signals. Subband signals are then *downsampled* by M to give the signals $d_i(n)$. Downsampling preserves the average data rates at the input ($x(n)$) and the output of the M subbands ($d_i(n)$). In *subband coding* the signals $d_i(n)$ are quantized (using standard scalar or vector quantization schemes) and encoded (using standard techniques like Huffman coding or arithmetic coding) so that interesting features of each subband (like energy level, sensitivity of the ear to that subband etc.) is exploited. For example, more bits (finer quantization) could be allotted to subbands that are more important perceptually. The quantized and encoded signals are transmitted. At the receiver they are decoded

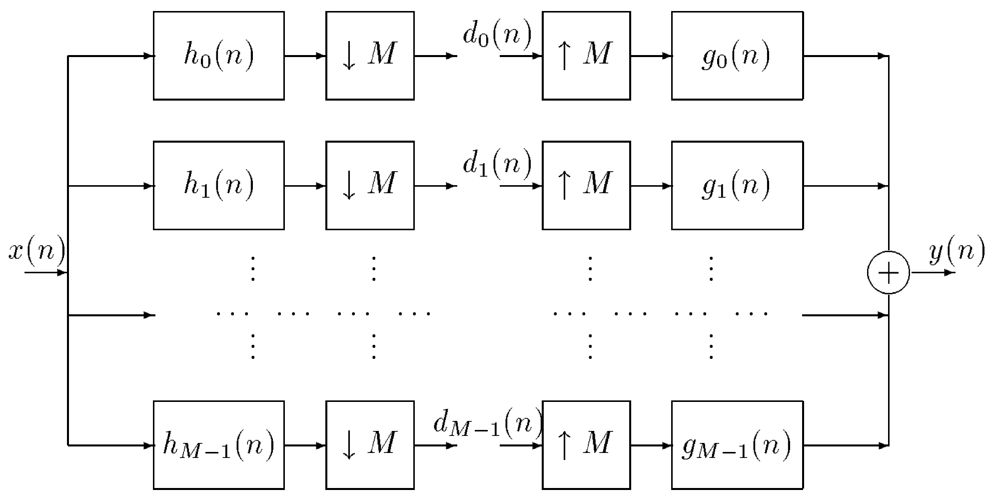

Figure 3.1: An M-channel Filter Bank

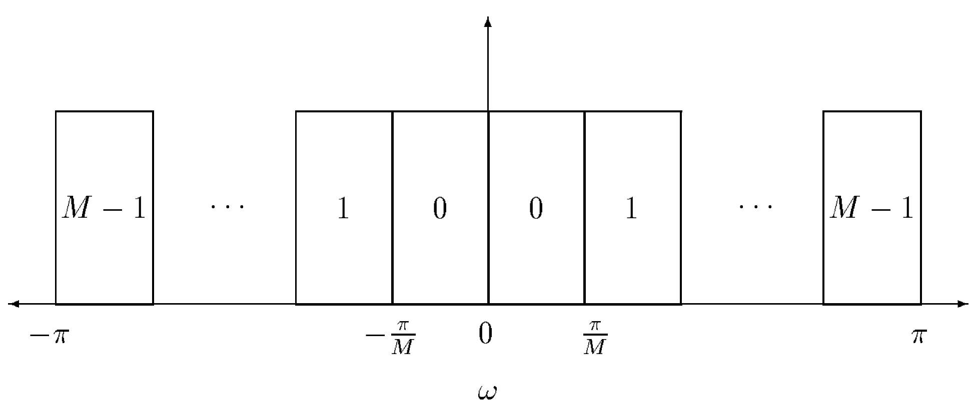

Figure 3.2: Ideal Frequency Responses in an M-channel Filter Bank

to produce approximations $\tilde{d}_i(n)$ to $d_i(n)$. These subband signals are then upsampled by M, and the i^{th} subband is passed through the filter $g_i(n)$. The outputs of these filters are added to give the reconstructed signal $y(n)$ which is an approximation to the input signal $x(n)$. The filters $\{h_i(n)\}$ are the *analysis* filters constituting the analysis filter bank and the filters $\{g_i(n)\}$ are the *synthesis* filters constituting the synthesis filter bank. The *filter bank problem* is to design the analysis and synthesis banks so that the subbands have desired frequency responses and $y(n)$ approximates $x(n)$. The non-linearities and lossiness introduced by quantization and encoding are difficult to analyze (we are unaware of any results in filter bank theory that incorporates quantization and encoding effects). Traditionally filter banks are designed by assuming that $d_i(n) = \tilde{d}_i(n)$. One desires that $y(n)$ is a scaled and delayed version of $x(n)$, i.e., $y(n) = ax(n - n_0)$. If $g_i(n)$ is replaced by $a^{-1}g_i(n + n_0)$, then the new output $y(n)$ is equal to $x(n)$ and therefore by perfect reconstruction we shall mean $y(n) = x(n)$. This may force the synthesis (or analysis) filters may become *finitely* non-causal (we will have more to say about this later).

In summary, the filter bank problem involves the design of the filters $h_i(n)$ and $g_i(n)$, with the following goals:

1. *Perfect Reconstruction* (i.e., $y(n) = x(n)$).

2. The filter responses must approximate ideal filter responses as shown in Fig. 3.2. In some applications other frequency responses may be desired.

3. The filters must be *real* and *realizable* and hence must be FIR filters or causal and stable IIR filters.

There are a number of variations of this problem.

1. The number of channels, say L, is different from the downsampling factor M.

2. A subset of the analysis filters is predetermined by an application. In this case PR may be impossible. One is interested in the necessary and sufficient

conditions on this set of filters such that PR is possible. If possible one desires to obtain a complementary set of filters.

3. The signals $x(n)$ (and hence $y(n)$) are periodic. In this case the causality or rationality of $H_i(z)$ and $G_i(z)$ is not important for realizability (since all filters can be implemented by circular convolution). It may still be useful to have rational IIR filters (in order to avoid long circular convolutions).

4. The filters are all linear-phase or have other symmetry restrictions. For instance linear-phase symmetry is useful in image processing applications [76].

5. The filters are allowed to have complex coefficients. The DFT filter bank [87] falls in this category. In this thesis filter coefficients are assumed to be real. The magnitude frequency responses of the filters will be symmetric about the origin.

6. PR may be weakened to alias-free reconstruction, where $Y(z) = T(z)X(z)$ for some filter $T(z)$.

7. The filter bank problem is cast in multiple dimensions where the sampling is by matrices M, and the downsampling factor is $|M|$. The PR conditions in this case is no more complex than in the 1-d case.

