where the filters are all of fixed length N, and the symmetry is about $\frac{N-1}{2}$. For these three symmetries, we will now study the form of $H_p(z)$. We will only consider the case of even M. For even M, if the filters are linear-phase half the filters are symmetric, while the other half are anti-symmetric [76].

Let J denote the exchange matrix with ones on the anti-diagonal. Post-multiplying a matrix A by J is equivalent to reversing the order of the columns of A, and pre-multiplying is equivalent to reversing the order of the rows of A. Let V denote the *sign-alternating* matrix, the diagonal matrix of alternating ± 1's. Post-multiplying by V, alternates the signs of the columns of A, while premultiplying alternates the signs of the rows of A. The polyphase components of $H(z)$ are related to the polyphase components of $H^R(z)$ by reflection and reversal of the ordering of the components. For, if $H(z)$ is of length Mm, and $H(z) = \sum_{l=0}^{M-1} z^{-l} H_l(z^M)$, then

$$\begin{aligned} H^R(z) &= z^{-Mm+1} \sum_{l=0}^{M-1} z^{l} H_l(z^{-M}) \\ &= \sum_{l=0}^{M-1} z^{-(M-1-l)} (z^{-mM+M} H_l(z^{-M})) \\ &= \sum_{l=0}^{M-1} z^{-l} H^R_{M-1-l}(z^M). \end{aligned} \tag{3.52}$$

Therefore,

$$(H^R)_l(z) = (H_{M-1-l})^R(z) \tag{3.53}$$

and for linear-phase $H(z)$, since $H^R(z) = \pm H(z)$

$$H_l(z) = \pm (H^R)_{M-1-l}(z). \tag{3.54}$$

Lemma 23 For even M, $H_p(z)$ is of the following forms for the different types of symmetry:

PS Symmetry:

$$H_p(z) = \begin{bmatrix} W_0(z) & W_1(z) \\ JW_0(z)V & (-1)^{M/2} JW_1(z)V \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0(z) & W_1(z) \\ W_0(z)V & (-1)^{M/2}W_1(z)V \end{bmatrix} \tag{3.55}$$

PCS Symmetry:

$$\begin{aligned} H_p(z) &= \begin{bmatrix} W_0(z) & W_1(z)J \\ JW_1^R(z)V & (-1)^{M/2}JW_0^R(z)JV \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0(z) & W_1(z)J \\ W_1^R(z)V & (-1)^{M/2}W_0^R(z)JV \end{bmatrix} \end{aligned} \tag{3.56}$$

Linear Phase:

$$\begin{aligned} H_p(z) &= \begin{bmatrix} W_0(z) & D_0W_0^R(z)J \\ W_1(z) & D_1W_1^R(z)J \end{bmatrix} \\ &= \begin{bmatrix} W_0(z) & D_0W_0^R(z) \\ W_1(z) & D_1W_1^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \end{aligned} \tag{3.57}$$

or

$$\begin{aligned} H_p(z) &= Q \begin{bmatrix} W_0(z) & W_0^R(z)J \\ W_1(z) & -W_1^R(z)J \end{bmatrix} \\ &= Q \begin{bmatrix} W_0(z) & W_0^R(z) \\ W_1(z) & -W_1^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}. \end{aligned} \tag{3.58}$$

Linear Phase and PCS:

$$\begin{aligned} H_p(z) &= \begin{bmatrix} W_0(z) & DW_0^R(z)J \\ JDW_0(z)V & (-1)^{M/2}JW_0^R(z)JV \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0(z) & DW_0^R(z)J \\ DW_0(z)V & (-1)^{M/2}W_0^R(z)JV \end{bmatrix}. \end{aligned} \tag{3.59}$$

Linear Phase and PS:

$$\begin{aligned} H_p(z) &= \begin{bmatrix} W_0(z) & DW_0^R(z)J \\ JW_0(z)V & (-1)^{M/2}JDW_0^R(z)JV \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & JD \end{bmatrix} \begin{bmatrix} W_0(z) & DW_0^R(z)J \\ DW_0(z)V & (-1)^{M/2}W_0^R(z)JV \end{bmatrix}. \end{aligned} \tag{3.60}$$

Proof: For PS symmetry, since $H_{M-1-i}(z) = H_i(-z)$, $H_{M-1-i,l}(z) = (-1)^l H_{i,l}(z)$. Thus if $[W_0(z)\ W_1(z)]$ is the first $M/2$ rows of $H_p(z)$, the rest is given by reversing the order of rows of $[W_0(z)\ W_1(z)]$ and alternating the signs of the columns. Hence the result follows. For PCS symmetry, since $H_{M-1-i}(z) = H_i^R(-z)$, from Eqn. 3.53

$$H_{i,l}(z) = (-1)^l H^R_{M-1-i,M-1-l}(z).$$

Therefore if $H_p(z)$ is a 2×2 block matrix, the 10 and 11 blocks are obtained by reversing the order of rows and columns of the 00 and 01 blocks respectively and alternating signs of columns, which is equivalent to Eqn. 3.56. For linear-phase, from Eqn. 3.54, $H_{i,l}(z) = \pm H_{i,M-1-l}(z)$, and therefore Eqn. 3.57 follows with D_0 and D_1 diagonal matrices of ± 1's to account for the type of symmetry. From the fact that half the filters are symmetric and the other half anti-symmetric, by a permutation Q, we can stack the symmetric filters in the first $M/2$ rows and the anti-symmetric filters in the rest, giving Eqn. 3.58 The combination of linear-phase and PCS (or PS) is then immediate. Notice also, that one needs to consider only one case (i.e., linear phase and PCS or linear phase and PS). □

Thus in order to *generate* $H_p(z)$ for all symmetries *other than* PS, we need a mechanism that generates a pair of matrices and their reflection (i.e., $W_0(z)$, $W_1(z)$ $W_0^R(z)$ and $W_1^R(z)$). In the scalar case, there are two well-known lattice structures that generate such pairs. The first case is the power-complementary (or orthogonal lattice) [86], while the second is the linear-prediction (or hyperbolic) lattice [71]. A K^{th} order (i.e., polynomial degree K) power-complementary lattice is generated by the product

$$\left\{ \prod_{i=1}^{K} \begin{bmatrix} a_i & z^{-1} b_i \\ -b_i & z^{-1} a_i \end{bmatrix} \right\} \begin{bmatrix} a_0 & b_0 \\ -b_0 & a_0 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} Y_0(z) & Y_1(z) \\ -Y_1^R(z) & Y_0^R(z) \end{bmatrix} \stackrel{\text{def}}{=} X(z).$$

This lattice is always invertible (unless a_i and b_i are both zero!), and the inverse is anti-causal since

$$\begin{bmatrix} a_i & z^{-1}b_i \\ -b_i & z^{-1}a_i \end{bmatrix}^{-1} = \frac{1}{a_1^2+b_i^2}\begin{bmatrix} a_i & -b_i \\ zb_i & za_i \end{bmatrix}.$$

This lattice plays a fundamental role in the theory of FIR unitary modulated filter banks, and many other areas. The hyperbolic lattice of order K (i.e., polynomial degree K) generates the product

$$\left\{\prod_{i=1}^{K}\begin{bmatrix} a_i & z^{-1}b_i \\ b_i & z^{-1}a_i \end{bmatrix}\right\}\begin{bmatrix} a_0 & b_0 \\ b_0 & a_0 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} Y_0(z) & Y_1(z) \\ Y_1^R(z) & Y_0^R(z) \end{bmatrix} \stackrel{\text{def}}{=} X(z),$$

where $Y_0(z)$ and $Y_1(z)$ are of order K. This lattice is invertible only when $a_i^2 \neq b_i^2$ (or equivalently $(a_i+b_i)/2$ and $(a_i-b_i)/2$ are non-zero), in which case, the inverse is non-causal since

$$\begin{bmatrix} a_i & z^{-1}b_i \\ b_i & z^{-1}a_i \end{bmatrix}^{-1} = \frac{1}{a_1^2-b_i^2}\begin{bmatrix} a_i & -b_i \\ -zb_i & za_i \end{bmatrix}.$$

Since the matrix $\begin{bmatrix} a_i & b_i \\ b_i & a_i \end{bmatrix}$ can be orthogonal iff $\{a_i, b_i\} = \{\pm 1, 0\}$, or $\{a_i, b_i\} = \{0, \pm 1\}$, the (2×2) matrix generated by the lattice can never be unitary.

