Notice the similarity between Householder reflection factors for V_0 and the factors of $H_p(z)$ or $V(z)$. Based on this similarity, the factorization of unitary matrices and vectors in this section is called the Householder factorization. Analogous to the Givens' factorization also one can obtain a factorization of unitary matrices $H_p(z)$ and unitary vectors $V(z)$ [26]. However, from the points of view of filter bank theory and wavelet theory, the Householder factorization is simpler to understand and implement except when $M = 2$.

3.4.1 Two Channel Unitary FIR Filter Bank

In this section we describe the Givens' factorization for unitary 2×2 matrices on the unit circle [86]. Since such a matrix could be chosen to be $H_p(z)$ of a unitary two channel filter bank, this will give a parameterization of two channel unitary filter banks. In the two channel case, this factorization is preferable to the Householder factorization. Moreover, this factorization plays an important role in the study of unitary filter banks with symmetry (Section 3.6) and in the study of modulated unitary filter banks (Section 3.5.2). The results are well known [86].

Fact 5 $H_p(z)$ of a two channel FIR unitary filter bank is of the form

$$H_p(z) = \begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \left\{ \prod_{i=0}^{K-2} \begin{bmatrix} \cos\theta_i & z^{-1}\sin\theta_i \\ \sin\theta_i & -z^{-1}\cos\theta_i \end{bmatrix} \right\} \begin{bmatrix} \cos\theta_{K-1} & \sin\theta_{K-1} \\ \sin\theta_{K-1} & -\cos\theta_{K-1} \end{bmatrix}. \tag{3.23}$$

The above parameterization is called the orthogonal (or unitary) lattice parameterization of unitary $H_p(z)$. The filters $H_0(z)$ and $H_1(z)$ are given by

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = H_p(z^2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}.$$

If $H^R_{0,j}(z)$ is the reflection of $H_{0,j}(z)$ (i.e., $H_{0,j}(z) = z^{-k+1}H_{0,j}(z^{-1})$) then

$$\begin{bmatrix} H_{0,0}(z) & \pm H_{1,0}(z) \\ H_{0,1}(z) & \pm H_{1,1}(z) \end{bmatrix} = \prod_{i=K-1}^{1} \begin{bmatrix} \cos\theta_i & z^{-1}\sin\theta_i \\ \sin\theta_i & -z^{-1}\cos\theta_i \end{bmatrix} \begin{bmatrix} \cos\theta_0 & \sin\theta_0 \\ \sin\theta_0 & -\cos\theta_0 \end{bmatrix}$$
$$= \begin{bmatrix} H_{0,0}(z) & (-1)^K H^R_{0,1}(z) \\ H_{0,1}(z) & -(-1)^K H^R_{0,0}(z) \end{bmatrix}. \tag{3.24}$$

3.5 Modulated Filter Banks

An important goal (besides PR) in filter bank design is that the filters $\{h_i\}$ and $\{g_i\}$ approximate ideal frequency responses (as shown in in Fig. 3.2 - in this section we will assume $L = M$) . The passbands of filters h_i and g_i must be the union of the intervals $[-(i+1)\frac{\pi}{M}, -i\frac{\pi}{M}]$ and $[i\frac{\pi}{M}, (i+1)\frac{\pi}{M}]$. If $h(n)$ is a real coefficient prototype filter with passband in $[-\frac{\pi}{2M}, \frac{\pi}{2M}]$, then $h(n)\cos\left(2\pi(2i+1)\frac{\pi}{2M}n + \epsilon_i\right)$, (where ϵ_i is an arbitrary phase), has a passband equal to the desired band for the i^{th} filter in Fig. 3.2. This technique gives rise to modulated filter banks (MFBs). For an MFB

$$h_i(n) = h(n)\cos\left(\frac{\pi}{2M}(2i+1)n + \epsilon_i\right) \tag{3.25}$$

and

$$g_i(n) = g(n)\cos\left(\frac{\pi}{2M}(2i+1)n + \gamma_i\right). \tag{3.26}$$

where ϵ_i and γ_i are phase factors. Several choices of ϵ_i and γ_i can be found in the literature [63, 51]. For example Malvar uses (see [63])

$$\epsilon_i = -\frac{\pi}{2M}(2i+1)\frac{N+M-1}{2} \quad \text{and} \quad \gamma_i = -\frac{\pi}{2M}(2i+1)\frac{N-M-1}{2},$$

while Koilpillai and Vaidyanathan (see [51]) use

$$\epsilon_i = -\frac{\pi(2i+1)}{2M}\frac{N+M-1}{2} + \frac{\pi}{2} \quad \text{and} \gamma_i = -\frac{\pi(2i+1)}{2M}\frac{N-M-1}{2} - \frac{\pi}{2},$$

where N is the length of the prototype filter. Note that while the dependence of the phase factors on M is natural, their dependence on N is quite artificial. Moreover, in both cases the authors assume $g(n) = h(n)$ (i.e., the analysis and synthesis prototypes are identical). Indeed in [63]

$$h_i(n) = h(n)\cos\left(\frac{\pi}{2M}(2i+1)(n - \frac{N+M-1}{2})\right), \tag{3.27}$$

$$g_i(n) = h(n)\cos\left(\frac{\pi}{2M}(2i+1)(n - \frac{N-M-1}{2})\right) \tag{3.28}$$

while in [51]

$$h_i(n) = h(n)\cos\left(\frac{\pi}{2M}(2i+1)(n - \frac{N-1}{2}) + (-1)^i\frac{\pi}{4},)\right) \tag{3.29}$$

and

$$g_i(n) = g(n)\cos\left(\frac{\pi}{2M}(2i+1)(n - \frac{N-1}{2}) - (-1)^i\frac{\pi}{4}\right). \tag{3.30}$$

What are the possible choices for ϵ_i and γ_i (assuming they are linear in i) so that PR may be possible? That answer can (without loss of generality - by shifting the filters if necessary) always be written in the form (see Appendix B)

$$\epsilon_i = -\frac{\pi}{2M}(2i+1)\frac{\alpha}{2} \quad \text{and} \gamma_i = \frac{\pi}{2M}(2i+1)\frac{\alpha}{2},$$

where $\alpha \in \mathbf{Z}$ will be called *the modulation phase*. Therefore for an MFB with modulation phase α

$$h_i(n) = h(n)\cos\left(\frac{\pi}{2M}(2i+1)(n - \frac{\alpha}{2})\right) \tag{3.31}$$

and

$$g_i(n) = g(n)\cos\left(\frac{\pi}{2M}(2i+1)(n + \frac{\alpha}{2})\right). \tag{3.32}$$

Notice that the modulation phase *does not depend* on the length of the filters. Hence the PR conditions we shall obtain for MFBs are true even when the filters are IIR. If

$$c_{i,\tau} = \cos\left(\frac{\pi}{2M}(2i+1)\tau\right) \quad \text{and} \quad s_{i,\tau} = \sin\left(\frac{\pi}{2M}(2i+1)\tau\right), \tag{3.33}$$

then $h_i(n) = h(n)c_{i,n-\frac{\alpha}{2}}$ and $g_i(n) = g(n)c_{i,n+\frac{\alpha}{2}}$. Note that

$$c_{i,n+2lM} = (-1)^l c_{i,n} \quad \text{and} \quad c_{i,M+2lM} = (-1)^l c_{i,M} = 0. \tag{3.34}$$

We now show that an MFB is PR iff pairs of appropriate polyphase components of $h(n)$ and $g(n)$ form a two-channel PR filter bank. An MFB is PR iff it satisfies the filter bank PR property (Eqn. 3.8):

$$f(n_1,n_2) \stackrel{\text{def}}{=} \sum_{i=0}^{M-1} \sum_n h_i(Mn+n_1)g_i(-Mn-n_2) = \delta(n_1-n_2).$$

Using the trigonometric identity $\cos a \cos b = \frac{1}{2}(\cos(a-b)+\cos(a+b))$,

$$\begin{aligned} f(n_1,n_2) &= \sum_{i=0}^{M-1} \sum_n h(Mn+n_1)g(-Mn-n_2)c_{i,Mn+n_1-\frac{\alpha}{2}}c_{i,Mn+n_2-\frac{\alpha}{2}} \\ &= \frac{1}{2}\sum_n h(Mn+n_1)g(-Mn-n_2)\left[\sum_{i=0}^{M-1} c_{i,n_1-n_2} + (-1)^n c_{i,n_1+n_2-\alpha}\right] \\ &\stackrel{\text{def}}{=} I_1 + I_2. \end{aligned} \tag{3.35}$$

Notice that

$$\sum_{i=0}^{M-1} \cos\left(\frac{\pi}{2M}(2i+1)(n_1-n_2)\right) = \begin{cases} M(-1)^l & \text{if } n_1-n_2 = 2Ml \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\sum_{i=0}^{M-1} \cos(\frac{\pi}{2M}(2i+1)(n_1+n_2-\alpha)) = \begin{cases} M(-1)^m & \text{if } n_1+n_2-\alpha = 2Mm \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $n_1 - n_2 \notin 2M\mathbf{Z}$, and $n_1+n_2-\alpha \notin 2M\mathbf{Z}$, both I_1 and I_2 are zero, as required for PR (i.e., no conditions on $h(n)$ and $g(n)$). However, if $n_1 - n_2 = 2Ml$, I_1 is non-zero and if $n_1+n_2-\alpha = 2Mm$, I_2 is non-zero. Now both of them can be non-zero simultaneously when

$$n_1 = M(m+l) + \frac{\alpha}{2} \quad \text{and} n_2 = M(m-l) + \frac{\alpha}{2}.$$

This occurs when α is even. Hence there is a dichotomy in the PR conditions depending on whether α is odd or even.