3.1.2 The Transmultiplexer Problem

The transmultiplexer problem is the *dual* of the filter bank problem [91]. A transmultiplexer is a device for converting time-domain-multiplexed (TDM) signals to frequency-domain-multiplexed signals (FDM). The basic structure of a transmultiplexer is shown in Fig. 3.3. Inputs $y_i(n)$ are the TDM signals which are upsampled, filtered and combined (by a synthesis bank of filters) to give the FDM signal $d(n)$, which is then transmitted. At the receiver $d(n)$ (assuming lossless transmission) the

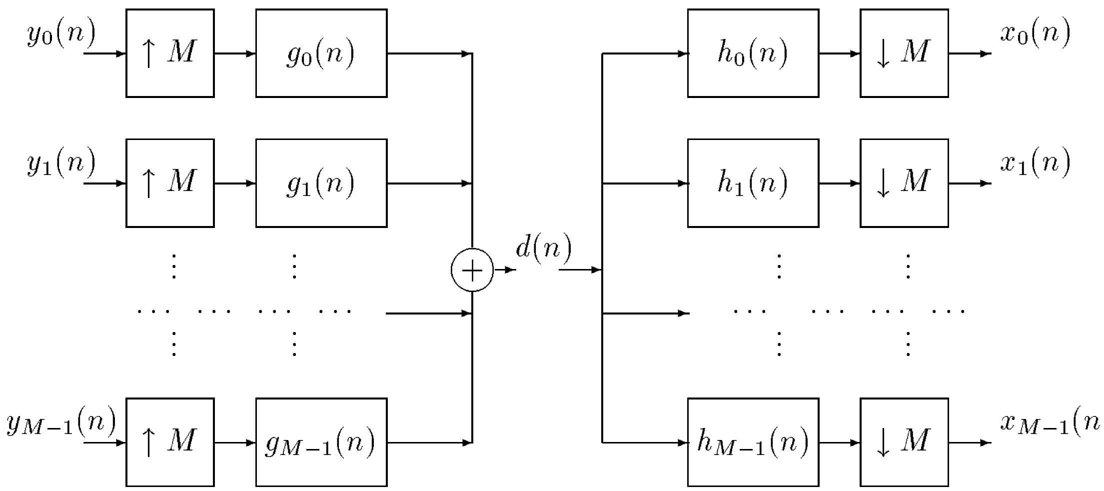

Figure 3.3: An M-channel Transmultiplexer

FDM signal is filtered and downsampled (by an analysis filter bank) to give the output TDM signals $x_i(n)$. The goal in a transmultiplexer is to design the synthesis and analysis filters so that PR is guaranteed (i.e., for all i, $x_i(n) = y_i(n)$), the filters approximate ideal bands, and that the filters are realizable. Just as in the filter bank problem there are variations of this standard problem.

1. The number of channels, say L, is not equal to the downsampling factor, M.

2. The signals are periodic. Here arbitrary choices of filters is realizable.

3. The signals and filters are multi-dimensional.

4. A subset of the filters are predetermined. The design problem is to find a complementary set of filters so that all the goals of a transmultiplexer are satisfied.

5. Only cross-talk free reconstruction is desired. That is, it is sufficient to have $X_i(z) = T_i(z)Y_i(z)$.

3.2 PR Filter Banks and Transmultiplexers

For PR the coefficients of the filters h_i and g_i have to satisfy a set of *algebraic* conditions that do not depend on whether the filters are rational, causal, etc. The PR conditions have the same form in the multidimensional case also. We characterize PR for the most general (multi-dimensional) filter bank and transmultiplexer. We assume that there are L filters in the analysis/synthesis banks and that upsampling and downsampling is by the integer matrix M.

Along each branch of a filter bank there is an analysis filter, a downsampler, an upsampler, and asynthesis filter. Using the Filter-Downsampler (FD) and Upsampler-Filter (UF) identities the analysis and synthesis filters can be brought in cascade giving explicit relations between the input and the subband signals, and the subband signals and the output.

Both the FD and UF identities require use of a generalized polyphase representation of the signals. Let $\mathcal{S}(M)$ be an *arbitrary* set of generalized representatives of $\mathcal{L}(M)$; that is $\mathcal{S}(M) = \{k_0, k_1, \ldots, k_{|M|-1}\}$. For $k \in \mathcal{S}(M)$, let $X_k(z)$, $Y_k(z)$ and $G_{i,k}$ be the components of the generalized polyphase representations of $X(z)$, $Y(z)$, and $G_i(z)$ respectively with respect to $\mathcal{S}(M)$. Let $H_{i,k}(z)$ be the components of the *dual* polyphase representation of $H_i(z)$ with respect to $\mathcal{S}(M)$.

$$X(z) = \sum_{k \in \mathcal{S}(M)} z^k X_k(z^M); \quad Y(z) = \sum_{k \in \mathcal{S}(M)} z^k Y_k(z^M).$$

$$H_i(z) = \sum_{k \in \mathcal{S}(M)} z^{-k} H_{i,k}(z^M). \tag{3.1}$$

$$G_i(z) = \sum_{k \in \mathcal{S}(M)} z^k G_{i,k}(z^M). \tag{3.2}$$

Along each branch of the analysis bank, the FD identity (Eqn. 2.30) implies

$$D_i(z) = \sum_{k \in \mathcal{S}(M)} H_{i,k}(z) X_k(z). \tag{3.3}$$

Along each branch of the synthesis filter bank, the UF identity (Eqn. 2.33) implies

$$Y_k(z) = \sum_{i=0}^{L-1} G_{i,k}(z) D_i(z). \tag{3.4}$$

For $i \in \{0, 1, L-1\} = \mathcal{R}(L)$ and $k \in \mathcal{S}(M)$, define the *polyphase* component matrices $(H_p(z))_{i,k} = H_{i,k}(z)$ and $(G_p(z))_{i,k} = G_{i,k}(z)$. There are $|M|$ columns in $H_p(z)$ and $G_p(z)$. Let the j^{th} column correspond to $k_j \in \mathcal{S}(M)$. Let $X_p(z)$ and $Y_p(z)$ denote the $\mathcal{Z}$-transforms of the polyphase signals $x_p(n)$ and $y_p(n)$, and let $D_p(z)$ be the vector whose components are $D_i(z)$. Eqn. 3.3 and Eqn. 3.4 can be written compactly as

$$D_p(z) = H_p(z) X_p(z), \tag{3.5}$$

$$Y_p(z) = G_p^T(z) D_p(z), \tag{3.6}$$

and

$$Y_p(z) = G_p^T(z) H_p(z) X_p(z). \tag{3.7}$$

Thus the analysis filter bank can be considered to be a multi-input multi-output (MIMO) linear-shift-invariant system $H_p(z)$, that takes in $X_p(z)$ and gives out $D_p(z)$. Similarly, the synthesis filter bank can be interpreted as the MIMO system $G_p^T(z)$, that maps $D_p(z)$ to $Y_p(z)$. Clearly we have PR iff $Y_p(z) = X_p(z)$. This occurs precisely when $G_p^T(z) H_p(z) = I$.