We now propose the following matrix generalizations of these two lattices which will play a fundamental role in the characterization of $H_p(z)$ for the various symmetries we considered earlier. In the matrix case it turns out that both the lattices can generate unitary matrices. This leads to a parameterization of FIR unitary $H_p(z)$ for PCS, linear-phase, and PCS+linear-phase symmetries. We prefer to call the generalization of the orthogonal lattice, the *anti-symmetric* lattice and the generalization of the hyperbolic lattice the *symmetric* lattice, which should be obvious from the form of the product. The reason for this is that the anti-symmetric lattice *may not* generate a unitary matrix transfer function (in the scalar case, the 2×2 transfer function generated is always unitary). The anti-symmetric lattice is defined by the product

$$X(z) \stackrel{\text{def}}{=} \left\{ \prod_{i=1}^{K} \begin{bmatrix} A_i & z^{-1}B_i \\ -B_i & z^{-1}A_i \end{bmatrix} \right\} \begin{bmatrix} A_0 & B_0 \\ -B_0 & A_0 \end{bmatrix}, \tag{3.61}$$

where A_i and B_i are constant square matrices of size $M/2 \times M/2$. It is readily verified that $X(z)$ is of the form

$$X(z) = \begin{bmatrix} Y_0(z) & Y_1(z) \\ -Y_1^R(z) & Y_0^R(z) \end{bmatrix}. \tag{3.62}$$

Given $X(z)$, its invertibility is equivalent to the invertibility of the constant matrices

$$\begin{bmatrix} A_i & B_i \\ -B_i & A_i \end{bmatrix} \quad \text{since} \quad \begin{bmatrix} A_i & z^{-1}B_i \\ -B_i & z^{-1}A_i \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ -B_i & A_i \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & z^{-1}I \end{bmatrix}. \tag{3.63}$$

This is related to the invertibility of the *complex* matrices $C_i = (A_i + \imath B_i)$ and $D_i = (A_i - \imath B_i)$ because

$$\frac{1}{2} \begin{bmatrix} I & I \\ \imath I & -\imath I \end{bmatrix} \begin{bmatrix} C_i & 0 \\ 0 & D_i \end{bmatrix} \begin{bmatrix} I & -\imath I \\ I & \imath I \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ -B_i & A_i \end{bmatrix}.$$

Moreover, the orthogonality of the matrix is equivalent to the unitariness of the complex matrix C_i (since D_i is just its Hermitian conjugate). Since an arbitrary complex matrix, of size $M/2 \times M/2$, is determined by precisely $2\binom{M/2}{2}$ parameters, each of the matrices $\begin{bmatrix} A_i & B_i \\ -B_i & A_i \end{bmatrix}$ has that many degrees of freedom. Clearly when these matrices are orthogonal, $X(z)$ is unitary (on the unit circle) and $X^T(z^{-1})X(z) = I$. For unitary $X(z)$, the converse is also true as we will shortly prove.

The symmetric lattice is defined by the product

$$X(z) \stackrel{\text{def}}{=} \left\{ \prod_{i=1}^{K} \begin{bmatrix} A_i & z^{-1}B_i \\ B_i & z^{-1}A_i \end{bmatrix} \right\} \begin{bmatrix} A_0 & B_0 \\ B_0 & A_0 \end{bmatrix} \tag{3.64}$$

Once again A_i and B_i are constant square matrices, and it is readily verified that $X(z)$ written as a product above is of the form

$$X(z) = \begin{bmatrix} Y_0(z) & Y_1(z) \\ Y_1^R(z) & Y_0^R(z) \end{bmatrix}. \tag{3.65}$$

The invertibility of $X(z)$ is equivalent to the invertibility of

$$\begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix} \quad \text{since} \quad \begin{bmatrix} A_i & z^{-1}B_i \\ B_i & z^{-1}A_i \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & z^{-1}I \end{bmatrix}. \tag{3.66}$$

This is equivalent to the invertibility of $C_i = (A_i + B_i)$ and $D_i = (A_i - B_i)$ because

$$\frac{1}{2} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} C_i & 0 \\ 0 & D_i \end{bmatrix} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix}.$$

Once again, the orthogonality of the constant matrix, is equivalent to the orthogonality of the *real* matrices C_i and D_i, and since each real orthogonal matrix of size $M/2 \times M/2$ is determined by $\binom{M/2}{2}$ parameters, the constant orthogonal matrices have $2\binom{M/2}{2}$ degrees of freedom. Clearly when the matrices are orthogonal $X^T(z^{-1})X(z) = I$. The converse is also true.

> **Theorem 14** Let $X(z)$ be an FIR $M \times M$, polynomial matrix of order (polynomial degree) K unitary on the unit circle. If $X(z)$ is of the form in Eqn. 3.62 or Eqn. 3.65, it is generated by an order K *anti-symmetric* or *symmetric* lattice.

Proof: In both cases, we prove the results by induction on the order K. First consider $X(z)$ as in Eqn. 3.62. When $K = 0$, the result is evident. For any $X(z)$ of order K, we reduce its order by 1 to complete the proof. Let

$$X(z) = \sum_{i=0}^{K} x(i)z^{-i} = \sum_{i=0}^{K} z^{-i} \begin{bmatrix} y_0(i) & y_1(i) \\ -y_1(K-i) & y_0(K-i) \end{bmatrix}.$$

It suffices to find an orthogonal matrix $\begin{bmatrix} A_K^T & -B_K^T \\ B_K^T & A_K^T \end{bmatrix}$ such that

$$\begin{bmatrix} A_K^T & -B_K^T \end{bmatrix} x(K) = \begin{bmatrix} A_K^T & -B_K^T \end{bmatrix} \begin{bmatrix} y_0(K) & y_1(K) \\ -y_1(0) & y_0(0) \end{bmatrix} = 0, \qquad (3.67)$$

for then $\begin{bmatrix} A_K^T & -B_K^T \\ zB_K^T & zA_K^T \end{bmatrix} X(z)$ is equal to

$$\sum_{i=0}^{K-1} z^{-i} \begin{bmatrix} A_K^T y_0(i) + B^T y_1(K-i) & A_K^T y_1(i+1) - B_K^T y_0(K-1-i) \\ -A_K^T y_1(K-i) + B_K^T y_0(i) & A_K^T y_0(K-1-i) + B_K^T y_1(i+1) \end{bmatrix},$$

and gives the desired order reduction. The right hand side is unitary (since it is a product of unitary matrices), of degree $K-1$ and of the form in Eqn. 3.62. Also $X(z)$ is $\begin{bmatrix} A_K & z^{-1}B_K \\ -B_K & z^{-1}A_K \end{bmatrix}$ times the reduced order unitary matrix. We now construct such an orthogonal matrix using the crucial assumption of *unitariness*. Since $X^T(z^{-1})X(z) = I$, the coefficient of z^K in $X^T(z^{-1})X(z)$ zero.

$$x^T(0)x(K) = \begin{bmatrix} y_0^T(0) & -y_1^T(K) \\ y_1^T(0) & y_0^T(K) \end{bmatrix} \begin{bmatrix} y_0(K) & y_1(K) \\ -y_1(0) & y_0(0) \end{bmatrix} = 0. \qquad (3.68)$$

Hence the rank of $x(K)$ (= rank of $x(0)$) is some $P \leq M/2$. Let T be an $M \times P$ matrix that orthogonalizes (Gram-Schmidt) the columns of this rank P matrix.