Modulation phase α is odd: If $n_1 - n_2 = 2Ml$, I_2 is zero. Hence for PR

$$\sum_n h(Mn + n_1)g(-Mn - n_1 + 2Ml) = \frac{2}{M}\delta(l) \quad \text{for all } n_1 \tag{3.36}$$

If $n_1 + n_2 - \alpha = 2Ml$, I_1 is zero and therefore for PR

$$\sum_n (-1)^n h(Mn + n_1)g(-Mn + n_1 - \alpha - 2Ml) = 0 \quad \text{for all } n_1. \tag{3.37}$$

Eqn. 3.36 and Eqn. 3.37 are necessary and sufficient for PR.

Modulation phase α is even: Let $n_1 - \frac{\alpha}{2} \notin M\mathbf{Z}$. If $n_1 - n_2 = 2Ml$ Eqn. 3.36 has to be satisfied and if $n_1 + n_2 - \alpha = 2Mm$ Eqn. 3.37 has to be satisfied. Let $n_1 - \frac{\alpha}{2} \in M\mathbf{Z}$. Then $I_1 + I_2 = \delta(n_1 - n_2)$ and hence

$$\sum_n h(Mn+Mm+Ml+\frac{\alpha}{2})g(-Mn-Mm+Ml-\frac{\alpha}{2})\left[(-1)^l + (-1)^{n+m}\right] = \frac{2}{M}\delta(l).$$

Therefore

$$\begin{aligned}\frac{1}{M}\delta(l) &= \frac{1}{2}\sum_n h(Mn + \frac{\alpha}{2})g(-Mn + 2Ml - \frac{\alpha}{2})[1 + (-1)^n] \\ &= \sum_n h(2Mn + \frac{\alpha}{2})g(-2Mn + 2Ml - \frac{\alpha}{2}).\end{aligned} \tag{3.38}$$

Eqn. 3.36 and Eqn. 3.37 can be identified with the *filter bank PR* property for appropriate pairs of polyphase components of the prototype filters. Eqn. 3.38 also gives specific conditions on a polyphase component of $h(n)$ and $g(n)$. For $l \in \mathbf{Z}$, let $p_l(n) = h(Mn + l)$ and let $q_l(n) = g(Mn - l)$. $p_l(n)$ and $q_l(n)$ are like polyphase components of $H(z)$ and $G(z)$ except that $l \in \mathbf{Z}$. Also for $j \in \{0, 1\}$, let $p_{l,j}(n) = h(2Mn + Mj + l)$ and $q_{l,j}(n) = g(2Mn - Mj - l)$.

Theorem 10 *(MFB PR Theorem)* An MFB satisfies PR iff

when α, the modulation phase, is an odd integer $P_l(z)$, $P_{\alpha-l}(-z)$, $Q_l(z)$ and $Q_{\alpha-l}(-z)$ form analysis and synthesis filters of a two-channel PR filter bank (scaled by $\sqrt{\frac{2}{M}}$), and

when α, the modulation phase, is an even integer if $l-\frac{\alpha}{2}(\bmod M) \neq 0$, $P_l(z)$, $P_{\alpha-l}(z)$, $Q_l(z)$ and $Q_{\alpha-l}(z)$ form analysis and synthesis filters of a two-channel PR filter bank (scaled by $\sqrt{\frac{2}{M}}$) and $P_{\frac{\alpha}{2},0}(z)Q_{\frac{\alpha}{2},0}(z) = \frac{1}{M}$.

Proof: When α is odd, Eqn. 3.36 and Eqn. 3.37 can be rewritten in the form:

$$\sum_n p_l(n)q_l(-2k-n) = \frac{2}{M}\delta(k). \tag{3.39a}$$

$$\sum_n (-1)^n p_{\alpha-l}(n)(-1)^{-2k-n} q_{\alpha-l}(-2k-n) = \frac{2}{M}\delta(k). \tag{3.39b}$$

$$\sum_n p_l(n)(-1)^{-2k-n} q_{\alpha-l}(-2k-n) = 0. \tag{3.39c}$$

$$\sum_n (-1)^n p_{\alpha-l}(n)q_l(-2k-n) = 0. \tag{3.39d}$$

Eqns. 3.39a-3.39d are (but for the scaling of $\frac{2}{M}$) the filter bank PR identity (actually transmultiplexer PR identity - but they are both equivalent - see Eqn. 3.9) for $p_l(n)$, $(-1)^n p_{\alpha-l}(n)$, $q_l(n)$, and $(-1)^n q_{\alpha-l}(n)$, or equivalently for $P_l(z)$, $P_{\alpha-l}(-z)$, $Q_l(z)$, and $Q_{\alpha-l}(-z)$. When α is even, and when $l-\frac{\alpha}{2} \notin M\mathbf{Z}$, Eqn. 3.36 and Eqn. 3.37 hold for PR and the result is the same as in the case of α odd. Moreover, from Eqn. 3.38 we also have

$$\sum_n p_{\frac{\alpha}{2}}(2n)q_{\frac{\alpha}{2}}(2k-2n) = \frac{1}{M}\delta(k)$$

which says $P_{\frac{\alpha}{2},0}(z)Q_{\frac{\alpha}{2},0}(z) = \frac{1}{M}$, where $P_{\frac{\alpha}{2},0}(z)$, $P_{\frac{\alpha}{2},1}(z)$, $Q_{\frac{\alpha}{2},0}(z)$, and $Q_{\frac{\alpha}{2},1}(z)$ are the two-channel polyphase components of $P_{\frac{\alpha}{2}}(z)$ and $Q_{\frac{\alpha}{2}}(z)$. □

Why is there a dichotomy between the PR conditions for different parities of the modulation phase α? Firstly notice that when α is odd there is no polyphase component corresponding to $\frac{\alpha}{2}$ (since its not an integer). Also when α is even the PR conditions imply $P_{\frac{\alpha}{2},0}(z)Q_{\frac{\alpha}{2},0}(z) = \frac{1}{M}$. PR does not depend on $P_{\frac{\alpha}{2},1}(z)$ and $Q_{\frac{\alpha}{2},1}(z)$. Therefore the odd powers of z in $P_{\frac{\alpha}{2}}(z)$ and $Q_{\frac{\alpha}{2}}(z)$ could be arbitrary as far as PR is concerned. In other words, for all n, $h(2Mn+M+\frac{\alpha}{2})$ and $g(-2Mn-M-\frac{\alpha}{2})$ could be arbitrary. This arbitrariness is illusory since the filter coefficients $h_i(2Mn + M - \frac{\alpha}{2})$ and $g_i(-2Mn - M - \frac{\alpha}{2})$ are always zero.

$$\begin{aligned} h_i(2Mn + M + \frac{\alpha}{2}) &= h(2Mn + M + \frac{\alpha}{2})c_{i,2Mn+M} \\ &= 0 \quad \text{from Eqn. 3.34.} \end{aligned}$$

3.5.1 Two Types of Modulated Filter Banks

The statement of Theorem 10 is redundant since $H(z)$ and $G(z)$ are uniquely determined by $P_l(z)$ and $Q_l(z)$, $l \in \mathcal{R}(M)$. Indeed for $k \in \mathbf{Z}$

$$P_l(z) = \sum_n h(Mn + l)z^{-n} = z^{-k} \sum_n h(Mn + Mk + l)z^{-n} = z^{-k} P_{l+Mk}(z) \qquad (3.40)$$

and

$$Q_l(z) = \sum_n g(Mn - l)z^{-n} = z^{k} \sum_n g(Mn - Mk - l)z^{-n} = z^{k} Q_{l+Mk}(z). \qquad (3.41)$$

Therefore we make two useful choices for α leading to Type 1 and Type 2 MFBs.

Consider l such that $\{l, \alpha - l\} \subset \mathcal{R}(M)$. The corresponding PR conditions depend directly on the polyphase components of $H(z)$ and $G(z)$. The best choice of α such that $\{l, \alpha - l\} \subset \mathcal{R}(M)$ is $\alpha = (M - 1)$. However, then α is even or odd depending whether M is odd or even. To obtain α with odd parity when M is odd (or even parity when M is even) the next best choice for α is $\alpha = (M - 2)$. These two choices will be called Type 1 and Type 2 MFBs. Once again note that the choice of α depends on M but not on the length of the filters.

Definition 7 An MFB is Type 1 if $\alpha = (M - 1)$ and Type 2 if $\alpha = (M - 2)$.

Type 1 and Type 2 MFBs are the only types of MFBs that need be studied. From now on only Type 1 or Type 2 MFBs will be considered (i.e., the modulation phase α is fixed by M). All other MFBs can be reduced to this case by appropriately shifting the filters or changing their sign. Theorem 10 can be reinterpreted for Type 1 and Type 2 MFBs. Define

$$J = \begin{cases} \frac{M}{2} & \text{Type 1, } M \text{ even} \\ \frac{M-1}{2} & \text{Type 1, } M \text{ odd} \\ \frac{M-2}{2} & \text{Type 2, } M \text{ even} \\ \frac{M-1}{2} & \text{Type 2, } M \text{ odd.} \end{cases} \tag{3.42}$$

We will show that a PR MFB decomposes into a set of J two-channel PR MFBs constructed from the polyphase components of the prototype filters.

Lemma 18 Filters $H_0(z)$, $-z^{-1}H_0(-z)$, $G_0(z)$ and $-zG_0(-z)$ form analysis and synthesis filters of a two channel PR filter bank iff

$$\begin{bmatrix} G_{0,0}(z) \\ G_{0,1}(z) \end{bmatrix} = \frac{1}{H_{0,0}^2(z) - z^{-1}H_{0,1}^2(z)} \begin{bmatrix} H_{0,0}(z) \\ z^{-1}H_{0,1}(z) \end{bmatrix}.$$

Furthermore, if all the filters are FIR, then, $H_0(z) = Az^n$ and $G_0(z) = \frac{1}{A}z^{-n}$, for $n \in \mathbf{Z}$ and non-zero A.