For the transmultiplexer problem let $Y_p(z)$ and $X_p(z)$ be vectorized versions of the input and output signals respectively and let $D_p(z)$ be the generalized polyphase representation of the signal $D(z)$. Now $D_p(z) = G_p^T(z) Y_p(z)$ and $X_p(z) = H_p(z) D_p(z)$. Hence $X_p(z) = H_p(z) G_p^T(z) Y_p(z)$ and for PR $H_p(z) G_p^T(z) = I$.

Theorem 5 A filter bank has the PR property iff $G_p^T(z) H_p(z) = I$ and a transmultiplexer has the PR property iff $H_p(z) G_p^T(z) = I$ where $H_p(z)$ and $G_p(z)$ are generalized polyphase representations with respect to any generalized set of representatives $\mathcal{S}(M)$ of the lattice $\mathcal{L}(M)$.

Remark: If $G_p^T(z)H_p(z) = I$ then $H_p(z)$ must have at least as many rows as columns (i.e., $L \geq |M|$). If $H_p(z)G_p^T(z) = I$ then $H_p(z)$ must have at least as many columns as rows (i.e., $|M| \geq L$). If $L = |M|$, $G_p^T(z)H_p(z) = I = H_p^T(z)G_p(z)$ and hence a filter bank is PR iff the corresponding transmultiplexer is PR.

The PR conditions in the $\mathcal{Z}$-transform domain are useful in the study of *unitary* filter banks. In some applications (theory of modulated filter banks, wavelet theory etc.) the the PR conditions in the time-domain are useful.

Theorem 6 A filter bank is PR iff

$$\sum_i \sum_n h_i(Mn + n_1)g_i(-Mn - n_2) = \delta(n_1 - n_2). \tag{3.8}$$

A transmultiplexer is PR iff

$$\sum_n h_i(n)g_j(-Ml - n) = \delta(l)\delta(i - j). \tag{3.9}$$

If the number of channels is equal to the downsampling factor (i.e., $L = |M|$) Eqn. 3.8 and Eqn. 3.9 are equivalent.

Proof:

Filter Bank PR : Since $G_p^T(z)H_p(z) = I$, for $k, l \in \mathcal{S}(M)$

$$\sum_i H_{i,k}(z)G_{i,l}(z) = \delta(k - l). \tag{3.10}$$

Consider $f(n_1, n_2)$ defined by

$$f(n_1, n_2) = \sum_i \sum_n h_i(Mn + n_1)g_i(-Mn - n_2).$$

For $n \in \mathbf{Z}^d$ $f(n_1 + Mn, n_2 + Mn) = f(n_1, n_2)$. One has to only consider $n_1 \in \mathcal{R}(M)$ and arbitrary n_2. Let $n_2 = -Ml + k$, be the $\mathcal{LU}$ decomposition of n_2 where $k \in \mathcal{R}(M)$. Then $f(n_1, n_2)$ is of the form $\sum_n \sum_i h_i(Mn+n_1)g_i(M(l-n)-k)$. Taking the $\mathcal{Z}$-transform of this expression as a sequence in l, say $a_{n_1,k}(l)$, for fixed n_1 and n_2, we get $\sum_i H_{i,n_1}(z)G_{i,k}(z) = A_{n_1,k}(z)$. But from Eqn. 3.10 $A_{n_1,k}(z) = \delta(n_1 - k)$ and hence is zero except when $n_1 = k$. When $n_1 = k$, $A_{n_1,n_1}(z) = 1$. Therefore $a_{n_1,n_1}(l) = \delta(l)$. This is zero except when $l = 0$ (in which case it is one). But when $l = 0$, and $k = n_1$, $n_1 = n_2$! Hence the result.

Transmultiplexer PR : Since $H_p(z)G_p^T(z) = I$

$$\sum_{k \in \mathcal{S}(M)} H_{i,k}(z)G_{j,k}(z) = \delta(i-j). \tag{3.11}$$

Define $\alpha(l) = \sum_n h_i(n)g_j(Ml - n)$. Then from Eqn. 3.1, Eqn. 3.2 and Eqn. 3.11

$$\begin{aligned}
\mathcal{Z}(a(l)) &= [\downarrow M]H_i(z)G_j(z) \\
&= [\downarrow M] \sum_{k,l \in \mathcal{S}(M)} z^{-k}H_{i,k}(z^M)z^l G_{j,l}(z^M) \\
&= \sum_{k \in \mathcal{S}(M)} H_{i,k}(z)G_{j,k}(z) = \delta(i-j)
\end{aligned}$$

Therefore, taking the inverse $\mathcal{Z}$-transform on both sides we get

$$\sum_n h_i(n)g_j(Ml - n) = \delta(l)\delta(i-j)$$

and replace l by $-l$ to get Eqn. 3.9.

□

The filter bank and transmultiplexer PR properties can be obtained directly in the time domain. Consider a PR filter bank. If the input is $x(n) = \delta(n - n_1)$, then $d_i(n) = h_i(Mn - n_1)$ and $y(n_2) = \sum_i g_i(n_2 - Mn)d_i(n)$. But by PR $y(n_2) = \delta(n_2 - n_1)$. The filter bank PR property is precisely a statement of this fact:

$$y(n_2) = \sum_i \sum_n g_i(n_2 - Mn)d_i(n) = \sum_i \sum_n g_i(n_2 - Mn)h_i(Mn - n_1) = \delta(n_2 - n_1).$$

Consider a PR transmultiplexer. If the input $x_i(n) = \delta(n)\delta(i-j)$ then $d(n) = g_j(n)$ and $y_i(l) = \sum_n h_i(n)d(Ml - n)$. But by PR $y_i(l) = \delta(l)\delta(i-j)$. The transmultiplexer PR property is precisely a statement of this fact:

$$y_i(l) = \sum_n h_i(n)d(Ml-k) = \sum_n h_i(n)g_j(Ml-n) = \delta(l)\delta(i-j).$$

PR filter banks and transmultiplexers can also be studied in an infinite matrix representation [90]. If $\mathbf{x} = [\cdots, x(0), x(1), \cdots]$, $\mathbf{y} = [\cdots, y(0), y(1), \cdots]$ and $\mathbf{d} = [\cdots, d_0(0), d_1(0), \cdots, d_{M-1}(0), d_0(1), d_1(1), \cdots]$, then there exist infinite matrices $\mathbf{G}$ and $\mathbf{H}$ such that

$$\mathbf{d} = \mathbf{H}\mathbf{x}, \quad \text{and} \quad \mathbf{y} = \mathbf{G}^T\mathbf{d}. \tag{3.12}$$