$$x(K)T \stackrel{\text{def}}{=} x(K) \begin{bmatrix} T_0 \\ T_1 \end{bmatrix} = \begin{bmatrix} y_0(K) & y_1(K) \\ -y_1(0) & y_0(0) \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} U_0 \\ U_1 \end{bmatrix},$$

and

$$\begin{bmatrix} U_0^T & U_1^T \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \end{bmatrix} = I.$$

By appending $M-P$ orthogonal columns appropriately to $\begin{bmatrix} U_0 \\ U_1 \end{bmatrix}$ one can unitarily complete it to give the desired orthogonal matrix. Since

$$x(0)\begin{bmatrix} T_1 \\ -T_0 \end{bmatrix} = \begin{bmatrix} y_0(0) & y_1(0) \\ -y_1(K) & y_0(K) \end{bmatrix}\begin{bmatrix} T_1 \\ -T_0 \end{bmatrix} = \begin{bmatrix} U_1 \\ -U_0 \end{bmatrix},$$

from Eqn. 3.68,

$$\begin{bmatrix} U_1^T & -U_0^T \end{bmatrix} x(K) = 0 = \begin{bmatrix} U_1^T & -U_0^T \end{bmatrix}\begin{bmatrix} U_1 \\ U_0 \end{bmatrix}.$$

Therefore, $U_e = (U_0 + \imath U_1)$ and $U_o = (U_0 - \imath U_1)$ are $M/2 \times P$, matrices of unitary columns, that are Hermitian conjugates. Let $[U_e V_e]$ and $[U_o V_o]$ be $M/2 \times M/2$ unitary matrices obtained by unitary completion (Gram-Schmidt again, though *complex*!!). Set $\left[\begin{array}{c|c} A_K & B_K \\ \hline -B_K & A_K \end{array}\right]$ to

$$\frac{1}{2}\begin{bmatrix} I & I \\ \imath I & -\imath I \end{bmatrix}\left[\begin{array}{cc|cc} U_e & V_e & 0 & 0 \\ \hline 0 & 0 & U_o & V_o \end{array}\right]\left[\begin{array}{cc|cc} I & 0 & -\imath I & 0 \\ 0 & I & 0 & -\imath I \\ \hline I & 0 & \imath I & 0 \\ 0 & I & 0 & \imath I \end{array}\right] = \left[\begin{array}{cc|cc} U_1 & V_1 & U_0 & V_0 \\ \hline -U_0 & -V_0 & U_1 & V_1 \end{array}\right].$$

Clearly this matrix is orthogonal, and moreover, satisfies Eqn. 3.67.

Now consider the case when $X(z)$ is of the form in Eqn. 3.65. Once gain for $K=0$ the result is evident. Again it suffices to reduce the order $X(z)$ by one to complete the proof. Let

$$X(z) = \sum_{i=0}^{K} x(i)z^{-i} = \sum_{i=0}^{K} z^{-i}\begin{bmatrix} y_0(i) & y_1(i) \\ y_1(K-i) & y_0(K-i) \end{bmatrix}.$$

It suffices to find an orthogonal matrix $\begin{bmatrix} A_K^T & B_K^T \end{bmatrix}$ such that

$$\begin{bmatrix} A_K^T & B_K^T \\ B_K^T & A_K^T \end{bmatrix} x(K) = \begin{bmatrix} A_K^T & B_K^T \\ B_K^T & A_K^T \end{bmatrix}\begin{bmatrix} y_0(K) & y_1(K) \\ y_1(0) & y_0(0) \end{bmatrix} = 0, \qquad (3.69)$$

because then $\begin{bmatrix} A_K^T & B_K^T \\ zB_K^T & zA_K^T \end{bmatrix} X(z)$ is equal to

$$\sum_{i=0}^{K-1} z^{-i} \begin{bmatrix} A_K^T y_0(i) + B^T y_1(K-i) & A_K^T y_1(i+1) + B_K^T y_0(K-1-i) \\ A_K^T y_1(K-i) + B_K^T y_0(i) & A_K^T y_0(K-1-i) + B_K^T y_1(i+1) \end{bmatrix}.$$

This gives the desired order reduction. The right hand side is unitary (since it is a product of unitary matrices), is of degree $K-1$ and is of the form in Eqn. 3.65. Also $X(z)$ is $\begin{bmatrix} A_K & z^{-1}B_K \\ B_K & z^{-1}A_K \end{bmatrix}$ times the reduced order unitary matrix. We now exhibit such an orthogonal matrix. Since $X^T(z^{-1})X(z) = I$, the coefficient of z^K in $X^T(z^{-1})X(z)$ is zero.

$$x^T(0)x(K) = \begin{bmatrix} y_0^T(0) & y_1^T(K) \\ y_1^T(0) & y_0^T(K) \end{bmatrix} \begin{bmatrix} y_0(K) & y_1(K) \\ y_1(0) & y_0(0) \end{bmatrix} = 0. \qquad (3.70)$$

Therefore the rank of $x(K)$ (= rank of $x(0)$) is some $P \leq M/2$. Let T be an $M \times P$ matrix that orthogonalizes (Gram-Schmidt) the columns of this rank P matrix.

$$x(K)T \stackrel{\text{def}}{=} x(K) \begin{bmatrix} T_0 \\ T_1 \end{bmatrix} = \begin{bmatrix} U_0 \\ U_1 \end{bmatrix} \text{ and } \begin{bmatrix} U_0^T & U_1^T \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \end{bmatrix} = I.$$

We will complete $\begin{bmatrix} U_0 \\ U_1 \end{bmatrix}$ by appending $M-P$ orthogonal columns to give the desired orthogonal matrix. From Eqn. 3.70 it follows that

$$x(0) \begin{bmatrix} T_1 \\ T_0 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_0 \end{bmatrix}$$

and therefore

$$\begin{bmatrix} U_1^T & U_0^T \end{bmatrix} x(K) = 0 = \begin{bmatrix} U_1^T & U_0^T \end{bmatrix} \begin{bmatrix} U_1 \\ U_0 \end{bmatrix}.$$

Now $U_e = (U_0 + U_1)$ and $U_o = (U_0 - U_1)$ are $M/2 \times P$, matrices of orthonormal columns. Let $[U_e V_e]$ and $[U_o V_o]$ be unitary completions to square $M/2 \times M/2$ orthogonal matrices (Gram-Schmidt again!). Set $\left[\begin{array}{c|c} A_K & B_K \\ \hline B_K & A_K \end{array}\right]$ to

$$\frac{1}{2}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}\left[\begin{array}{cc|cc} U_e & V_e & 0 & 0 \\ \hline 0 & 0 & U_o & V_o \end{array}\right]\left[\begin{array}{cc|cc} I & 0 & I & 0 \\ 0 & I & 0 & I \\ \hline I & 0 & -I & 0 \\ 0 & I & 0 & -I \end{array}\right] = \left[\begin{array}{cc|cc} U_1 & V_1 & U_0 & V_0 \\ \hline U_0 & V_0 & U_1 & V_1 \end{array}\right].$$

Clearly this matrix is orthogonal, and moreover, satisfies Eqn. 3.69 □

3.6.1 PS Symmetry

The form of $H_p(z)$ for PS symmetry in Eqn. 3.55 can be simplified by a permutation. Let P be the permutation matrix that exchanges the first column with the last column, the third column with the last but third etc. That is,

$$P = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}.$$

Then the matrix $\begin{bmatrix} W_0(z) & W_1(z) \\ W_0(z)V & (-1)^{M/2}W_1(z)V \end{bmatrix}$ in Eqn. 3.55 can be rewritten as $\frac{1}{\sqrt{2}}\begin{bmatrix} W_0'(z) & W_1'(z) \\ -W_0'(z) & W_1'(z) \end{bmatrix} P$, and therefore

$$H_p(z) = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} W_0(z) & W_1(z) \\ W_0(z)V & (-1)^{M/2}W_1(z)V \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0'(z) & W_1'(z) \\ -W_0'(z) & W_1'(z) \end{bmatrix} P$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \begin{bmatrix} W_0'(z) & 0 \\ 0 & W_1'(z) \end{bmatrix} P$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -J & J \end{bmatrix} \begin{bmatrix} W_0'(z) & 0 \\ 0 & W_1'(z) \end{bmatrix} P.$$

For PS symmetry one has the following parameterization of unitary filter banks.