Proof: Let $H_0(z)$, $H_1(z)$, $G_0(z)$ and $G_1(z)$ be the filters. Also let $H_0(z) = H_{0,0}(z^2) + z^{-1}H_{0,1}(z^2)$ and $G_0(z) = G_{0,0}(z^2) + zG_{0,1}(z^2)$. Now

$$H_1(z) = -z^{-1}H_0(-z) = z^{-2}H_{0,1}(z^2) - z^{-1}H_{0,0}(z^2) \stackrel{\text{def}}{=} H_{1,0}(z^2) + z^{-1}H_{1,1}(z^2),$$

$$G_1(z) = -zG_0(-z) = z^2G_{0,1}(z^2) - zG_{0,0}(z^2) \stackrel{\text{def}}{=} G_{1,0}(z^2) + zG_{1,1}(z^2),$$

and therefore

$$\begin{bmatrix} H_{1,0}(z) \\ H_{1,1}(z) \end{bmatrix} = \begin{bmatrix} z^{-1}H_{0,1}(z) \\ -H_{0,0}(z) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} G_{1,0}(z) \\ G_{1,1}(z) \end{bmatrix} = \begin{bmatrix} zG_{0,1}(z) \\ -G_{0,0}(z) \end{bmatrix}.$$

The result now follows from the PR condition $G_p^T(z)H_p(z) = I$:

$$\begin{bmatrix} G_{0,0}(z) & zG_{0,1}(z) \\ G_{0,1}(z) & -G_{0,0}(z) \end{bmatrix} \begin{bmatrix} H_{0,0}(z) & H_{0,1}(z) \\ z^{-1}H_{0,1}(z) & -H_{0,0}(z) \end{bmatrix} = I.$$

If the filters are FIR then (since $H_{0,0}^2(z) - H_{0,1}^2(z)$ has to be a constant times some power of z) $H_0(z) = Az^n$ and $G_0(z) = \frac{1}{A}z^{-n}$ for some integer n and non-zero A. □

Theorem 11 A Type 1 MFB is PR iff for $l \in \mathcal{R}(J)$ $(P_l(z), P_{M-1-l}(-z))$ and $(Q_l(z), Q_{M-1-l}(-z))$ form a PR filter bank (scaled by $\sqrt{\frac{2}{M}}$) and additionally for odd M $P_{J,0}(z)Q_{J,0}(z) = \frac{1}{M}$. If $P_{J,0}(z)$ and $Q_{J,0}(z)$ are FIR the last condition becomes $P_{J,0}(z) = A\sqrt{\frac{1}{M}}z^n$ and $Q_{J,0}(z) = \frac{1}{A}\sqrt{\frac{1}{M}}z^{-n}$, for some integer n and non-zero A.

A Type 2 MFB is PR iff

$$\begin{bmatrix} Q_{M-1,0}(z) \\ Q_{M-1,1}(z) \end{bmatrix} = \frac{2/M}{P_{M-1,0}^2(z) - z^{-1}P_{M-1,1}^2(z)} \begin{bmatrix} P_{M-1,0}(z) \\ z^{-1}P_{M-1,1}(z) \end{bmatrix}$$

(if $P_{M-1}(z)$ and $Q_{M-1}(z)$ are FIR this is equivalent to $P_{M-1}(z) = A\sqrt{\frac{2}{M}}z^n$ and $Q_{M-1}(z) = \frac{1}{A}\sqrt{\frac{2}{M}}z^{-n}$ for some integer n and non-zero A), for $l \in \mathcal{R}(J)$ $(P_l(z), P_{M-2-l}(-z))$ and $(Q_l(z), Q_{M-2-l}(-z))$ form a PR filter bank, and additionally for even M, $P_{J,0}(z)Q_{J,0}(z) = \frac{1}{M}$. If $P_{J,0}(z)$ and $Q_{J,0}(z)$ are FIR the last condition becomes $P_{J,0}(z) = A\sqrt{\frac{1}{M}}z^n$ and $Q_{J,0}(z) = \frac{1}{A}\sqrt{\frac{1}{M}}z^{-n}$ for some integer n and non-zero A.

Proof: For a Type 1 MFB with M even, $\alpha(= M-1)$ is odd and hence from Theorem 10, for $l \in \mathcal{R}(J)$, $P_l(z), P_{M-1-l}(-z), Q_l(z)$ and $Q_{M-1-l}(z)$ form a two-channel PR filter bank. For M even, α is even and hence additionally we have for $l = J$, $P_{J,0}(z)Q_{J,0}(z) = \frac{1}{M}$, a solution in the FIR case being $P_{J,0}(z) = A\sqrt{\frac{1}{M}}z^n$ and $Q_{J,0}(z) = \frac{1}{A}\sqrt{\frac{1}{M}}z^{-n}$ for some integer n and non-zero A. For a Type 2 MFB be Type 2 (for both M even and M odd) $P_{M-1}(z)$, $P_{-1}(-z)$, $Q_{M-1}(z)$ and $Q_{-1}(-z)$ forms a two channel PR filter bank (scaled by $\sqrt{2}M$). From Eqn. 3.40 and Eqn. 3.41 $P_{-1}(-z) = -z^{-1}P_{M-1}(-z)$ and $Q_{-1}(z) = -zQ_{M-1}(z)$. Therefore from Lemma 18, $P_{M-1}(z) = A\sqrt{\frac{2}{M}}z^n$ and $Q_{M-1}(z) = \frac{1}{A}\sqrt{\frac{2}{M}}z^{-n}$ for some integer n and non-zero A.

For $l \in \mathcal{R}(J)$, and for $l = J$, the results follow as in the Type 1 case. □

In general a PR MFB has J associated two-channel PR filter banks. Pairs of polyphase components of the prototype filters that form two channel PR filter banks is depicted in Fig. 3.4.

Augmenting and Decomposing MFBs

From Fig. 3.4 one sees that a number of PR MFBs can be constructed from any given PR MFB - by choosing appropriate pairs of polyphase components and forming a new prototype filter. A Type 1 PR MFB with M even can be decomposed into a set of Type 1 PR MFBs. Each PR MFB has an even number of channels and the total number of channels in all the PR MFBs taken together is M. A Type 1 MFB with odd number of channels can be decomposed into a set of Type 1 MFBs with an even number of channels, and one Type 1 MFB with an odd number of channels. Similar decompositions can be described for Type 2 MFBs also.

In the special case of a Type 2 MFB with M even, the odd and even samples of $H(z)$ and $G(z)$ themselves give rise to a Type 1 and Type 2 MFB respectively with $M/2$ channels. For a Type 2 MFB with M even, define the following sub-sequences of the prototype filter:

$$\check{h}(n) = h(2n), \quad \breve{h}(n) = h(2n+1), \quad \check{g}(n) = g(2n) \quad and \breve{g}(n) = g(2n+1).$$

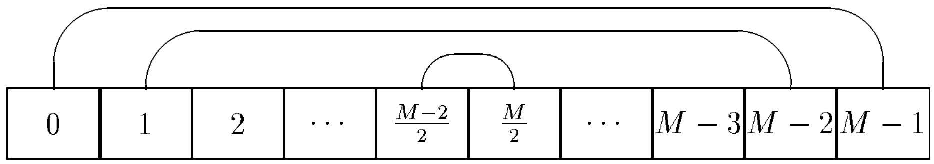
Since

$$H(z) = \sum_{l=0}^{\frac{M}{2}} z^{-2l} P_{2l}(z^M) + \sum_{l=0}^{\frac{M}{2}} z^{-2l-1} P_{2l+1}(z^M),$$

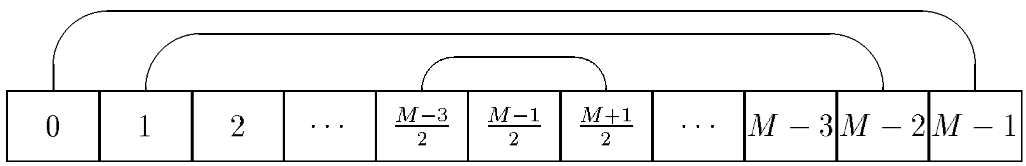
$$\check{H}(z) = \sum_{l=0}^{\frac{M}{2}-1} z^{-l} P_{2l}(z^{M/2}), \quad \text{and} \breve{H}(z) = \sum_{l=0}^{\frac{M}{2}-1} z^{-l} P_{2l+1}(z^{M/2}).$$

Similarly

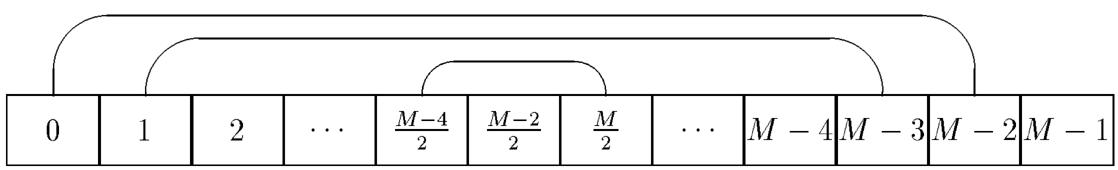
$$\check{G}(z) = \sum_{l=0}^{\frac{M}{2}-1} z^{-l} Q_{2l}(z^{M/2}) \quad \text{and} \breve{G}(z) = \sum_{l=0}^{\frac{M}{2}-1} z^{-l} Q_{2l+1}(z^{M/2}).$$