For example in the FIR case if the filter lengths are N we have

$$\mathbf{d} = \mathbf{H}\mathbf{x} \stackrel{\text{def}}{=} \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & h_0(N-1) & \dots & h_0(N-M-1) & \dots & h_0(0) & 0 & \\ & \vdots & \dots & \dots & \dots & \dots & \dots & \\ \ddots & h_{M-1}(N-1) & \dots & h_{M-1}(N-M-1) & \dots & h_{M-1}(0) & 0 & \ddots \\ & 0 & 0 & h_0(N-1) & \dots & \dots & \dots & \\ & \vdots & \vdots & \vdots & \dots & \dots & \dots & \\ & 0 & 0 & h_{M-1}(N-1) & \dots & \dots & \dots & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \mathbf{x}. \tag{3.13}$$

In this case we have the following theorem:

Theorem 7 A filter bank if PR iff $\mathbf{G}^T\mathbf{H} = \mathbf{I}$ and a transmultiplexer if PR iff $\mathbf{H}\mathbf{G}^T = \mathbf{I}$.

The three forms of the PR conditions obtained above are the most useful. Another form of it based on the *modulated representation* can also be found in the literature [91]. This thesis is written with the strong belief that as far as PR filter bank theory is concerned the modulated representation is dispensable. In the frequency domain the transmultiplexer PR property becomes (take the Fourier transform of Eqn. 3.9)

$$\frac{1}{|M|} \sum_{k \in \mathcal{R}(M^T)} H_i(M^{-T}(\omega - 2\pi k))G_j(M^{-T}(\omega - 2\pi k)) = \delta(i-j). \tag{3.14}$$

There is no elegant frequency domain characterization of the filter bank PR property. However the filter bank PR property implies that

$$\begin{aligned}\frac{1}{|M|}\sum_i H_i(z)G_i(z) &= \frac{1}{|M|}\left[\sum_{l,m\in\mathcal{S}(M)} z^{m-l}\sum_i H_{i,l}(z^M)G_{i,m}(z^M)\right]\\ &= \frac{1}{|M|}\sum_{l,m\in\mathcal{S}(M)} z^{m-l}\delta(l-m)\\ &= \frac{1}{|M|}\sum_{l\in\mathcal{S}(M)} 1 = 1.\end{aligned}$$

and hence in the frequency domain

$$\frac{1}{|M|}\sum_i H_i(\omega)G_i(\omega) = 1. \tag{3.15}$$

We now give several algebraic properties of PR filter banks (FBs) and transmultiplexers (TMs).

Lemma 6 PR property is preserved under interchange of the analysis and synthesis filters (along each branch).

Proof: Readily seen from Eqn. 3.8 for the filter bank case, and Eqn. 3.9 for the transmultiplexer case. An alternate proof illustrating the use of the generalized polyphase representation is as follows: For the filter bank case $G_p^T(z)H_p(z) = I$ (assume that the polyphase representation is taken with respect to $\mathcal{S}(M)$). If the filters are interchanged and if we use polyphase representations with respect to $-\mathcal{S}(M)$, PR is equivalent to $H_p^T(z)G_p(z) = I$, which is the transpose of $G_p^T(z)H_p(z) = I$! Similarly, for the transmultiplexer case also. □

Lemma 7 PR property is preserved under reflection of all the analysis and synthesis filters. That is, PR is preserved if $h_i(n)$ is replaced by $h_i(-n)$ and $g_i(n)$ is replaced by $g_i(-n)$ for all i.

Proof: Readily seen from Eqn. 3.8 and Eqn. 3.9. □

Given a filter bank, for $l \in \mathbf{Z}^d$, consider a new filter bank with filters given by $\breve{h}_i(n) = h_i(n-l)$ and $\breve{g}_i(n) = g(n+l)$. Notice that if $\mathcal{S}(M)$ is a generalized set of representatives of $\mathcal{L}(M)$, then necessarily, for $l \in \mathbf{Z}^d$, $S(M)+l$ is also a generalized set of representatives. Now by representing $\breve{h}_i$ and $\breve{g}_i$ in polyphase with respect to this set of representatives, $\breve{H}_p(z) = H_p(z)$ and $\breve{G}_p(z) = G_p(z)$.

> **Lemma 8** PR property is preserved under fixed shifts of all the analysis and synthesis filters in *opposite* directions.

> **Lemma 9** PR property is preserved if analysis/synthesis filter pairs are shifted in opposite directions by points in $\mathcal{L}(M)$.

Proof: The result will be proved for the transmultiplexer PR property and the filter bank case follows similarly. Given a PR transmultiplexer consider another with the following filters

$$\breve{h}_i(n) = \begin{cases} h_i(n) & \text{for } i \neq i_0 \\ h_{i_0}(n - Mn_0) & \text{for } i = i_0. \end{cases}$$

$$\breve{g}_i(n) = \begin{cases} g_i(n) & \text{for } i \neq i_0 \\ g_{i_0}(n + Mn_0) & \text{for } i = i_0. \end{cases}$$

For i and j both not equal to i_0 the PR condition Eqn. 3.9 holds without change. When one of them is i_0

$$\begin{aligned} \sum_n \breve{h}_{i_0}(n)\breve{g}_j(-Ml-n) &= \sum_n h_{i_0}(n-Mn_0)g_j(-Ml-n) \\ &= \sum_n h_{i_0}(n)g_j(-M(l-n_0)-n) \\ &= 0. \end{aligned}$$

When both are i_0

$$\sum_n \breve{h}_{i_0}(n)\breve{g}_{i_0}(-Ml-n) = \sum_n h_{i_0}(n-Mn_0)g_{i_0}(-Ml-n+Mn_0)$$

$$
\begin{aligned}
&= \sum_n h_{i_0}(n) g_{i_0}(-Ml-n) \\
&= \delta(l).
\end{aligned}
$$

Thus all the PR conditions are satisfied, and hence the new filter bank is PR. □

In FIR case one can (by appropriately shifting the filters) assume that all the analysis filters $h_i(n)$ are supported in the first-orthant (i.e., causal in the one dimensional case). Using Lemma 9, one can also ensure that the non-zero coefficients of $h_i(n)$ are as close to $n = 0$ as possible. In *most* of the one dimensional results in this thesis $h_i(n)$ will be assumed to be causal with its support as close to the origin as possible. This is useful in obtaining minimal parameterizations of unitary filter banks and transmultiplexers.