Theorem 15 *(Unitary PS Symmetry)* $H_p(z)$ of order K forms a unitary PR filter bank with PS symmetry iff there exist unitary, order K, $M/2 \times M/2$ matrices $W_0'(z)$ and $W_1'(z)$, such that

$$H_p(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -J & J \end{bmatrix} \begin{bmatrix} W_0'(z) & 0 \\ 0 & W_1'(z) \end{bmatrix} P. \qquad (3.71)$$

A unitary H_p, with PS symmetry is determined by precisely $2(M/2 - 1)K + 2\begin{pmatrix} M/2 \\ 2 \end{pmatrix}$ parameters.

3.6.2 PCS Symmetry

In this case

$$\begin{aligned} H_p(z) &= \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0(z) & W_1(z)J \\ W_1^R(z)V & (-1)^{M/2}W_0^R(z)JV \end{bmatrix} \\ &\stackrel{\text{def}}{=} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0' & W_1'J \\ -(W_1')^R & (W_0')^R J \end{bmatrix} P \\ &= \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0' & W_1' \\ -(W_1')^R & (W_0')^R \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} P. \end{aligned}$$

Hence from Lemma 23 $H_p(z)$ of unitary filter banks with PCS symmetry can be parameterized as follows:

Theorem 16 $H_p(z)$ forms an order K, unitary filter bank with PCS symmetry iff

$$H_p(z) = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \left\{ \prod_{i=1}^{K} \begin{bmatrix} A_i & z^{-1}B_i \\ -B_i & z^{-1}A_i \end{bmatrix} \right\} \begin{bmatrix} A_0 & B_0 \\ -B_0 & A_0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} P \tag{3.72}$$

where $\begin{bmatrix} A_i & B_i \\ -B_i & A_i \end{bmatrix}$ are constant orthogonal matrices. $H_p(z)$ is characterized by $2K \begin{pmatrix} M/2 \\ 2 \end{pmatrix}$ parameters.

3.6.3 Linear Phase

For the linear-phase case,

$$\begin{aligned} H_p(z) &= Q \begin{bmatrix} W_0(z) & W_0^R(z) \\ W_1(z) & -W_1^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \\ &= \frac{1}{2} Q \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} W_0(z)+W_1(z) & W_0^R(z)-W_1^R(z) \\ W_0(z)-W_1(z) & W_0^R(z)+W_1^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \\ &\stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} Q \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} W_0'(z) & W_1'(z) \\ (W_1')^R(z) & (W_0')^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}. \end{aligned}$$

Therefore, we have the following Theorem:

Theorem 17 $H_p(z)$ of order K, forms a unitary filter bank with linear-phase filters iff

$$H_p(z) = \frac{1}{\sqrt{2}} Q \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \left\{ \prod_{i=1}^{K} \begin{bmatrix} A_i & z^{-1}B_i \\ B_i & z^{-1}A_i \end{bmatrix} \right\} \begin{bmatrix} A_0 & B_0 \\ B_0 & A_0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}, \tag{3.73}$$

where $\begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix}$ are constant orthogonal matrices. $H_p(z)$ is characterized by $2K \begin{pmatrix} M/2 \\ 2 \end{pmatrix}$ parameters.

3.6.4 Linear Phase and PCS

In this case, $H_p(z)$ is given by,

$$\begin{aligned} H_p(z) &= \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0(z) & DW_0^R(z)J \\ DW_0(z)V & (-1)^{M/2}W_0^R(z)JV \end{bmatrix} \\ &\stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} W_0'(z) & D(W_0')^R(z)J \\ -DW_0'(z) & (W_0')^R(z)J \end{bmatrix} P \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} I & D \\ -D & I \end{bmatrix} \begin{bmatrix} W_0'(z) & 0 \\ 0 & W_0')^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} P. \end{aligned}$$

Therefore we have proved the following Theorem:

Theorem 18 $H_p(z)$ of order K forms a unitary filter bank with linear-phase and PCS filters iff there exists a unitary, order K, $M/2 \times M/2$ matrix $W_0'(z)$ such that

$$H_p(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} I & D \\ -JD & J \end{bmatrix} \begin{bmatrix} W_0'(z) & 0 \\ 0 & (W_0')^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} P. \quad (3.74)$$

In this case $H_p(z)$ is determined by precisely $(M/2-1)K + \binom{M/2}{2}$ parameters.

3.6.5 Linear Phase and PS Symmetry

From the previous result we have the following result:

Theorem 19 $H_p(z)$ of order K forms a unitary filter bank with linear-phase and PS filters iff there exists a unitary, order K, $M/2 \times M/2$ matrix $W_0'(z)$ such that

$$H_p(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} I & D \\ -J & JD \end{bmatrix} \begin{bmatrix} W_0'(z) & 0 \\ 0 & (W_0')^R(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} P. \quad (3.75)$$

H_p is determined by precisely $(M/2-1)K + \binom{M/2}{2}$ parameters.

3.7 Completion of Filter Banks and Transmultiplexers

Consider the following problem: If a set of $L (L < M)$ filters in a filter bank are chosen a priori, how does one design a set of $M - L$ filters to give rise to an M channel PR filter bank? This section solves this problem for unitary and causal filter banks. The completion problem for filter banks is related to the PR problem for transmultiplexers and vice versa. This section deals with the completion problem for filter banks only (similar results for the transmultiplexer can also be obtained).

3.7.1 Unitary Completion Theory

Theorem 20 *(FIR Unitary Completions)* A set of L filters can be completed to a unitary filter bank iff the filters can be considered to be filters of a unitary transmultiplexer. If the McMillan degree of $H_p(z)$ and the McMillan degree of the polyphase matrix of the completed filter bank are equal, then the completing set is parameterized by $\binom{M-L}{2}$ parameters (and is independent of the length of the filters).

3.7.2 Causal (or unimodular) Completion Theory

For causal filter banks PR is equivalent to the (first-orthant) polyphase component matrix $H_p(z)$ being unimodular over the ring of polynomials or stable proper rational functions (depending on the FIR or IIR cases respectively). Therefore the completion problem for causal filter banks becomes one of unimodular completions. This fact brings a connection between the celebrated Youla (YJBK) parameterization of compensators in control systems theory [30] and the completion problem.

Polynomial matrices U and V with the same number of columns are said to be left-coprime if $\begin{bmatrix} U & V \end{bmatrix}$ is right invertible [49]. This can be generalized to the case where U and V are matrices in any Euclidean domain [92], and therefore in particular for matrices of stable proper rational entries. Notice that left coprimeness is not a property of U or V individually but an aggregate property of the rows of $\begin{bmatrix} U & V \end{bmatrix}$. Since right-invertibility is equivalent to a transmultiplexer being PR, it readily follows that a causal transmultiplexer is PR iff the corresponding $H_p(z)$ is coprime over the ring of polynomials (in the FIR case) and the ring of stable proper rational functions (in the IIR case).

It turns out that coprimeness is precisely the property that is related to unimodular completions [92]:

Fact 6 A matrix H over a Euclidean domain has a unimodular completion iff the rows of H are coprime.

Therefore we have the following results:

Lemma 24 *(Causal FIR Completions)* Given $\{h_i\}$, $i\,\{0, 1, \ldots, L-1\}$, all causal FIR filters, they can be completed to form a *causal FIR* filter bank, iff the rows of $H_p(z)$ are (left) coprime over the ring of polynomials in z^{-1} (or equivalently if the filters form a causal FIR PR transmultiplexer).