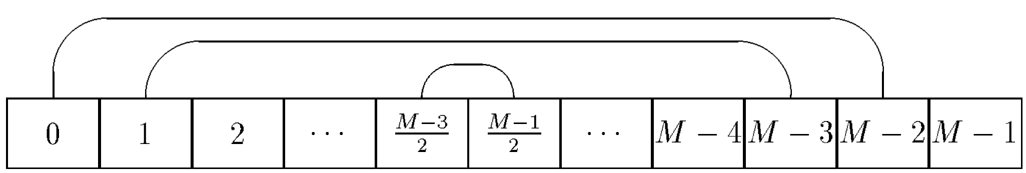
(a) Type 1 MFB - M even



(b) Type 1 MFB - M odd



(c) Type 2 MFB - M even



(d) Type 2 MFB - M odd

Figure 3.4: Two Channel PR Pairs in a PR MFB

Since $H(z)$ and $G(z)$ form prototype filters of a Type 2 PR MFB, $(\check{P}_l, \check{P}_{M/2-1-l})$ and $(\check{Q}_l, \check{Q}_{M/2-1-l})$ form a two channel PR filter bank. Therefore $\check{H}$ and $\check{G}$ form the prototype filters of an $M/2$ channel Type 1 PR MFB. Similarly $\breve{H}(z)$ and $\breve{G}(z)$ form analysis and synthesis prototype filters for a Type 2, $M/2$ channel filter PR MFB. If $M/2$ one can further decompose the Type 2 $M/2$ channel MFB into two $M/4$ channel MFBs. In summary every Type 2 PR MFB of 2^{K+1} channels can be decomposed into a set of $K-1$ Type 1 PR MFBs of 2^{K+1-i} channels, $i \in \{1, \ldots, K-1\}$, and one Type 2 PR MFB of two channels.

Existence of Causal MFBs

The minimal delay with which an M channel PR filter bank can be implemented is $(M-1)$ and occurs (without loss of generality) when the filter bank is causal. Do there exist causal MFBs which can be implemented with this minimal delay? To answer this question it suffices to only consider the J two channel PR pairs in an PR MFB. The polyphase components of $H(z)$ and $G(z)$ than are not part of the J PR pairs are assumed to be such that they *do not* make the MFB non-causal. Each of the J PR pairs can be implemented with a minimal delay of 1 (since $M=2$) when $p_l(n)$, $p_{\alpha-l}(n)$, $q_l(n-1)$ and $q_{\alpha-l}(n)$ are causal, $l \in \mathcal{R}(J)$; in other words when $h(Mn+l)$, $h(Mn+\alpha-l)$, $g(Mn-M-l)$ and $g(Mn-M-\alpha+l)$ are causal, or equivalently when $h(n)$ and $g(n-M-\alpha)$ are causal. Therefore the smallest delay with which an MFB can be implemented is $M+\alpha$, $\alpha \in \{M-1, M-2\}$.

Lemma 19 The minimum delay with which an MFB can be implemented is $M+\alpha$, ($\alpha = M-1$ for Type 1 and $\alpha = M-2$ for Type 2) and this occurs when all the J associated two channel filter banks are causal. Moreover, since $M+\alpha > M-1$, there are no causal MFBs.

3.5.2 Unitary Modulated Filter Banks

A PR filter bank is unitary iff $g_i(n) = h_i(-n)$ (Lemma 17). For unitary MFBs there is a similar relationship between the prototype filters.

Lemma 20 A PR MFB (Type 1 or Type 2) is unitary iff $g(n) = h(-n)$.

Proof: When $\alpha = (M-1)$ or $\alpha = (M-2)$, unitariness of MFB is equivalent to

$$g_i(n) = h_i(-n) = h(-n)c_{i,-n-\frac{\alpha}{2}} = h(-n)c_{i,n+\frac{\alpha}{2}} = g(n)c_{i,n+\frac{\alpha}{2}}.$$

Since the last expression is true for all i and n it is equivalent to the statement $g(n) = h(-n)$ for all n. □

A unitary MFB is therefore determined by a single prototype filter $h(n)$, and the synthesis prototype filter is the time-reverse of the analysis prototype filter. The minimal delay in an implementation of a unitary MFB is $N-1$, where N is the length of the prototype filter $h(n)$. It turns out that for a sub-class of unitary FIR MFBs, the prototype filter *can be chosen* to be linear-phase. The unitary MFBs discussed in the literature ([63, 51]) fall in this class. We give examples of prototype filters of unitary MFBs that are not linear-phase.

Unitariness of an MFB is equivalent to the J PR pairs being unitary. This is attractive in the FIR case the parameterization of unitary two-channel filter banks gives a parameterization of all FIR unitary MFBs.

For α odd the PR conditions for a unitary MFB are (from Eqn. 3.36 and Eqn. 3.37 with $g(n) = h(-n)$)

$$\sum_n h(Mn+n_1)h(Mn+n_1+2Ml) = \frac{2}{M}\delta(l) \quad \text{for all } n_1 \tag{3.43}$$

and

$$\sum_n (-1)^n h(Mn+n_1)h(Mn-n_1+\alpha+2Ml) = 0 \quad \text{for all } n_1. \tag{3.44}$$

Eqn. 3.43 and Eqn. 3.44 imply that $h(Mn+n_1)$ and $h(Mn-n_1+\alpha)$, considered as sequences in n, form analysis filters of a two-channel *unitary* filter bank. For even α, the same is true if $n_1 - \frac{\alpha}{2} \notin M\mathbf{Z}$. Otherwise from Eqn. 3.38

$$\sum_n h(2Mn+\frac{\alpha}{2})(2Mn+\frac{\alpha}{2}+2Ml) = \frac{1}{M}\delta(l). \tag{3.45}$$

Theorem 12 A Type 1 MFB is unitary iff for $l \in \mathcal{R}(J)$, $(P_l(z), P_{M-1-l}(-z))$ form analysis filters of a *unitary* filter bank (scaled by $\sqrt{\frac{2}{M}}$) and additionally for odd M, $P_{J,0}(z)$ is allpass. If the MFB is FIR the last condition becomes $P_{J,0}(z) = \pm\sqrt{\frac{1}{M}}z^n$ for some n. A Type 2 MFB is unitary iff $P^2_{M-1,0}(z) - z^{-1}P^2_{M-1,1}(z)$ is allpass (when $P_{M-1}(z)$ is FIR this becomes $P_{M-1}(z) = \pm\sqrt{\frac{2}{M}}z^n$ for some n), for $l \in \mathcal{R}(J)$, $(P_l(z), P_{M-2-l}(-z))$ form analysis filters of a *unitary* filter bank (scaled by $\sqrt{\frac{2}{M}}$), and additionally for even M, $P_{J,0}(z)$ is allpass (when $P_{J,0}(z)$ is FIR this becomes $P_{J,0}(z) = \pm\sqrt{\frac{1}{M}}z^n$ for some n).

For FIR unitary MFBs (once the delays are fixed) the only degrees of freedom are in the choice of the J unitary PR pairs.

Symmetry of Unitary MFB Prototype Filters

For the unitary FIR MFBs in [63] and [51], $h(n)$ is even symmetric and of length $2Mk$. We now show that $h(n)$ could be even-symmetric, odd-symmetric (hence not lowpass any more) or not symmetric at all. We also show that Type 2 MFBs cannot have odd symmetric prototypes.

Lemma 21 For a two-channel unitary filter bank $H_1(z) = A(z^2)z^{-1}H_0(-z^{-1})$, where $A(z)$ is an allpass function. In the FIR case $H_1(z) = \pm z^{-2k+1}H_0(-z^{-1})$ for some $k \in \mathbf{Z}$.

Proof: Since $H_p(z)$ is unitary on the unit circle, $\det H_p(z^{-1})$ must be an allpass function, say $1/A(z)$. Also $H_p^T(z^{-1})H_p(z) = I$ and hence

$$\begin{bmatrix} H_{0,0}(z^{-1}) & H_{1,0}(z^{-1}) \\ H_{0,1}(z^{-1}) & H_{1,1}(z^{-1}) \end{bmatrix} \begin{bmatrix} H_{0,0}(z) & H_{0,1}(z) \\ H_{1,0}(z) & H_{1,1}(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

implying that $H_{1,0}(z) = -A(z)H_{0,1}(z^{-1})$ and $H_{1,1}(z) = A(z)H_{0,0}(z^{-1})$. Therefore

$$\begin{aligned} H_1(z) &= H_{1,0}(z^2) + z^{-1}H_{1,1}(z^2) \\ &= A(z^2)z^{-1}\left[H_{0,0}(z^{-2}) - zH_{0,1}(z^{-2})\right] \\ &= A(z^2)z^{-1}H_0(-z^{-1}). \end{aligned}$$

If the filters are FIR $A(z) = \pm z^n$ for $n \in Z$, and therefore the result follows. □

Lemma 22 If $H(z) = A(z) + z^{-N}A(z^{-1})$, then, $h(n)$ is even-symmetric about $\frac{N}{2}$, and if $H(z) = A(z) - z^{-N}A(z^{-1})$, then $h(n)$ is odd symmetric about $\frac{N}{2}$.

For a unitary MFB (from Lemma 21) there exists allpass functions $A_l(z)$, $l \in \mathcal{R}(J)$, such that $P_l(z) = A_l(z^2)z^{-1}P_{\alpha-l}(z^{-1})$. Let $A_l(z^2) = \pm z^{-2k_l+2}$ for some integers $k_l \in \mathbf{Z}$ (note that includes all FIR unitary MFBs and IIR unitary MFBs where the determinant of the polyphase component matrix is $\pm z^{k_l-1}$).