3.3 Some PR Results in One Dimension

There are two goals other than PR in the design of filter banks and transmultiplexers: that the filters approximate ideal responses, and that the filters are constrained to be *realizable*. If the input is periodic, then, any rational transfer function can be *realized*. However, if the input is not periodic, the only transfer functions that can be realized are FIR and finitely non-causal IIR transfer functions (that are also stable). This section tries to discuss these problems. Moreover, for filters constrained to be in a variety of classes we give necessary and sufficient conditions for PR. Throughout this section we will assume that $G_p(z)$ is made of first-orthant polyphase components and that $H_p(z)$ are made of the corresponding dual polyphase components.

Definition 5 A filter bank is said to be *causal (or finitely non-causal)* iff the entries in $H_p(z)$ and $G_p(z)$ are causal (or finitely non-causal)

If $H_p(z)$ is causal, then the filters $h_{i,j}$ are causal, and hence necessarily the filters h_i are causal. This is because $H_p(z)$ is a first-orthant polyphase representation of the filters: $h_{i,j}(n) = h_i(Mn - k)$ for $k \in \{0, 1, \ldots, M-1\}$. If $G_p(z)$ is causal, the filters

$g_i(n)$ may be non-causal since $g_{i,j}(n) = g_i(Mn-k)$, $k \in \{0, 1, \ldots, M-1\}$. However, $g_i(n-M-1)$ will be causal. *Thus the structure of a filter bank or transmultiplexer naturally introduces a delay of $M-1$.*

Clearly all finitely-causal filter banks are realizable. We now show that the class of all possible PR filter banks is quite large. First let $L = M$ and let $GL^{\infty}_{M \times M}$ be the set of all $M \times M$ matrix functions of z that have an inverse on the unit circle. Then any $H_p(z) \in GL^{\infty}_{M \times M}$ can be interpreted as the polyphase component matrix of the analysis bank of a PR filter. One can set $G_p(z) = (H_p^{-1}(z))^T$ to get the synthesis bank. This allows one to *construct* a PR filter banks, not *design* them. There is explicit control only over $h_{i,j}(n)$; not over the filters $h_i(n)$. However, realizability can be easily imposed. Let RH^{∞} denote the set of real, causal (stable), rational functions; that is real, rational functions, analytic in the exterior of the closed unit disc in the complex plane. Given $H(z) \in RH^{\infty}$, by reflecting all the exterior zeros of the function inside the unit circle one can obtain the well-known *allpass/minimum-phase* factorization: $H(z) = H_{all}(z)H_{min}(z)$. The function $H_{min}(z)$ has all its poles and zeros on or inside the unit circle and is called *minimum phase* because for all other functions in RH^{∞} that have the same magnitude on the unit circle, $H_{min}(z)$ has the least phase. The function $H_{all}(z)$ is *allpass* because its magnitude on the unit circle is unity: $\left[H_{all}^T(z^{-1})H_{all}(z)\right]_{|z|=1} = 1$. The minimum-phase function is invertible on the unit circle if it has no zeros on the unit circle. The minimum-phase function always has an analytic inverse outside the unit circle and therefore is also in in RH^{∞} Minimum phase rational functions with no zeros on the unit circle, have minimum-phase rational inverses. Notice that these functions are causal and have causal inverse. Similarly, the all-pass function is also invertible on the unit circle and has an all-pass inverse. Now if we can find an equivalent all-pass/minimum-phase factorizations for matrices in $GL^{\infty}_{M \times M}$, with entries in $RH^{\infty}_{M \times M}$, then we can construct *causal* filter banks, by choosing $H_p(z)$ to be the minimum phase factor. Also we would get another filter bank by choosing H_p to be the all-pass factor. A matrix $U(z) \in RH^{\infty}_{M \times M}$ is said to be *unitary*, if it is unitary on the unit circle (sometimes referred to as inner or lossless functions).

$$U^T(z^{-1})U(z) = I \tag{3.16}$$

This generalizes th notion of an *allpass* function. A matrix $Q \in RH^\infty_{M\times M}$, is said to be *minimum phase*, if its inverse is also in $RH^\infty_{M\times M}$. This occurs precisely when $\det Q$ is a minimum phase function and generalizes the notion of a minimum-phase function. Matrices in $RH^\infty_{M\times M}$ also have an *allpass/minimum-phase* factorization [30, 45].

> **Fact 3** Let $H \in RH^\infty_{M\times M}$. Then there exists an all pass $H_{all}(z)$ and minimum phase $H_{min}(z)$ such that $H(z) = H_{all}(z)H_{min}(z)$.

In a state-space representation we can construct $H(z) \in RH^\infty_{M\times M}$. This is equivalent to choosing the system matrix (the A matrix) in the state-space description to have all its eigenvalues insides the unit circle. Moreover, almost all random choices of $H(z)$ will also be in $GL^\infty_{M\times M}$. Therefore, from $H_{all}(z)$ and $H_{min}(z)$ we can construct two filter banks.

3.3.1 Causal FIR and Causal IIR filter banks

This section discusses necessary and sufficient conditions for PR for classes of filter banks. For example, one might require that the filters in a filter bank (transmultiplexer) are FIR, or that the filter bank is causal and the filters are IIR etc. Since the PR conditions are essentially the invertibility of $H_p(z)$ (or $G_p(z)$) one naturally expects an algebraic structure to the problem.

Let $\mathcal{A}$ be an commutative ring with identity [44]. An element in $\mathcal{A}$ is said to be a *unit* if it is invertible over $\mathcal{A}$. A matrix H over $\mathcal{A}$ is invertible iff $\det H$ is a unit in $\mathcal{A}$. This follows from the fact that the inverse is the reciprocal of the determinant times the adjoint, which has entries in $\mathcal{A}$ being made up of sums of products of elements in $\mathcal{A}$. An invertible matrix over $\mathcal{A}$ is said to be *unimodular* over $\mathcal{A}$. In a filter bank (or transmultiplexer) with $L = M$ if $H_p(z)$ is a unimodular matrix with elements from an appropriate ring then it is invertible with inverse $G_p^T(z)$ and moreover $G_p^T(z)$ also has entries from the same ring.

For FIR filter banks $H_i(z)$ and $G_i(z)$ are Laurent polynomials (polynomials in z and z^{-1}) which form a commutative ring with identity the units of which are given by the Laurent polynomials $Az^n, n \in \mathbf{Z}, A \in \mathbb{R}\setminus\{0\}$.