In the FIR case the Euclidean algorithm can be used to completely parameterize all possible completions. To see this, let the rows of $H_p(z)$ be coprime. Then by elementary column operations ([49]) one can construct a unimodular matrix $U(z)$ such that

$$H_p(z) = \begin{bmatrix} I & 0 \end{bmatrix} U(z) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} U_{11}(z) & U_{12}(z) \\ U_{21}(z) & U_{22}(z) \end{bmatrix}, \qquad (3.76)$$

where I is the $L \times L$ identity matrix. Let $V = U^{-1}$ be also partitioned similarly. Then

$$\begin{aligned} I &= \begin{bmatrix} U_{11}(z) & U_{12}(z) \\ U_{21}(z) & U_{22}(z) \end{bmatrix} \begin{bmatrix} V_{11}(z) & V_{12}(z) \\ V_{21}(z) & V_{22}(z) \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ Q(z) & I \end{bmatrix} \begin{bmatrix} U_{11}(z) & U_{12}(z) \\ U_{21}(z) & U_{22}(z) \end{bmatrix} \begin{bmatrix} V_{11}(z) & V_{12}(z) \\ V_{21}(z) & V_{22}(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ -Q(z) & I \end{bmatrix} \\ &= \begin{bmatrix} U_{11}(z) & U_{12}(z) \\ U_{21}(z) + Q(z)U_{11}(z) & U_{22}(z) + Q(z)U_{12}(z) \end{bmatrix} \begin{bmatrix} V_{11} - V_{12}(z)Q(z) & V_{12}(z) \\ V_{21}(z) - V_{22}(z)Q(z) & V_{22}(z) \end{bmatrix} \end{aligned}$$

From this it readily follows that all possible completions are given explicitly by the formula

$$\breve{H}_p(z) = \begin{bmatrix} U_{11}(z) & U_{12}(z) \\ U_{21}(z) + Q(z)U_{11}(z) & U_{22}(z) + Q(z)U_{12}(z) \end{bmatrix}$$

where $Q(z)$ is an arbitrary polynomial matrix. In particular the smallest degree completion (i.e the one with filters of smallest possible length) is given by $U(z)$.

> **Lemma 25** *(Causal IIR Completions)* Given $\{h_i\}$, $i \in \{0, 1, \ldots, L-1\}$, all causal FIR filters, they can be completed to form a *causal FIR* filter bank, iff the rows of $H_p(z)$ are (left) coprime over the ring of stable proper rational functions (or equivalently if the filters form a causal stable IIR PR transmultiplexer).

In the IIR case also a complete parametrization is given as above. There is a cleaner approach that gives a complete parameterization of all completions that have the same McMillan degree. This follows directly from the Youla parameterization of controllers given in [30].

The so called *standard problem* is control theory is as shown in Fig. 3.9. The plant $\mathcal{P}$ is a stable proper MIMO shift-invariant system. That is, its transfer function $P(z)$ is a matrix with entries from RH^∞. The goal is to design a compensator $\mathcal{C}$, whose transfer function $C(z)$ is a matrix with entries in RH^∞, *and* such that the the

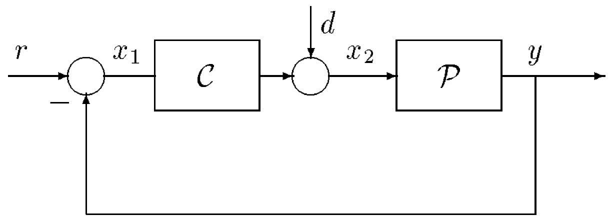

Figure 3.9: The Standard Control Problem

closed loop system is *internally stable*. The final design goal is to choose among all internally stabilizing compensators, the one that minimizes some desired objective function. Thus a parameterization of internally stabilizing controllers reduces the control problem to an unconstrained optimization problem.

Fact 7 Let $P(z) = N(z)D^{-1}(z)$ and $C(z) = \tilde{N}(z)\tilde{D}^{-1}(z)$ left coprime factorizations of the plant and the controller. Then $\mathcal{C}$ is internally stabilizing iff $\begin{bmatrix} D(z) & \tilde{N}(z) \\ N(z) & \tilde{D}(z) \end{bmatrix}$ is unimodular.

By taking the transpose one readily sees that $H_p(z)$ in the completion problem is analogous to being given the plant and the parameterization of completions is equivalent to the parameterization of internally stablizing compensators. In the controller problem state-space formulae for the parameterization of all compensators with the same McMillan degree are well-known. This directly leads to a parameterization of all causal IIR (rational stable) filter bank completions [27].

3.8 Rational Sampling Rate Filter Banks

Consider the rational sampling rate filter bank problem (Fig. 3.10) where the signal is spectrally split into nonuniform bins. The sampling rate change is different in each branch of the analysis bank. In one dimension this problem has been studied recently [53, 65]. Using the tools developed in Chapter 2 an algebraic reduction of the multidimensional rational sampling rate filter bank problem to a multidimensional uniform sampling rate filter bank problem is given.

3.8.1 Multidimensional Rational Sampling Filter Banks

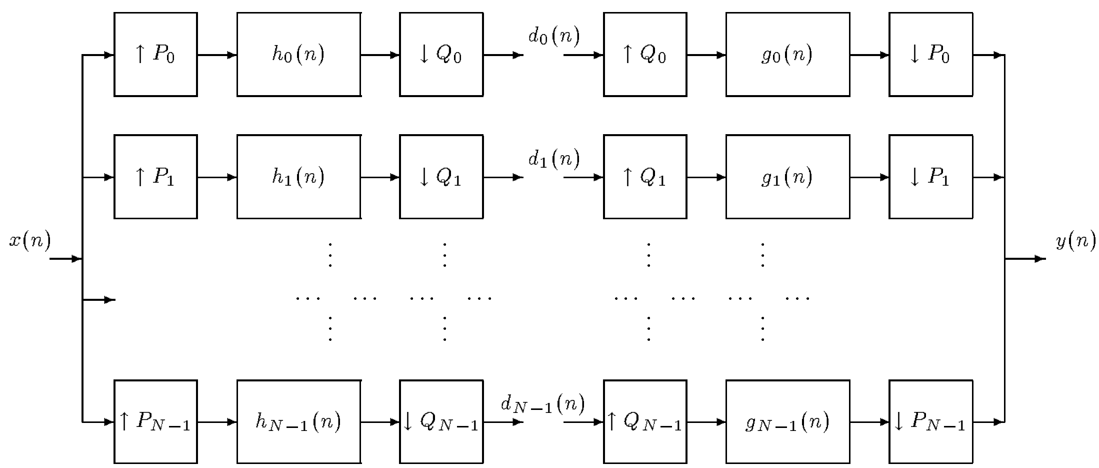

Figure 3.10: A Rational Sampling Rate Filter Bank

In Fig. 3.10 let P_i and Q_i be left coprime such that the "sampling-feasibility" condition, $\sum_i |P_i| \,/\, |Q_i| = 1$, holds. This condition is not sufficient for the existence of perfect reconstruction rational sampling rate filter banks even in one dimension [53]. In any case, we algebraically reduce the problem to a $|Q|$-channel filter bank problem for some appropriate Q. The result is a direct generalization of the one dimensional result in [53]. The reduction is algebraic and says nothing about the relationship between the responses of the original and reduced filters. The reduction is a two step procedure, one to slide the upsampler in each branch past the filter and the downsampler and the other to then replace the downsampler by the least common left multiple of the downsamplers (that are different from the original downsamplers due to the first step).