If the MFB is Type 1 with M even

$$\begin{aligned} H(z) &= \sum_{l=0}^{M-1} P_l(z^M)z^{-l} \\ &= \sum_{l=0}^{J-1} P_l(z^M)z^{-l} \pm z^{-2k_lM+M}P_l(z^{-M})z^{-(M-1-l)} \\ &= \sum_{l=0}^{J-1} P_l(z^M)z^{-l} \pm z^{-2k_lM+1}P_l(z^{-M})z^{l}. \end{aligned}$$

If $k_l = k$ (independent of l), $H(z)$ is even or odd symmetric about $\frac{2Mk-1}{2}$ depending on whether $A_l(z) = +z^{-2k+2}$ or $A_l(z) = -z^{-2k+1}$ (from Lemma 22). However, if k_l

is not independent of l, $h(n)$ is *not* symmetric about any point. For a Type 1 MFB with M odd

$$H(z) = \sum_{l=0}^{J-1} P_l(z^M)z^{-l} \pm z^{-2k_lM+1}P_l(z^{-M})z^l + z^{-J}P_J(z^M).$$

In this case (for symmetry) one additionally requires $z^{-J}P_J(z^M)$ is symmetric about $\frac{2Mk-1}{2}$. This can always be accomplished since $P_{J,1}(z)$ can be chosen arbitrarily (see Theorem 12). Indeed if $P_{J,1}(z) = \pm z^{-k+1}P_{J,0}(z^{-1})$,

$$\begin{aligned} z^{-J}P_J(z^M) &= z^{-J}(P_{J,0}(z^{2M}) + z^{-M}P_{J,1}(z^{2M})) \\ &= z^{-J}P_{J,0}(z^{2M}) \pm z^{-M-J}z^{-2Mk+2M}P_{J,0}(z^{-2M})) \\ &= z^{-J}P_{J,0}(z^{2M}) \pm z^{-2Mk+1}z^{J}P_{J,0}(z^{-2M})), \end{aligned}$$

and hence is even or odd symmetric about $\frac{2Mk-1}{2}$. If $P_J(z)$ is FIR, then for symmetry it must be of the form

$$P_J(z) = \sqrt{\frac{1}{M}}z^{-2Mn} \pm \sqrt{\frac{1}{M}}z^{-2(k-1-n)M}.$$

If the MFB is Type 2 and M odd

$$\begin{aligned} H(z) &= \sum_{l=0}^{M-1} P_l(z^M)z^{-l} \\ &= \sum_{l=0}^{J-1} \left[P_l(z^M)z^{-l} \pm z^{-2k_lM+M}P_l(z^{-M})z^{-(M-2-l)}\right] + z^{-M+l}P_{M-1}(z^M) \\ &= \sum_{l=0}^{J-1} \left[P_l(z^M)z^{-l} \pm z^{-2k_lM+2}P_l(z^{-M})z^{l}\right] + z^{-M+l}P_{M-1}(z^M). \end{aligned}$$

If $k_l = k$ (independent of l), but for the last term we have even or odd symmetry about $kM-1$. Since $h(kM-1)$ is a point in $P_{M-1}(z^M)$ and $P_{M-1}(z^M)$ is a delay one cannot have *odd* symmetry about $kM-1$. For even symmetry we must have $P_{M-1}(z^M) = \pm\sqrt{\frac{2}{M}}z^{-(k-1)M}$. However, if M is even

$$H(z) = \sum_{l=0}^{J-1} \left[P_l(z^M)z^{-l} \pm z^{-2k_lM+2}P_l(z^{-M})z^{l}\right] + z^{-M+1}P_{M-1}(z^M) + z^{-J}P_J(z^M).$$

One cannot have odd symmetry. For even symmetry ($k_l = k$ independent of l and) we must choose $P_{M-1}(z^M) = \pm\sqrt{\frac{2}{M}}z^{-(k-1)M}$ and $P_{J,1}(z) = z^{-k+1}P_{J,0}(z^{-1})$. With this choice $z^{-J}P_J(z)$ is even symmetric about $kM-1$.

$$\begin{aligned}
z^{-J}P_J(z^M) &= z^{-J}(P_{J,0}(z^{2M}) + z^{-M}P_{J,1}(z^{2M})) \\
&= z^{-J}P_{J,0}(z^{2M}) + z^{-M-J}z^{-2Mk+2M}P_{J,0}(z^{-2M})) \\
&= z^{-J}P_{J,0}(z^{2M}) + z^{-2Mk+2}z^{J}P_{J,0}(z^{-2M}))
\end{aligned}$$

Furthermore in the FIR case $P_J(z) = \sqrt{\frac{1}{M}}z^{-2Mn} \pm \sqrt{\frac{1}{M}}z^{-2(k-1-n)M}$.

We summarize the above results for unitary FIR MFBs. For MFBs based on a lowpass prototype filter only cases corresponding to even symmetric prototype filters are possible.

Type 1 and M even, even symmetry about $\frac{2Mk-1}{2}$

$$P_l(z) = z^{-2k+1}P_{M-1-l}(z^{-1}) \text{ for } l \in \mathcal{R}(J)$$

Type 1 and M even, odd symmetry about $\frac{2Mk-1}{2}$

$$P_l(z) = -z^{-2k+1}P_{M-1-l}(z^{-1}) \text{ for } l \in \mathcal{R}(J)$$

Type 1 and M odd, even symmetry about $\frac{2Mk-1}{2}$

$$P_l(z) = z^{-2k+1}P_{M-1-l}(z^{-1}) \text{ for } l \in \mathcal{R}(J)$$

$$P_J(z) = \pm\sqrt{\frac{1}{M}}(z^{-2Mn} + z^{-2(k-1-n)M})$$

Type 1 and M odd, odd symmetry about $\frac{2Mk-1}{2}$

$$P_l(z) = -z^{-2k+1}P_{M-1-l}(z^{-1}) \text{ for } l \in \mathcal{R}(J)$$

$$P_J(z) = \pm\sqrt{\frac{1}{M}}(z^{-2Mn} - z^{-2(k-1-n)M})$$

Type 2 and M even, even symmetry about $Mk-1$

$$P_l(z) = z^{-2k+1}P_{M-2-l}(z^{-1}) \text{ for } l \in \mathcal{R}(J)$$

$$P_{M-1}(z) = \pm\sqrt{\frac{2}{M}}z^{-(k-1)}$$

$$P_J(z) = \pm\sqrt{\frac{1}{M}}(z^{-2Mn} + z^{-2(k-1-n)M})$$

Type 2 and M odd, even symmetry about $Mk-1$

$$P_l(z) = z^{-2k+1}P_{M-2-l}(z^{-1}) \text{ for } l \in \mathcal{R}(J)$$

$$P_{M-1}(z) = \pm\sqrt{\frac{2}{M}}z^{-(k-1)}$$

Parameterization of FIR Unitary Modulated Filter Banks

All FIR unitary MFBs can be parameterized by parameterizing the J two-channel unitary PR pairs using Fact 5. Each of the J unitary PR pairs are independent of each other and hence the length of the filters in each PR pair could be different. However, the filters in any one given pair has to have even length (since the two-channel filter bank is unitary). Moreover, using Lemma 8 and Lemma 9 the filters in the two-channel PR pairs could be shifted giving rise to a rich set of possibilities for the length of the prototype filter $h(n)$. In fact by appropriate choice of the shifts and the lengths one can construct $h(n)$ of any prescribed length N (by length here we mean the size of the support of $h(n)$ - i.e., the support of $h(n)$ is of the form $[K_1, K_1+N-1]$). This result has been obtained independently for MFBs with the choice of modulation given in Eqn. 3.29 (see [66]).

For the purposes of the parameterization we will assume that the prototype filter is lowpass (i.e., this precludes the case of odd symmetry considered earlier). More precisely for $l \in \mathcal{R}(J)$ we make the choice $P_{\alpha-l}(z) = z^{-2k_l+1}P_l(z^{-1})$, where $2k_l$ is the length of the filter $p_l(n)$. This implies that $P_{\alpha-l}(z)$ is the reflection of $P_l(z)$. Therefore

$P_{\alpha-l,0}(z)$ is the reflection of $P_{l,1}(z)$ and $P_{\alpha-l,1}(z)$ is the reflection of $P_{l,0}(z)$. That is

$$P_{\alpha-l,0}(z) = z^{-k_l+1}P_{l,1}(z) \quad \text{and} \quad P_{\alpha-l,1}(z) = z^{-k_l+1}P_{l,0}(z). \tag{3.46}$$

The polyphase component matrix of the l^{th} two channel unitary PR pair is given by

$$\begin{bmatrix} P_{l,0}(z) & P_{l,1}(z) \\ P_{\alpha-l,0}(z) & -P_{\alpha-l,1}(z) \end{bmatrix} = \begin{bmatrix} P_{l,0}(z) & P_{l,1}(z) \\ P_{l,1}^R(z) & -P_{l,0}^R(z). \end{bmatrix}$$

The degree $k_l - 1$ polynomial matrix above, when scaled by $\sqrt{\frac{M}{2}}$ is unitary on the unit circle. From Eqn 3.24 (Fact 5) there exists k_l Givens' rotation parameters $\theta_{l,k}$ such that

$$\begin{bmatrix} P_{l,0}(z) & (-1)^{k_l}P_{l,1}^R(z) \\ P_{l,1}(z) & -(-1)^{k_l}P_{l,0}^R(z) \end{bmatrix} = \sqrt{\frac{2}{M}}\left\{\prod_{k=k_l-1}^{1}\begin{bmatrix} \cos\theta_{l,k} & z^{-1}\sin\theta_{l,k} \\ \sin\theta_{l,k} & -z^{-1}\cos\theta_{l,k} \end{bmatrix}\right\} \times \begin{bmatrix} \cos\theta_{l,0} & \sin\theta_{l,0} \\ \sin\theta_{l,0} & -\cos\theta_{l,0} \end{bmatrix}. \tag{3.47}$$

A unitary MFB is determined by $N_p = \sum_{l=0}^{J-1} k_l$ angle parameters. For simplicity assume that that $P_l(z)$ is a polynomial in z^{-1}. With this assumption we will characterize the filter lengths for the resulting unitary MFBs. By relaxing this assumption one can obtain FIR unitary MFBs with filters of any given length N.