Lemma 10 *(FIR PR)* An FIR filter bank (with $L = M$) is PR iff $\det H_p(z)$ (or $\det G_p(z)$) is of the form Az^n for some $n \in \mathbf{Z}$ and non-zero $A \in \mathbb{R}$

Now consider a *causal* FIR filter bank. Then (from Definition 5) $H_p(z)$ and $G_p(z)$ have entries from the ring of polynomials in z^{-1}. The units of this ring are of the form $A \in \mathbb{R}\setminus\{0\}$.

Lemma 11 *(Causal FIR PR Lemma)* An FIR causal filter bank (transmultiplexer) is PR iff $\det H_p(z)$ (or $\det G_p(z)$) is a constant $A \in \mathbb{R}\setminus\{0\}$.

An interesting result for FIR filter banks is that $H_p(z)$ for an FIR filter bank can always be factored into a *causal* part and an *allpass* part.

Lemma 12 Let $H(z)$ be invertible over the Laurent polynomials. Then $H(z)$ is of the form $z^N H_1(z)H_2(z)$, where $H_1(z)$ is allpass and $H_2(z)$ is unimodular over the ring of polynomials in z^{-1}.

Proof: From Lemma 10 $H(z)$ is a Laurent polynomial. By multiplication by z^{-N} for large enough N, $H(z)$ can be converted into a polynomial in z^{-1}. From Fact 3 we can do an allpass/minimum phase factorization of this matrix so that $H(z) = z^N H_1(z)H_2(z)$. □

For designing filter banks with respect to some criterion it is desirable to be able to parameterize all invertible $H_p(z)$. For FIR filter banks, the above result states that this is equivalent to parameterizing FIR allpass and FIR minimum phase functions. Unfortunately there is no known parameterization of FIR minimum phase functions, while FIR allpass functions can be parameterized as we will see in Section 3.4.

Consider the case that the filters have rational transfer functions. In this case $H_p(z)$ and $G_p(z)$ are rational matrix functions of z. Every rational function is a unit in the ring of rational functions. Hence one would expect that a filter bank with rational filters is always PR. However for PR we require invertibility of $H_p(z)$ on the unit circle. The only rational functions that are invertible on the unit circle are the ones with neither zeros nor poles on the unit circle.

> **Lemma 13** *(IIR PR Lemma)* An IIR filter bank is PR iff $\det H_p(z)$ has no zeros or poles on the unit circle.

Remark: Notice that we have not discussed about the stability of the filters (i.e., ℓ^1 summability of the sequences $h_i(n)$ and $g_i(n)$). If $\det H_p(z)$ has no zeros or poles on the unit circle, then by analytical continuation there is an open annulus around the unit circle where $H_p(z)$ is invertible. This implies the existence of a unique Laurent series expansions for the entries in $H_p(z)$ and $G_p(z)$. The coefficients in these sequences are absolutely summable and therefore stability is guaranteed. However, the filters could be two sided sequences (since a Laurent series has terms with both positive and negative powers of z).

Since IIR filters are in general not realizable, we have to impose causality in order for the filter bank to be realizable. For causal IIR filter banks, it should be clear that $H_p(z) \in RH^\infty_{M\times M}$. The units in this ring are precisely the minimum-phase functions that have no zeros on the unit circle.

> **Lemma 14** *(Causal IIR PR Lemma)* An IIR causal filter bank is PR iff $\det H_p(z)$ is a minimum phase function with no zeros on the unit circle.

3.3.2 Alias-Free/Cross-Talk-Free Reconstruction

Sometimes PR is not required and then it may be possible to design better filters for the same computational cost if only alias-free reconstruction is demanded. Similarly, for a transmultiplexer one sometimes only desires that the channels are cross-talk free. We give the necessary and sufficient conditions for these two situations in the one dimensional case. The results are well known [84]. The novelty in our presentation of the proof (which uses the polyphase representation) which is in the spirit of the rest of the thesis. The multidimensional problem is relatively more complex [84]. For alias-free reconstruction $Y(z) = T(z)X(z)$. Let $T(z) = \sum_{k=0}^{M-1} z^k T_k(z)$ be the polyphase representation of $T(z)$. Then

$$\begin{aligned}
Y(z) &= \sum_{l=0}^{M-1} z^l X_l(z^M) \sum_{k=0}^{M-1} z^k T_k(z) \\
&= \sum_{l=0}^{M-1} X_l(z^M) \left[\sum_{k=l}^{M-1+l} z^k T_{k-l}(z) \right] \\
&= \sum_{l=0}^{M-1} X_l(z^M) \left[\sum_{k=0}^{l-1} z^k (z^M T_{M-1+k-l}) + \sum_{k=l}^{M-1} z^k T_{k-l}(z) \right]
\end{aligned}$$

and can be written in the form

$$Y_p(z^M) = \begin{bmatrix} T_0(z^M) & z^M T_{M-1}(z^M) & z^M T_{M-2}(z^M) & \dots & z^M T_1(z^M) \\ T_1(z^M) & T_0(z^M) & z^M T_{M-1}(z^M) & \dots & z^M T_2(z^M) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ T_{M-2}(z^M) & T_{M-3}(z^M) & T_{M-4}(z^M) & \dots & z^M T_{M-1}(z^M) \\ T_{M-1}(z^M) & T_{M-2}(z^M) & T_{M-3}(z^M) & \dots & T_0(z^M) \end{bmatrix} X_p(z^M),$$

and hence

$$Y_p(z) = \begin{bmatrix} T_0(z) & zT_{M-1}(z) & zT_{M-2}(z) & \dots & zT_1(z) \\ T_1(z) & T_0(z) & zT_{M-1}(z) & \dots & zT_2(z) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ T_{M-2}(z) & T_{M-3}(z) & T_{M-4}(z) & \dots & zT_{M-1}(z) \\ T_{M-1}(z) & T_{M-2}(z) & T_{M-3}(z) & \dots & T_0(z) \end{bmatrix} X_p(z)$$

$$= G_p^T(z)H_p(z)X_p(z).$$

The matrix in the above equation is referred to as *right pseudocirculant* because the rows are right shifts and there is a z in some of the terms (hence *pseudo*). The corresponding time domain conditions are obvious.

Lemma 15 A filter bank provides alias-free reconstruction iff $G_p^T(z)H_p(z)$ is right pseudocirculant.

A transmultiplexer satisfies cross-talk free reconstruction if the input signals and output signals are related as $X_i(z) = T_i(z)Y_i(z)$. Therefore

$$H_p^T(z)G_p(z) = \begin{bmatrix} T_0(z) & 0 & \dots & 0 \\ 0 & T_1(z) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & T_{L-1}(z) \end{bmatrix}.$$

Lemma 16 A transmultiplexer satisfies cross-talk free reconstruction iff $H_p(z)G_p^T(z)$ is diagonal.