Rational Filter Bank Reduction: Transform 1

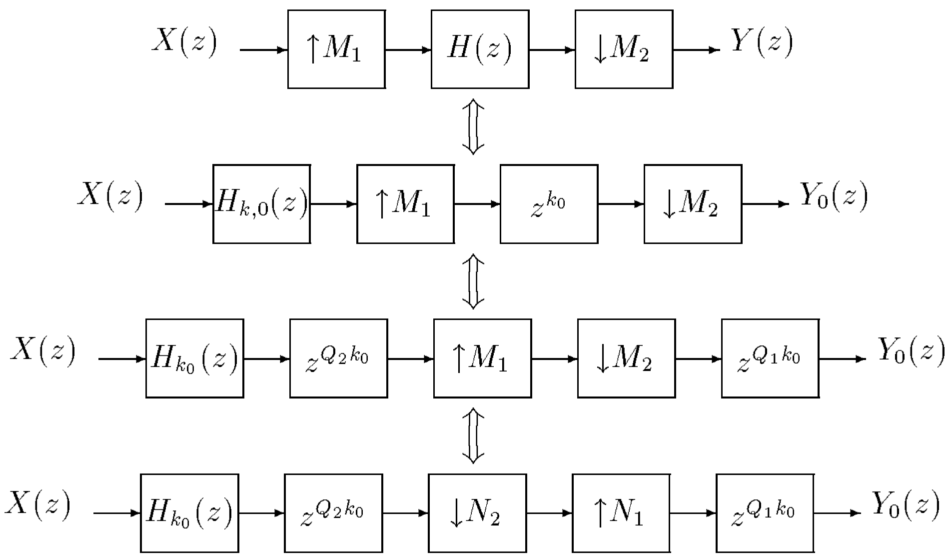

Figure 3.11: Rational Filter Bank Reduction: Steps in Transform 1

Along any branch in Fig. 3.10 if the matrices corresponding to the upsampler and downsampler are left coprime one can transform the structure. For example, consider the cascade: [$\uparrow M_1$], filter, [$\downarrow M_2$] with M_1, M_2 left coprime. This can be converted to the following cascade: filter, delay, [$\downarrow N_1$], [$\uparrow N_2$] and delay with N_1, N_2 right coprime and $M_1N_2 = M_2N_1$. An outline of this reduction is as follows (see Fig. 3.11): (1) Use the UF identity to swap [$\uparrow M_1$] and $H(z)$, (2) Use the UΔD identity (Lemma 5) to slide delays introduced in Step 1 along each polyphase branch and (3) Use the Swapping Theorem (Theorem 2) to introduce [$\downarrow N_1$] and [$\uparrow N_2$] along each branch.

The delays $z^{Q_2k_i}$ in Fig. 3.11 can be absorbed into the filters $H_{k_i}(z)$. Moreover the delays $z^{Q_1k_i}$ can be thought of as implementing a generalized polyphase representation of $Y(z)$ with respect to N_2, the upsampler. This fact is important since the original problem reduces to the problem of design of the modified filters in Fig. 3.12 with only

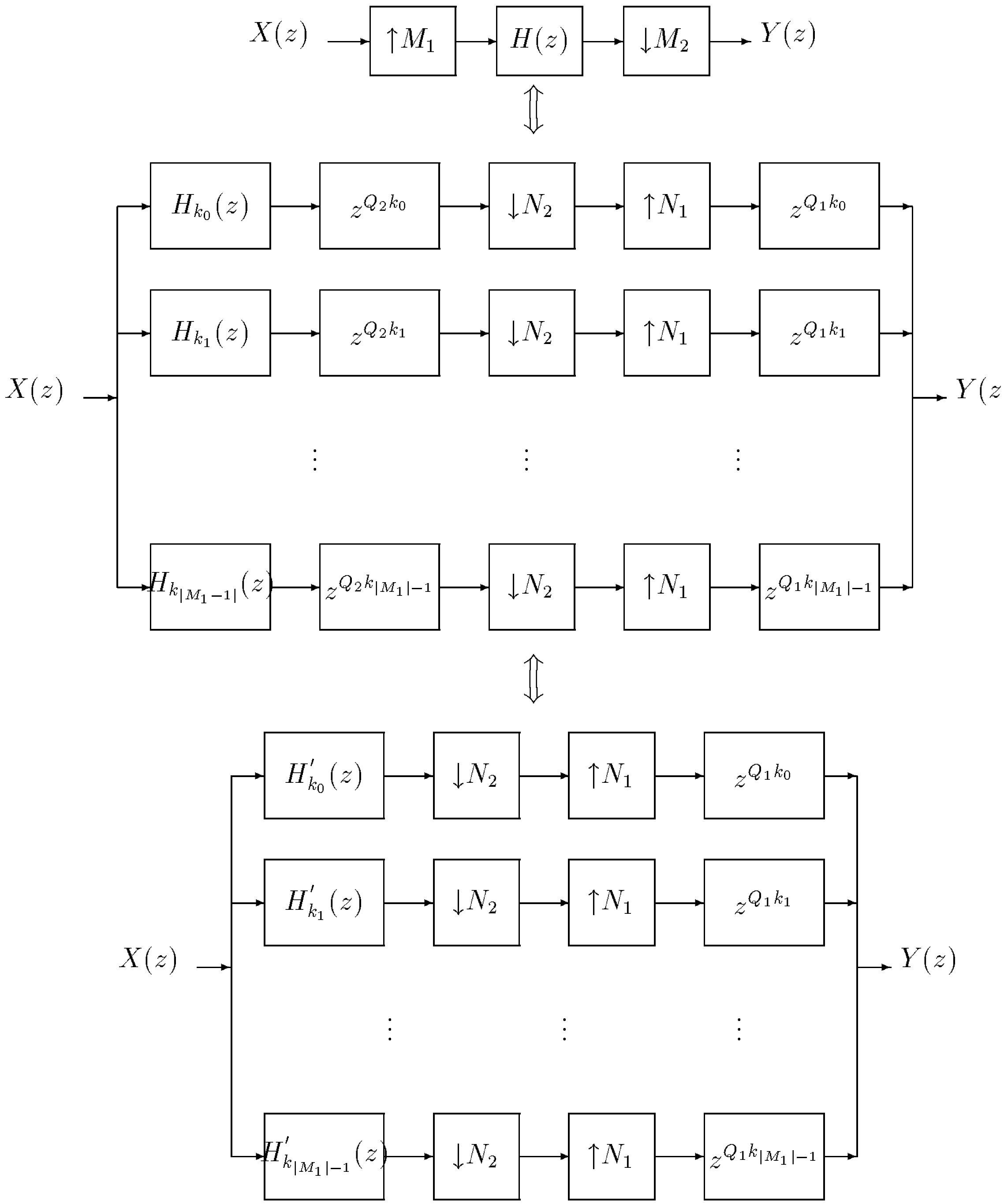

Figure 3.12: Rational Filter Bank Reduction: Transform 1

downsamplers. So see this notice the following Aryabhatta/Bezout identity:

$$\begin{bmatrix} N_2 & Q_2 \\ N_1 & -Q_1 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ M_1 & -M_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Since N_1 and Q_1 are left coprime, from Theorem 1 (Eqn. 2.9b) $Q_1\mathcal{R}(M_1) \bmod N_1 = \mathcal{R}(N_1)$ showing that the generalized representatives in question is $\mathcal{S}(N_1) = Q_1\mathcal{R}(M_1)$.

Rational FB Reduction: Transform 2

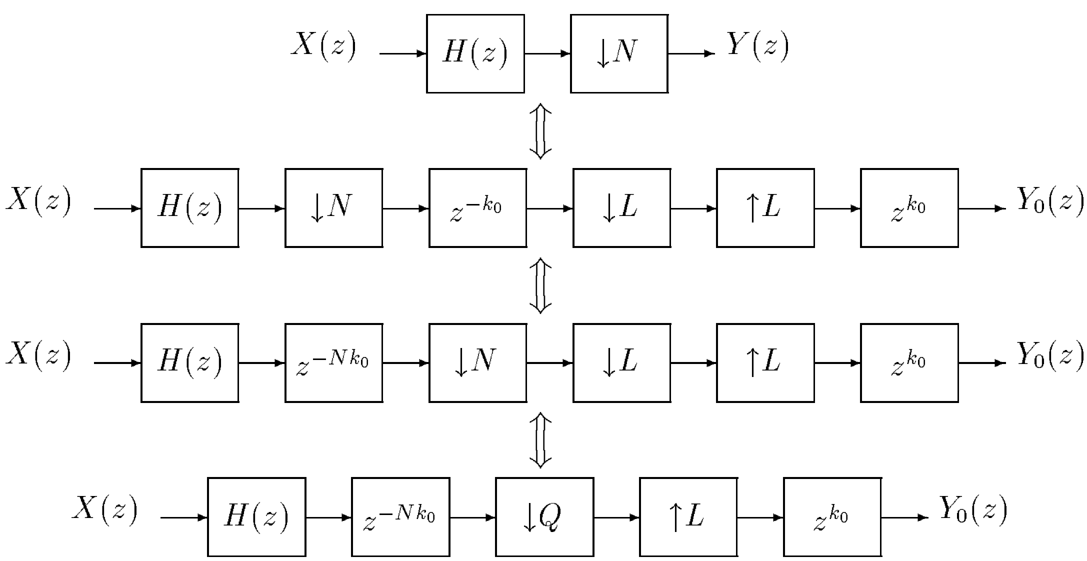

Figure 3.13: Rational FB Reduction: Steps in Transform 2

After Transform 1 the downsampler in each branch of the original filter bank are not the same. By making these downsamplers the same the reduction to a uniform sampling rate filter bank would be complete. The idea is to replace $[\downarrow N]$ by $[\downarrow NL]$, for some L, followed by an inverse polyphase transform corresponding to L as shown in Fig. 3.14. The steps in transform 2 are as follows: (1) Use GPIP identity with respect to L as in Fig. 2.10 and (2) Use DF identity to slide the delays after the downsampler to before it. This derivation of Transform 2 is much simpler than even corresponding result in the one dimensional case in [53].