Type 1 FIR MFBs: First consider the case M even. Since $p_l(n) = h(Mn + l)$ and $p_{M-1-l}(2k_l - 1)$ is non-zero, $h(2k_lM - 1 - l)$ is non-zero. Therefore (since we assume $h(n)$ starts at $n = 0$)

$$N = \max_{l\in\mathcal{R}(J)} 2k_lM - l. \tag{3.48}$$

If we choose $k_l = k - 1$, $l \in \left\{0, 1, \ldots, \hat{l} - 1\right\}$ and $k_l = k$, $l \in \left\{\hat{l}, \ldots, J - 1\right\}$ then $N = 2Mk - \hat{l}$ and $N_p = Jk - \hat{l}$. When M is odd, we have additionally $P_J(z)$ which is not part of the J lattices. From Theorem 12 $P_J(z)$ is a pure delay. Therefore N could made arbitrarily large by appropriate choice of $P_J(z)$.

Type 2 FIR MFBs: For Type 2 unitary FIR MFBs, $P_{M-1}(z)$ is a delay. If M is odd the rest of the polyphase components of $h(n)$ are part of the J lattices. Since $p_l(n) = h(Mn+l)$ and $p_{M-2-l}(2k_l - 1)$ is non-zero, $h(2k_lM - 2 - l)$ is non-zero. Therfore

$$N = \max_{l \in \mathcal{R}(J)} 2k_lM - 1 - l. \tag{3.49}$$

if we choose $k_l = k-1,\ l \in \left\{0, 1, \ldots, \hat{l}-1\right\}$ and $k_l = k,\ l \in \left\{\hat{l}, \ldots, J-1\right\}$ then $N = 2Mk - 1 - \hat{l}$ and $N_p = Jk - \hat{l}$. Furthermore if M is even $P_{J,0}(z)$ is a pure delay an can be chosen to increase N arbitrarily (while still having N_p fixed).

If we assume that the delays (i.e., $P_{M-1}(z)$ in the Type 2 case and $P_J(z)$ in the Type 1 case with M odd or Type 2 case with M even) do not determine N, the lengths and the number of parameters for MFBs such that $P_l(z)$ is a polynomial in z^{-1} is given in Table 3.1.

Table 3.1: Unitary FIR MFBs - Lengths and # of. Parameters

M	α	N	N_p
even	$M-1$	$2Mk_{\hat{l}} - \hat{l}$	$Jk_{\hat{l}} - \hat{l}$
odd	$M-1$	$2Mk_{\hat{l}} - \hat{l}$	$Jk_{\hat{l}} - \hat{l}$
even	$M-2$	$2Mk_{\hat{l}} - 1 - \hat{l}$	$Jk_{\hat{l}} - \hat{l}$
odd	$M-2$	$2Mk_{\hat{l}} - 1 - \hat{l}$	$Jk - \hat{l}$

We now define a class of MFBs that we call canonical MFBs. These correspond to a specific choice of the delays and the constraint that $P_l(z)$ is a polynomial in z^1. The MFBs in [63] and [51] fall under this class.

Definition 8 An MFB is canonical if

1. The number of lattice parameters is Jk, for some k, with each lattice having k parameters.
2. If the MFB is Type 1 with M odd,

$$P_J(z) = \sqrt{\frac{1}{M}}\left[z^{-Mk+J+1} + z^{-Mk-J}\right]$$

3. If the MFB is Type 2 with M even,

$$P_J(z) = \sqrt{\frac{1}{M}}\left[z^{-Mk+J+2} + z^{-Mk-J}\right]$$

4. If the MFB is Type 2,

$$P_{M-1}(z) = \sqrt{\frac{2}{M}} z^{-Mk+1}.$$

The relationships between α, M and N and N_p for canonical MFBs is shown in Table 3.2. The prototype filter of a canonical MFB is always even symmetric and

Table 3.2: Canonical FIR MFBs - Lengths and # of. Parameters

M	α	N	N_p
even	$M-1$	$2Mk$	$\frac{M}{2}k$
odd	$M-1$	$2Mk$	$\frac{M-1}{2}k$
even	$M-2$	$2Mk-1$	$\frac{M-2}{2}k$
odd	$M-2$	$2Mk-1$	$\frac{M-3}{2}k$

therefore lowpass.

We now show that for unitary FIR MFBs, though the prototype filter can be chosen to be linear-phase (even or odd symmetric), the filters themselves cannot be chosen to be linear-phase. This implies that despite their many advantages, unitary FIR MFBs *cannot* be used in certain image processing applications where the the filters h_i (not the prototype) have to be linear phase.

Theorem 13 There do not exist linear-phase, unitary FIR MFBs

Proof: The result is proved for canonical MFBs only. For non-canonical MFBs a similar proof can be employed. For a canonical MFB Type 1 MFB of length $N = 2Mk$, $h(n) = h(2Mk-1-n)$. Now

$$h_0(0) = h(0)c_{0,\frac{M-1}{2}} = h(0)\cos(\frac{\pi}{4} - \frac{\pi}{4M})$$

and

$$\begin{aligned} h_0(2Mk-1) &= h(0)c_{0,2Mk-1-\frac{M-1}{2}} \\ &= (-1)^k h(0)c_{0,\frac{M-1}{2}+1} \\ &= (-1)^k h(0)\cos(\frac{\pi}{4}+\frac{\pi}{4M}). \end{aligned}$$

Hence for $M \geq 2$ $h_0(0) \neq h_0(N-1)$. Therefore, the MFB does not have linear-phase filters. When the MFB is Type 2, $N = 2Mk-1$ and one can similarly show that

$$h_0(0) = h(0)\cos(\frac{\pi}{4}-\frac{\pi}{2M}) \quad \text{and} \quad h_0(2Mk-2) = (-1)^k h(0)\cos(\frac{\pi}{4}+\frac{\pi}{2M}).$$

Once again for $M \geq 2$, $h_0(0) \neq h_0(N-1)$ and therefore the MFB cannot be linear phase. □

Remark: If $M = 2$ and $N = 2Mk-1$, $h(N-1) = (-1)^k h(0)\cos(\frac{\pi}{2}) = 0$. This shows that the length is less than N, and that there exists no Type 2 MFBs for $M = 2$. This is in accordance with the fact that a two channel unitary MFB must have filters that are necessarily even in length, and from Table 3.2 this has to be Type 1.

3.5.3 Efficient Design of Unitary Modulated Filter Banks

The design problem is to minimize the stopband energy of the prototype filter. If $\Omega = \left\{\omega \mid \pi \geq |\omega| \geq \frac{\pi}{2M}+\delta\right\}$, then the problem is

$$\min_{\vartheta} \frac{1}{2\pi}\int_{\Omega} |H(\omega)|^2 \, d\omega.$$

Here $\delta > 0$, is a parameter controlling the transition width of the prototype filter (and hence $h_i(n)$) and $\vartheta \in \mathbb{R}^{N_p}$ is a vector that is obtained by stacking the N_p angle parameters $\theta_{l,k}$ of the MFB. The design problem reduces to an unconstrained minimization problem.

Computation of the function

Let h be the length N vector of the coefficients $h(n)$. Then since

$$H(\omega)H^{\star}(\omega) = \left[\sum_k h(k)e^{-\imath\omega k}\right]\left[\sum_l h(l)e^{\imath\omega l}\right]$$

the objective function can be written in the form

$$f(\vartheta) = \frac{1}{2}h(\vartheta)^T P h(\vartheta),$$

where P is a positive definite matrix with entries given by

$$(P)_{i,j} = \begin{cases} -\frac{\sin(\omega_s(i-j))}{i-j} & \text{if } i \neq j \\ \pi - \omega_s & \text{if } i = j, \end{cases}$$

with $\omega_s = \frac{\pi}{2M} + \delta$ (the stopband edge). The cost of computing $f(\vartheta)$ is equivalent to the cost of computing h (J lattices), a symmetric Toeplitz matrix-vector multiplication Ph and an inner-product. Also, since P is Toeplitz, only one row of P need be stored. Moreover, if the MFB is canonical, then h is a symmetric vector and therefore the quadratic form $h^T Ph$ can be reduced to another quadratic form with roughly half the dimension. However the Toeplitz structure of P becomes a Toeplitz plus Hankel structure of the matrix in the new quadratic form. Typically the l^{th} lattice is implemented using the normalized lattice. One can also use the denormalized lattice with

$$\begin{bmatrix} P_{l,0}(z) \\ P_{l,1}(z) \end{bmatrix} = \beta \prod_{k=k_l-1}^{1} \begin{bmatrix} \gamma_{l,k} & z^{-1} \\ 1 & -z^{-1}\gamma_{l,k} \end{bmatrix} \begin{bmatrix} \gamma_{l,0} & 1 \\ 1 & -\gamma_{l,0} \end{bmatrix}.$$

Here $\beta = \prod_{k=k_l-1}^{1} \sin(\theta_{l,k})$ and $\gamma_{l,k} = \cot(\theta_{l,k})$. This lattice is the most efficient way to compute h (see Section 5.1) and hence $f(\vartheta)$. Let $\zeta \in \mathbb{R}^{N_p}$ be the vector of γ parameters. Then $h(\zeta)$ is another parameterization of h. With this parameterization trigonometric evaluations are eliminated, reducing computational complexity.