3.4 Unitary Filter Banks

Unitary filter banks (in all dimensions) are a very important class of PR filter banks. In one dimension unitary filter banks can be parameterized. One can *walk* over this parameter space to get to an optimal point with respect to any given design criterion. Unitary filter banks are also important from a wavelets perspective since they give rise to wavelet tight frames.

Definition 6 A filter bank (or transmultiplexer) is said to be *unitary* if $H_p^T(z^{-1})H_p(z) = I$ (or $H_p(z)H_p^T(z^{-1}) = I$).

Lemma 17 For unitary filter banks $g_i(n) = h_i(-n)$.

Proof: Since $G_p(z) = H_p(z^{-1})$, from Eqn. 3.2 and Eqn. 3.1 we have

$$G_i(z) = \sum_{k\in\mathcal{S}(M)} z^k G_{i,k}(z) = \sum_{k\in\mathcal{S}(M)} z^k H_{i,k}(z^{-1}) = H_i(z^{-1}).$$

□

Therefore one readily has the following result (from Eqn. 3.8 and Eqn. 3.9):

Theorem 8 A filter bank is unitary iff

$$\sum_i \sum_n h_i(Mn+n_1)h_i(Mn+n_2) = \delta(n_1-n_2). \tag{3.17}$$

A transmultiplexer is unitary iff

$$\sum_n h_i(n)h_j(Ml+n) = \delta(l)\delta(i-j). \tag{3.18}$$

If the number of channels is equal to the downsampling factor then a filter bank is unitary iff the corresponding transmultiplexer is unitary.

Since $g_i(n) = h_i(-n)$, in the infinite matrix representation $\mathbf{G} = \mathbf{H}$. and therefore for unitary filter banks it follows (from the PR condition) that $\mathbf{H}^T\mathbf{H} = \mathbf{I}$ or equivalently $\mathbf{H}$ is a (left) unitary matrix. Similarly for a unitary transmultiplexer $\mathbf{H}$ is a (right) unitary matrix and $\mathbf{H}\mathbf{H}^T = \mathbf{I}$. The frequency domain version of the transmultiplexer PR property now becomes (see Eqn. 3.14)

$$\frac{1}{|M|}\sum_{k\in\mathcal{R}(M^T)} H_i(M^{-T}(\omega-2\pi k))H_j^*(M^{-T}(\omega-2\pi k)) = \delta(i-j), \tag{3.19}$$

and the filter bank PR property implies (in the frequency domain - see Eqn. 3.15)

$$\frac{1}{|M|}\sum_{i=0}^{L-1} |H_i(\omega)|^2 = 1 \tag{3.20}$$

For a unitary filter bank, if $H_p(z)$ is causal, then necessarily $G_p(z)$ is anti-causal. Thus the only *realizable* unitary filter banks are FIR (unless the inputs are periodic). Moreover, the minimal delay in the implementation of an FIR unitary filter bank is precisely $N-1$, where N is the maximum length of any filter. This follows immediately from Lemma 17 The importance of unitary filter banks in filter bank theory stems from the fact that (in one dimension) $H_p(z)$ for FIR and IIR (not irrational) unitary filter banks can be parameterized [85]. We will give a new parameterization of unitary FIR $H_p(z)$ and obtain the factorization in [85] as a corollary. Using this parameterization, unitary filter banks can be numerically *designed* so that the filters approximate ideal frequency responses (or satisfy any other optimality criterion). Moreover, unitary filter bank theory is a natural starting point for the theory of wavelet tight frames and orthonormal bases. It must be emphasized that FIR unitary $H_p(z)$ is well understood only in one dimensions. Very little is known about parameterizations of multidimensional unitary matrices on the unit poly-circle. Recently it has been shown that all two dimensional FIR unitary matrices on the unit circle can be parameterized [2]. The precise result is that every two dimensional FIR lossless system can be obtained by terminating some one dimensional lossless two-port with a set of delays in the second dimension.

The parameterization of unitary filter banks is intimately related to a certain *energy preserving property* of unitary filter banks. For a real scalar signal $x(n)$ its energy is given by

$$\|x\|^2 = \sum_n |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{-\pi} d\omega\, |X(\omega)|^2 = \frac{1}{2\pi}\int_{-\pi}^{-\pi} d\omega \left[X(z^{-1})X(z)\right]_{z=e^{\imath\omega}}.$$

Now consider a vector signal $x(n)$, with components $x_i(n)$, $i \in \mathcal{R}(M)$. If $\|x(n)\|$ denotes the Euclidean norm of $x(n)$, then the energy of the signal is given by

$$\|x\|^2 = \sum_{i=0}^{M-1} \|x_i\|^2 = \sum_n \|x(n)\|^2.$$

Just as in the scalar case it is easily verified that

$$\|x\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \left[X^T(z^{-1})X(z)\right]_{z=e^{\imath\omega}}.$$

Consider a unitary filter bank. The energy of the subband signals is given by

$$\begin{aligned}\|d_p\|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \left[D_p^T(z^{-1})D_p(z)\right]_{z=e^{\imath\omega}} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \left[(H_pX_p)^T(z^{-1})(H_pX_p)(z)\right]_{z=e^{\imath\omega}} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \left[X_p^T(z^{-1})X_p(z)\right]_{z=e^{\imath\omega}} \\ &= \|x_p\|^2 .\end{aligned}$$

Similarly, $\|d_p\|^2 = \|y_p\|^2$. Therefore, the input, output and subband energies of a unitary filter bank are the same. Moreover, unitariness is characterized by this energy preservation property.

If U is an orthogonal matrix, then the signals $x(n)$ and $Ux(n)$ have the same energy. If P denotes the orthogonal projection onto a subspace of $\mathbb{R}^M$, then $I - P$ denotes the orthogonal projection onto its orthogonal complement and hence

$$\|x\|^2 = \|Px\|^2 + \|(I-P)x\|^2 .$$

For any give $X(z)$, $X(z)$ and $z^{-1}X(z)$ have the same energy. Using the above facts we have for any projection matrix P,

$$D_p(z) = \left[I - P + z^{-1}P\right] X_p(z) \stackrel{\text{def}}{=} T(z)X_p(z)$$

has the same energy as $X_p(z)$. That is $T(z)$ is unitary. Moreover $T(z)$ is also a first order polynomial matrix in z^{-1}. For a set of $K-1$ projection matrices P_i of rank δ_i we have

$$T(z) = \prod_{k=1}^{K-1}\left[I - P_i + z^{-1}P_i\right] V_0,$$

is unitary. We now prove that every polynomial matrix of polynomial degree $(K-1)$ can be factored in the above form. If the length N of the filters in a filter bank satisfy $M(K-1) < N \leq MK$ then $H_p(z)$ is a polynomial matrix of degree $(K-1)$.