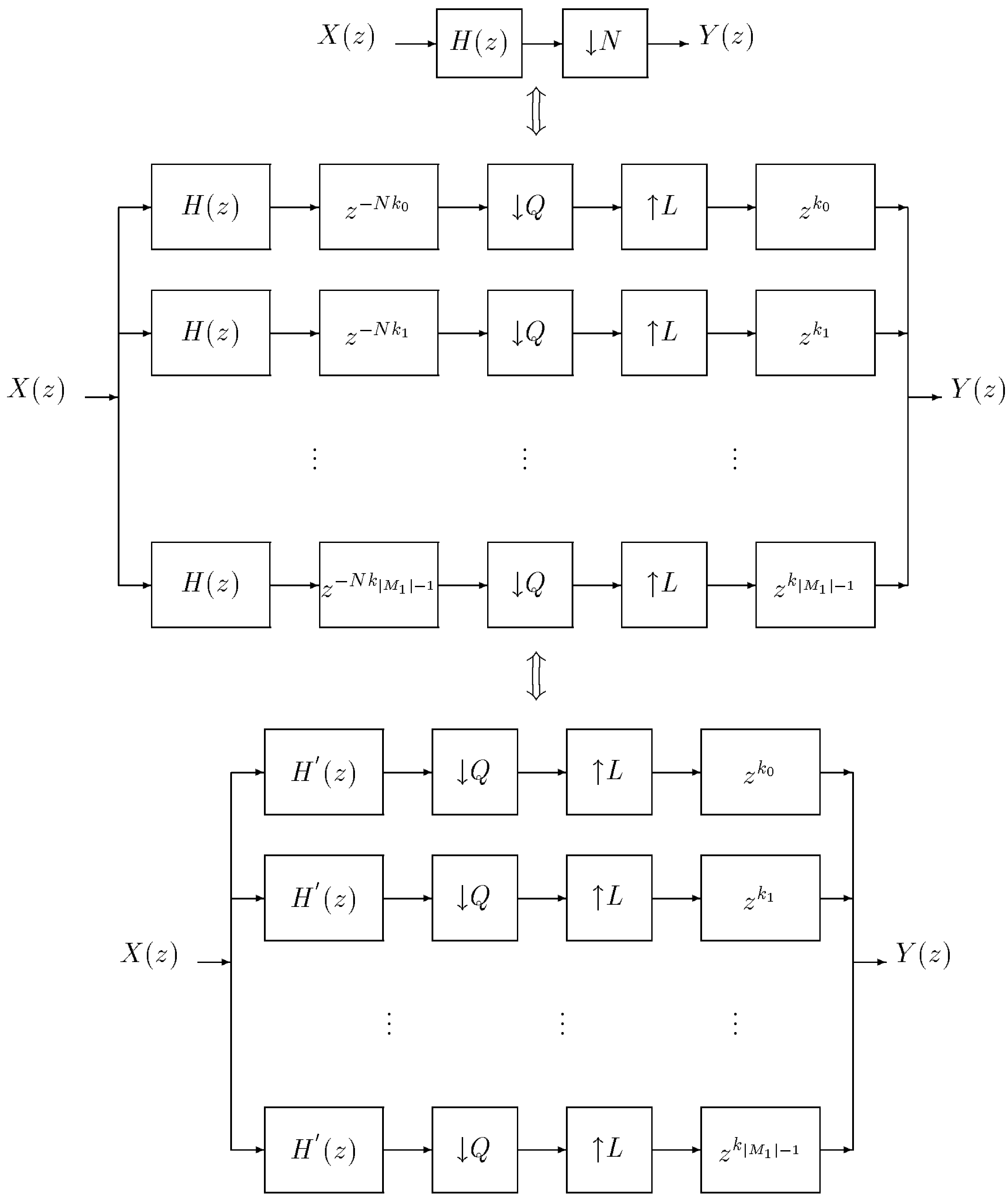

Figure 3.14: Rational FB Reduction: Transform 2

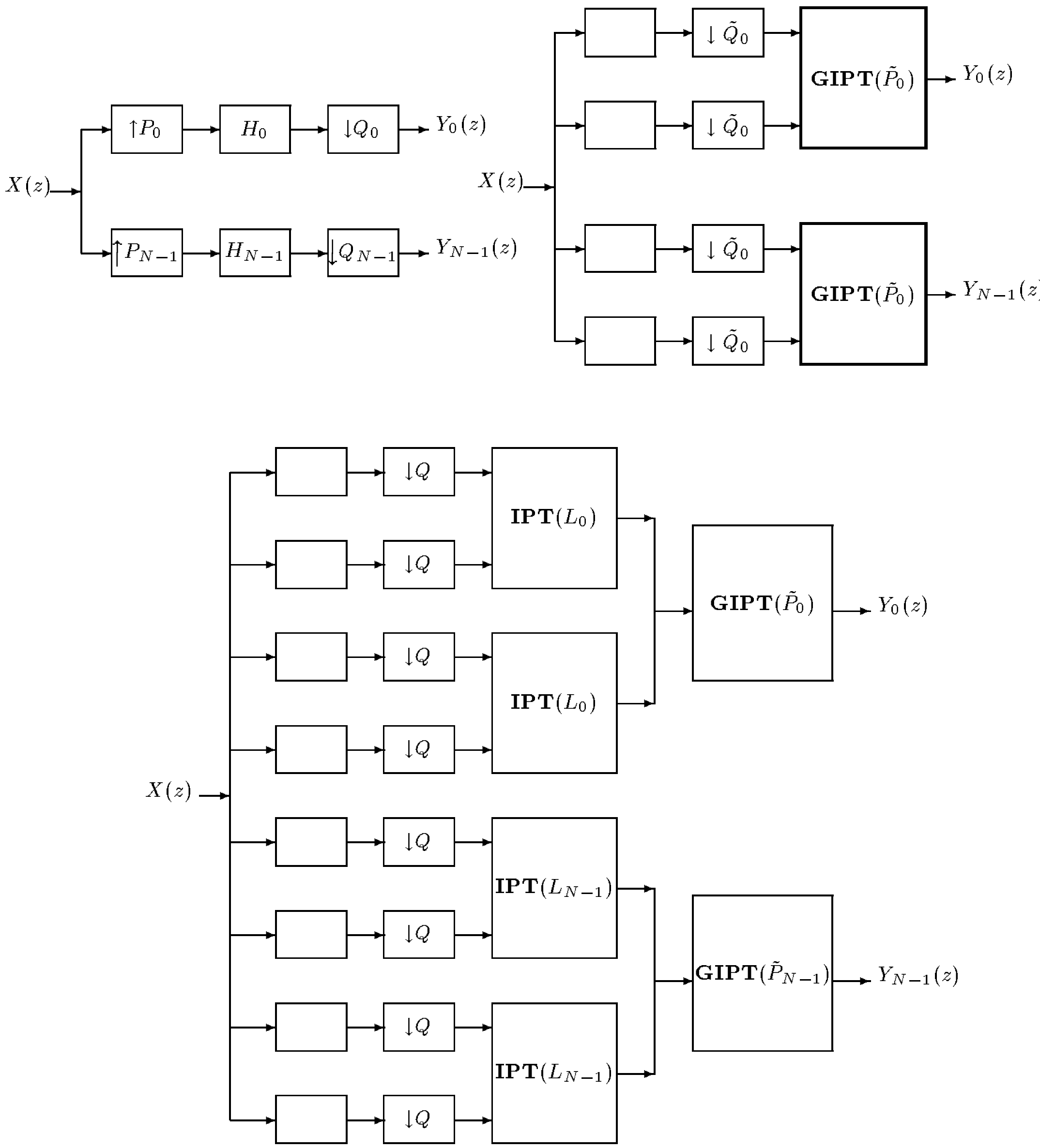

Figure 3.15: Rational FB - Reduction to a Uniform Filter Bank

The Reduction to a Uniform Filter Bank

Consider the original problem in Fig. 3.10. Transform 1 is applied in each branch. This expands the i^{th} branch into $|P_i|$ branches giving an analysis bank with $\sum_i |P_i|$ branches. The downsampler in each new branch is, say $\tilde{Q}_i$ (which is different from Q_i). Let Q be an lcrm of these matrices with $Q = \tilde{Q}_i L_i$ for suitable matrices L_i. Apply Transform 2 in each branch. The branch corresponding to $\tilde{Q}_i$ expands into $|L_i|$ branches giving a (uniform) analysis filter bank with $\sum_{i=0}^{N-1} |P_i| |L_i|$ branches. Since $|L_i| = |Q| / \left|\tilde{Q}_i\right|$ and (from Corollary 1) $\left|\tilde{Q}_i\right| = |P_i|$, $\sum_{i=0}^{N-1} |P_i| |L_i| = |Q|$. In summary one has a Q channel uniform band filter bank corresponding to the downsampling matrix Q. A final question is whether Transform 2 can be modified so that the generalized inverse polyphase transform blocks in Fig. 3.15 can be replaced by an ordinary **IPT** block. The answer is in the affirmative and is an easy consequence of the GPIP identity.

Chapter 4

Wavelet Theory

4.1 Bases, Frames and Generalized Frame Pairs

A set of vectors $\{\phi_j\}$ in a separable Hilbert space $\mathcal{H}$ is a *basis* if, $\forall f \in \mathcal{H}$, there exists a unique set of scalars, $\{f_j\}$, such that $f = \sum_j f_j \phi_j$. A set of vectors $\{\phi_j\}$ is *complete* if every vector $f \in \mathcal{H}$ can be approximated (in the Hilbert space norm) by finite linear combinations of ϕ_j. A set of vectors $\{\phi_j\}$ is *minimal* if it ceases to be complete on removal of any one vector. Every basis is both minimal and complete. A basis is *bounded* if $\sup_j \|\phi_j\|$ and $\inf_j \|\phi_j\|$ are bounded above and below respectively by positive constants. For example, an orthonormal basis is bounded above and below by one. A wide class of bounded bases called *Riesz bases* (or biorthogonal bases) can be generated from a orthonormal basis. To define Riesz bases we need the notion of a *bounded* linear operator. An operator $T : \mathcal{H}_1 \mapsto \mathcal{H}_2$ is *bounded* if $\forall f \in \mathcal{H}_1$, $\|Tf\| \leq A \|f\|$ for some A. The least such A is called the norm of T, $\|T\|$. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional, T is a matrix, and $\|T\|$ is just the maximum singular value of T (square root of the maximum eigenvalue of $T^\star T$, where $T^\star$ is the (conjugate) transpose of T).

> **Definition 9** If $T : \mathcal{H} \mapsto \mathcal{H}$ is bounded and has a bounded inverse T^{-1}, then any set $\{Te_j\} \stackrel{\text{def}}{=} \{\phi_j\}$, where $\{e_j\}$ is an orthonormal basis, is said to be a Riesz basis.

Every Riesz basis is bounded. Since $\|\phi_j\| = \|Te_j\| \leq \|T\| \|e_j\| = \|T\|$ and $1 = \|e_j\| = \|T^{-1}\phi_j\| \leq \|T^{-1}\| \|\phi_j\|$, we have