Computation of the gradient

The gradient of f is given by

$$\frac{\partial f}{\partial \vartheta_i} = \frac{\partial h^T}{\partial \vartheta_i} Ph.$$

Since Ph is available from the evaluation of f it suffices to compute $\frac{\partial h^T}{\partial \vartheta_i}$. If $\vartheta_i = \theta_{l,k}$, then for components of h that do not depend on the l^{th} lattice $\frac{\partial h^T}{\partial \vartheta_i} = 0$ and hence need not be computed. Furthermore, the same lattice that computed h can compute $\frac{\partial h^T}{\partial \vartheta_i}$ (i.e., the same code does the job). This follows from the observation

$$\frac{\partial}{\partial \theta} \begin{bmatrix} \cos(\theta) & z^{-1}\sin(\theta) \\ \sin(\theta) & -z^{-1}\cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta') & z^{-1}\sin(\theta') \\ \sin(\theta') & -z^{-1}\cos(\theta') \end{bmatrix},$$

where $\theta' = \frac{\pi}{2} + \theta$. One merely has to replace the angle corresponding to ϑ_i, say $\theta_{l,k}$ by $\theta_{l,k} + \frac{\pi}{2}$. However, if the ζ parameterization (more efficient) is used, then, the same code cannot be used. In fact, for the derivative evaluation with respect to $\gamma_{l,k}$, one has to replace β by $\beta\sqrt{1+\gamma_{l,k}^2}$, and $\begin{bmatrix} \gamma_{l,k} & z^{-1} \\ 1 & -z^{-1}\gamma_{l,k} \end{bmatrix}$ by $\begin{bmatrix} 1 & -\gamma_{l,k}z^{-1} \\ -\gamma_{l,k} & -z^{-1} \end{bmatrix}$.

Computation of Hessian

It turns out that the Hessian of the objective function can also be computed efficiently. In fact,

$$\frac{\partial^2 f}{\partial \vartheta_i \partial \vartheta_j} = \frac{\partial h^T}{\partial \vartheta_i \partial \vartheta_j} Ph + \frac{\partial h^T}{\partial \vartheta_i} P \frac{\partial h}{\partial \vartheta_j}.$$

Again Ph is available from the evaluation of f and furthermore the first derivatives of h are available from the evaluation of the gradient. Only the second derivatives have to be evaluated. If ϑ_i and ϑ_j are from different lattices, $\frac{\partial h^T}{\partial \vartheta_i \partial \vartheta_j} = 0$. Moreover, in the same lattice, if $i \neq j$, one has the corresponding θ's by $\theta + \frac{\pi}{2}$, and if $i = j$, the corresponding angle has to be replace by $\theta + \pi$. In summary, the second derivatives are sparse, and the non-zero entries can be evaluated by the *same* lattice, with appropriate angles replaced. If the ζ parameterization is used, then if $i \neq j$, the parameters may be replaced as in the gradient computation. However, if $i = j$, then if $\gamma_{l,k}$

is the corresponding parameter, it turns out we have to replace β by $\beta\sqrt{1+\gamma_{l,k}^2}$ and $\begin{bmatrix} \gamma_{l,k} & z^{-1} \\ 1 & -z^{-1}\gamma_{l,k} \end{bmatrix}$ by $\begin{bmatrix} \tau_{l,k} & z^{-1} \\ 1 & -z^{-1}\tau_{l,k} \end{bmatrix}$ where, $\tau_{l,k} = \frac{2\gamma_{l,k}^2-1}{-3\gamma_{l,k}}$. A program that implements the above algorithm for Type 1 *canonical* unitary MFBs can be obtained from the author.

3.5.4 Example Designs of Unitary FIR MFBs

Example 4 (Type 1, M even) This example designs an eight channel Type 1 canonical MFB. The filter length is $N = 48$, the number of lattices is $J = 4$, and there are three parameters $\gamma_{l,k}$ per lattice. The stopband edge was set at $\omega_s = .0909\pi$. Fig. 3.5 gives the frequency response of the MFB filters, Table 3.3 and Table 3.4 give the γ parameters and the prototype filter for this design. Since the prototype filter is even symmetric only the first 24 coefficients are given.

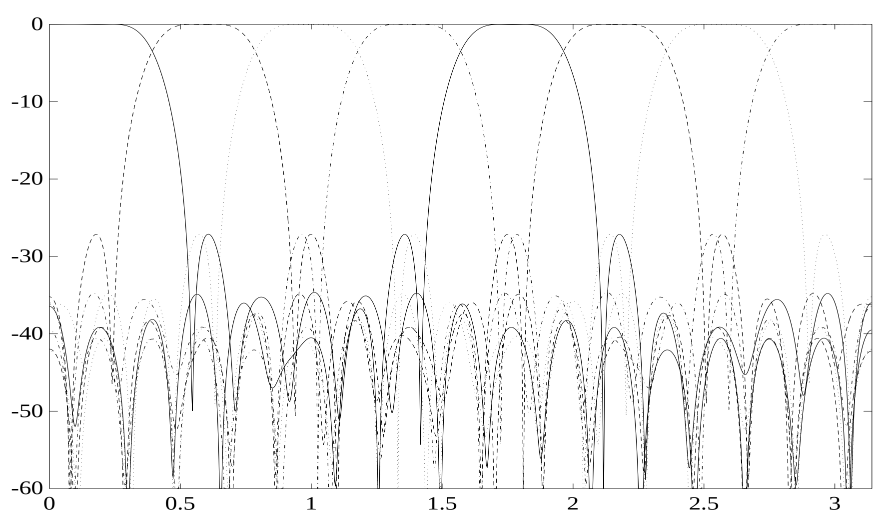


Figure 3.5: Type 1 MFB Design: $M = 8$, $N = 48$

Example 5 (Type 1, M odd) For this example $M = 11$ and $N = 88$. Therefore $J = 5$ and $k = 4$. The frequency response of the designed MFB with $\omega_s = .0667\pi$ is shown

Table 3.3: Gamma Parameters - Type 1 MFB Example : $M = 8$, $N = 48$

	$l=0$	$l=1$	$l=2$	$l=3$
$\gamma_{l,0}$	0.22106691396256	0.19045044278156	0.13283939888893	0.04844106726267
$\gamma_{l,1}$	-0.40508567351029	-0.31272550929392	-0.23205728663771	-0.16428828280244
$\gamma_{l,2}$	0.28065212094880	0.40800918265943	0.58280081466750	0.83453449768568

Table 3.4: Prototype Filter - Type 1 MFB Example : $M = 8$, $N = 48$

n	$h(n)$	n	$h(n)$	n	$h(n)$
0	-0.01094940111461	8	-0.03901413991666	16	0.21858078571915
1	-0.01054751991073	9	-0.02585118266698	17	0.25975862557855
2	-0.00749423902911	10	-0.01285900575376	18	0.29852452479804
3	-0.00251288230331	11	-0.00301111854606	19	0.33408443766633
4	-0.05187504002916	12	0.06216045013480	20	0.36306647567327
5	-0.05641578546568	13	0.09680114379708	21	0.38484852624944
6	-0.05538196581056	14	0.13573705731223	22	0.40031763857113
7	-0.04952980488279	15	0.17648113513392	23	0.40863388869049

in Fig. 3.6. The corresponding gamma parameters and prototype filter coefficients are given in Table 3.5 and Table 3.6 respectively.

Example 6 (Type 2, M even) This example designs a canonical 8 channel MFB with $N = 47$. There are $J = 3$ lattices with $k = 3$ free parameters each. This filter is designed with the same specifications as Example 4 (i.e., $\omega_s = .0909\pi$). Fig. 3.7 gives the frequency response of the MFB, while Table 3.7 and Table 3.8 give the gamma parameters and the prototype filter coefficients.

Table 3.5: Gamma Parameters - Type 1 MFB Example : $M = 11$, $N = 88$

l	$\gamma_{l,0}$	$\gamma_{l,1}$	$\gamma_{l,2}$	$\gamma_{l,3}$
0	0.95922501350723	0.33292575590657	-0.36308534180717	-0.78813894287151
1	1.14848782997282	0.34025833513668	-0.39733580508322	-0.67370722594167
2	1.34356464811682	0.35548249255366	-0.41421181508292	-0.58079953157983
3	1.55523708095043	0.37307621658359	-0.41910162231552	-0.49866071123177
4	1.79403865277665	0.39026980112040	-0.41525386051437	-0.42248710413672