Theorem 9 Every unitary polynomial matrix $H_p(z)$ of (polynomial) degree $(K-1)$ can be uniquely factored in the form

$$H_p(z) = \prod_{i=K-1}^{1} \left[I - P_i + z^{-1}P_i\right] V_0 \tag{3.21}$$

where P_i are projection matrices of rank δ_i and V_0 is a constant unitary matrix.

Proof: Since $H_p(z) \stackrel{\text{def}}{=} \sum_{n=0}^{K-1} h_p(n)z^{-n}$ is unitary $h_p^T(0)h_p(K-1) = 0$ and therefore $h_p(0)$ is singular. Let P_{K-1} be the *unique* projection matrix onto the nullspace of $h_p(0)$ (say of dimension δ_{K-1}). That is, $h_p(0)^T P_{K-1} = 0 = P_{K-1}h_p(0)$ (because P_{K-1} is symmetric). Also, since $h_p(K-1)$ is in the nullspace, $P_{K-1}h_p(K-1) = h_p(K-1)$ and hence $(I - P_{K-1})h_p(K-1) = 0$. Therefore $[I - P_{K-1} + zP_{K-1}]\, H_p(z)$ is a matrix polynomial of degree at most degree $K-2$. However if $h_p(0)$ and $h_p(K-1)$ are not zero (an assumption one makes without loss of generality) then the degree is *precisely* $K-2$. Moreover, it is unitary since $I - P_{K-1} + zP_{K-1}$ is unitary on the unit circle (complementary projections taken together preserve energy). Repeated application of this procedure $(K-1)$ times gives a degree zero unitary matrix V_0. □

A rank n projection matrix is of the form $w_1w_1^T + \ldots + w_nw_n^T$, where w_i are unit norm M-vectors that are mutually orthogonal. Therefore if $L = \sum_{i=1}^{K-1} \delta_i$, one has the (non-unique) Householder factorization [83]

$$H_p(z) = \left[\prod_{i=L-1}^{1} \left[I - v_iv_i^T + z^{-1}v_iv_i^T\right]\right] V_0. \tag{3.22}$$

L is the McMillan degree of $H_p(z)$ [84]. The unit M-vectors v_i are known as Householder parameters, and each is determined by $(M-1)$ scalar parameters. Moreover the unitary matrix V_0 is determined by $\binom{M}{2}$ parameters. Therefore a polyphase matrix $H_p(z)$ of polynomial degree $(K-1)$ is completely determined by $\binom{M}{2} + (M-1)(L-1)$

parameters where L is the McMillan degree of $H_p(z)$. Factorization of polynomial matrices unitary on the unit circle is also a consequence of classical results in network theory [3, 84].

Another result that we find useful when we discuss wavelet theory is the following representation theorem for *unitary* vectors on the unit circle. An $M \times 1$ vector $V(z)$ is said to be unitary on the unit circle if $V^T(z^{-1})V(z) = 1$. Let $(K-1)$ be the polynomial degree of $V(z)$ (in this case the polynomial degree is equal to the McMillan degree). We have the following factorization theorem for unitary vectors of degree $(K-1)$ analogous to the result for unitary matrices [83].

> **Fact 4** Every polynomial vector of (polynomial) degree $(K-1)$ is uniquely determined by $(K-1)$ projection matrices P_i, $i \in \{1, \ldots, K-1\}$, each of rank one (i.e., $P_i = v_i v_i^T$) and the vector v_0
>
> $$V(z) = \left[\prod_{i=1}^{K-1} \left[I - v_i v_i^T + z^{-1} v_i v_i^T\right]\right] v_0,$$

The McMillan degree of this vector polynomial is precisely $(K-1)$. Therefore, the McMillan degree of any one filter in an M-channel filter bank with filters of length MK is always $K-1$. However, the McMillan degree of $H_p(z)$ could be $L \geq K$.

Therefore in the FIR case unitary filter banks may be designed from Eqn. 3.22 by appropriate choice of the Householder parameters v_i and V_0 [67]. Another advantage of unitary filter banks is that once the analysis filters are given the synthesis filters are just time-reverses of the analysis filters.

In many applications it is desirable to have h_i and g_i to be linear-phase filters, so that there is no dispersion. Unitary FIR filter banks with filters that are linear phase have been constructed in [67]. Recently, a complete parameterization of unitary FIR linear phase filter banks has been obtained in [76]. Later in Section 3.6, we will give a complete parameterization of FIR unitary filter banks with various types of symmetry, that includes the results in [76]. An important property of linear phase unitary filter banks is that $\lceil \frac{M}{2} \rceil$ filters have to be even-symmetric and $\lfloor \frac{M}{2} \rfloor$ filters have to be odd-symmetric [76].

We now describe the parameterization of V_0 and $\{v_i\}$ using *angle* parameters helpful for design purposes. First consider $v_i \in \mathbb{R}^M$, with $\|v_i\| = 1$. Clearly, v_i has $(M-1)$ degrees of freedom. One way to parameterize v_i using $(M-1)$ angle parameters $\theta_{i,k}$, $k \in \{0, 1, \ldots, M-2\}$ would be to define the components of v_i as follows:

$$(v_i)_j = \begin{cases} \left\{ \prod_{l=0}^{j-1} \sin(\theta_{i,l}) \right\} \cos(\theta_{i,j}) & \text{for } j \in \{0, 1, \ldots, M-2\} \\ \left\{ \prod_{l=0}^{M-1} \sin(\theta_{i,l}) \right\} & \text{for } j = M-1. \end{cases}$$

As for V_0, it being an $M \times M$ orthogonal matrix, it has $\binom{M}{2}$ degrees of freedom. There are two well known parameterizations of constant orthogonal matrices, one based on Givens' rotation (well known in QR factorization etc. [25]), and another based on Householder reflections. In the Householder parameterization

$$V_0 = \prod_{i=0}^{M-1} \left[I - 2 v_i v_i^T \right],$$

where v_i are unit norm vectors with the first i components of v_i being zero. Each matrix factor $\left[I - 2 v_i v_i^T \right]$ when multiplied by a vector q, reflects q about the plane perpendicular to v_i, hence the name Householder reflections. Since the first i components of v_i is zero, and $\|v_i\| = 1$, v_i has $M - i - 1$ degrees of freedom. Each being a unit vector can be parameterized as before using $M - i - 1$ angles. Therefore, the total degrees of freedom is

$$\sum_{i=0}^{M-1} (M - 1 - i) = \sum_{i=0}^{M-1} i = \binom{M}{2}.$$

In summary, any orthogonal matrix can be factored into a cascade of M reflections about the planes perpendicular to the vectors v_i.