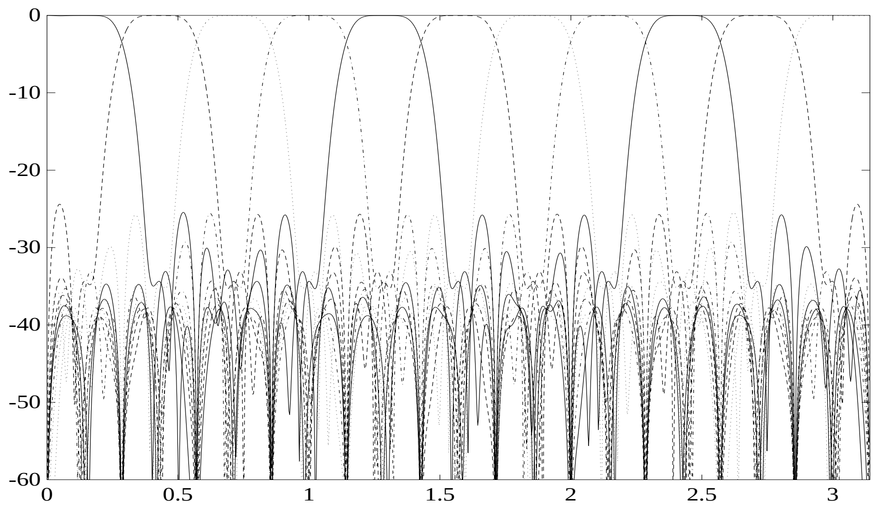


Figure 3.6: Type 1 MFB Design: $M = 11$, $N = 88$

Table 3.6: Prototype Filter - Type 1 MFB Example : $M = 11$, $N = 88$

	$h(n)$	$h(n)$	$h(n)$	$h(n)$
0	0.01969721844827	-0.02499206342540	-0.03243647593309	0.18274079922414
1	0.02137204874066	-0.03172305107874	-0.02355121968944	0.20725358238202
2	0.02202020794328	-0.03791361174722	-0.01191145557027	0.23126687556456
3	0.02162443215865	-0.04336502088812	0.00243882358615	0.25420491156901
4	0.02021000555374	-0.04783579275168	0.01935524752086	0.27551917038051
5	0.00000000000000	0.00000000000000	0.00000000000000	0.30151134457776
6	-0.01126508925683	-0.02666374700324	0.05708538434071	0.31422831958023
7	-0.01390426734517	-0.02788322206259	0.08035807284866	0.32774339534784
8	-0.01638939218455	-0.02821867321410	0.10512528845819	0.33786590214143
9	-0.01860885956551	-0.02762158226743	0.13093700150514	0.34437278957197
10	-0.02053451293587	-0.02605443256116	0.15728011192233	0.34716241186711

Table 3.7: Gamma Parameters - Type 2 MFB Example : $M = 8$, $N = 47$

l	$l = 0$	$l = 1$	$l = 2$
$\gamma_{l,0}$	0.20988792781530	0.17210537565669	0.10039481526196
$\gamma_{l,1}$	-0.34379243295169	-0.25485497854146	-0.17794166411200
$\gamma_{l,2}$	0.32742692839520	0.47119431442124	0.68225141219292

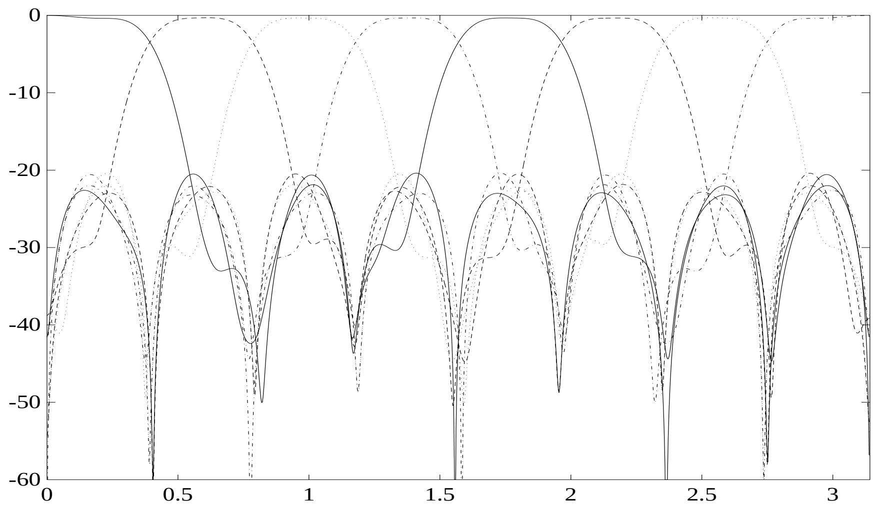


Figure 3.7: Type 1 MFB Design: $M = 8$, $N = 47$

Table 3.8: Protype Filter - Type 2 MFB Example : $M = 8$, $N = 47$

n	$h(n)$	n	$h(n)$	n	$h(n)$
0	-0.01039043657002	8	-0.03173360426080	16	0.23630041086624
1	-0.00892718132451	9	-0.01894585959823	17	0.27786900073705
2	-0.00493138246397	10	-0.00722810151190	18	0.31666554094133
3	0.00000000000000	11	0.00000000000000	19	0.35355339059327
4	-0.04911989181013	12	0.07199676091875	20	0.37689530886127
5	-0.05187043862195	13	0.11008290430172	21	0.39691483978902
6	-0.04950468889839	14	0.15119308952694	22	0.40955716141868
7	0.00000000000000	15	0.00000000000000	23	0.50000000000000

Example 7 (Type 2, M odd) Here we design a Type 2, 11 channel MFB. In this case $J = 5$ and $k = 4$ making it a total of 20 free parameters. Fig. 3.8 shows the frequency response of the designed filters, while Table 3.9 and Table 3.10 shows the gamma parameters and the prototype filter coefficients.

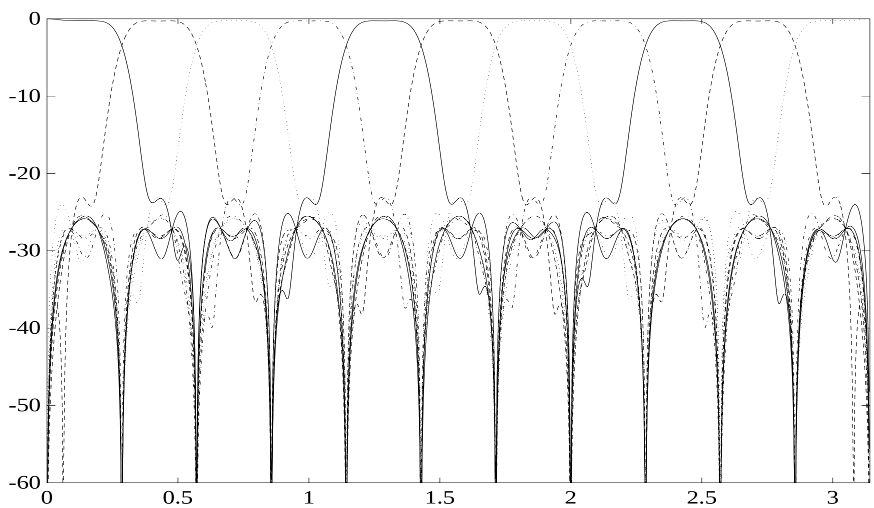


Figure 3.8: Type 1 MFB Design: $M = 11$, $N = 87$

3.6 Unitary FIR Filter Banks and Symmetry

In a filter bank the passbands of the filters must approximate ideal filters. Since the ideal filters occupy adjacent bands, a number of transformations relate these filter

Table 3.9: Gamma Parameters - Type 2 MFB Example : $M = 11$, $N = 87$

l	$\gamma_{l,0}$	$\gamma_{l,1}$	$\gamma_{l,2}$	$\gamma_{l,3}$
0	1.01917226433288	0.31848223733700	-0.36370279791256	-0.70406341146672
1	1.20775811153539	0.33052799038495	-0.38889232047644	-0.60554032549770
2	1.40874687517846	0.34722794932289	-0.40037388511515	-0.52151939318310
3	1.63125386160919	0.36528515505720	-0.40176767683512	-0.44573164422369
4	1.88604229340477	0.38258303194103	-0.39561233194538	-0.37461368409693

Table 3.10: Protype Filter - Type 2 MFB Example : $M = 11$, $N = 87$

	$h(n)$	$h(n)$	$h(n)$	$h(n)$
0	0.01817412116120	-0.02581318793905	-0.02993489301986	0.19329229654630
1	0.01935076601691	-0.03195619713849	-0.01989651910233	0.21831701402564
2	0.01960276112542	-0.03758778941234	-0.00704939334058	0.24249394973393
3	0.01893120798774	-0.04247220997897	0.00848937401420	0.26524202548555
4	0.01737197198091	-0.04637303098736	0.02651701802418	0.28603129000684
5	-0.00921080722403	-0.02458748202494	0.04411244636111	0.30694233355932
6	-0.01160531075713	-0.02603654218300	0.06609778677902	0.32256627043644
7	-0.01391503432647	-0.02668171981398	0.08990604889853	0.33493196634007
8	-0.01602205427734	-0.02645910371727	0.11515456321124	0.34372514780063
9	-0.01783223680355	-0.02532760048758	0.14137457806476	0.34873672792452
10	0.00000000000000	0.00000000000000	0.00000000000000	0.42640143271122

responses. For example, the filters could be related by modulation as seen in the theory of modulated filter banks. In Fig. 3.2, the response of the $(M-1-i)^{th}$ filter can be obtained by shifting the response of the i^{th} filter by π. Therefore, they could be related as

$$h_{M-1-i}(n) = (-1)^n h_i(-n) \quad ; \quad H_{M-1-i}(z) = H_i(-z) \quad ; \quad H_{M-1-i}(\omega) = H_i(\omega+\pi) \tag{3.50}$$

or

$$h_{M-1-i}(n) = (-1)^n h_i(N-1-n) \quad ; \quad H_{M-1-i}(z) = H_i^R(-z) \quad ; \quad H_{M-1-i}(\omega) = H_i^*(\omega+\pi), \tag{3.51}$$

where for causal $H(z)$, $H^R(z)$ denotes its reflection (i.e., causal time-reversed sequence). The former will be called *pairwise-shift* (or PS) symmetry, while the latter will be called *pairwise-conjugated-shift* (or PCS) symmetry. Both these symmetries relate pairs of filters in the filter bank. Another type of symmetry occurs when the filters themselves are symmetric or linear-phase. The only type of linear-phase symmetry we will consider is

$$h_i(n) = \pm h_i(N-1-n) \quad ; \quad H_i(z) = \pm H_i^R(z)$$