$$\frac{1}{\|T^{-1}\|} \leq \inf_j \|\phi_j\| \leq \sup_j \|\phi_j\| \leq \|T\|$$

Though most useful (familiar) bounded bases are Riesz bases, there do exist bounded bases that are not Riesz bases [96]. The *adjoint* operator $T^\star$ is defined by $\langle f, Tg \rangle = \langle T^\star f, g \rangle, \forall f, g \in \mathcal{H}$. $T^\star$ satisfies the following properties [28]: $\|T\| = \|T^\star\|$ and (if T is invertible) $(T^{-1})^\star = (T^\star)^{-1}$. Hence for a Riesz basis there exists a unique *dual* Riesz basis, $\left\{\tilde{\phi}_j\right\}$, defined by $\tilde{\phi}_j = (T^\star)^{-1} e_j$.

$$\left\langle \tilde{\phi}_i, \phi_j \right\rangle = \left\langle (T^\star)^{-1} e_i, Te_i \right\rangle = \left\langle e_i, T^{-1} Te_j \right\rangle = \delta(i-j) \tag{4.1}$$

Because of Eqn. 4.1 Riesz bases are also known as *biorthogonal bases.* Expansion coefficients in Riesz bases are easily evaluated since (from Eqn. 4.1)

$$f = \sum_j \left\langle f, \tilde{\phi}_j \right\rangle \phi_j = \sum_j \langle f, \phi_j \rangle \tilde{\phi}_j. \tag{4.2}$$

The vector $\tilde{\phi}_j$ can be characterized geometrically: $\tilde{\phi}_j$ is that vector in the orthogonal complement of $Span \{\phi_i \mid i \neq j\}$ scaled so that $\left\langle \tilde{\phi}_j, \phi_j \right\rangle = 1$. If $\mathcal{H}$ is finite dimensional every non-singular square matrix T is bounded with a bounded inverse (in fact the bounds are the maximum and minimum singular values of T). Hence there is a one-to-one correspondence between invertible square matrices and Riesz bases. One can *read off* $\{\phi_j\}$ and $\left\{\tilde{\phi}_j\right\}$ from the matrix T: ϕ_j is the j^{th} column of T, and $\tilde{\phi}_j$ is the j^{th} column of $(T^{-1})^\star$. Moreover, from Eqn. 4.2, $\sum_j \tilde{\phi}_j \rangle\langle \phi_j = I$ just says that $T^{-1}T = I$! In finite dimensions every basis is a Riesz basis.

In finite dimensions given a non-minimal set of vectors one can always throw out some of them to get a basis. In infinite dimensions given a complete but non-minimal (redundant) set of vectors, one *cannot* always throw out some of them and obtain a basis (minimal set). One encounters sets of vectors that are *complete, but not minimal* leading to the notion of *frames.*

Definition 10 A set $\{\phi_j\}$ forms a frame in a separable Hilbert space $\mathcal{H}$ if there exist constants $A, B > 0$, such that $\forall f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_j |\langle f, \phi_j \rangle|^2 \leq B\|f\|^2 \tag{4.3}$$

The lower bound ensures that $\{\phi_j\}$ is complete, since if f is orthogonal to $\{\phi_j\}$ it implies that $A\|f\| = 0$, or equivalently that $f = 0$. Define the self-adjoint *frame operator* $T : \mathcal{H} \mapsto \mathcal{H}$ given by $Tf = \sum_j \langle f, \phi_j \rangle \phi_j$ The upper and lower bounds in Eqn. 4.3 imply that T is invertible and that both T and T^{-1} are bounded:

$$A\|f\| \leq \|Tf\| \leq B\|f\| \Leftrightarrow A \leq \|T\| \leq B \Leftrightarrow \frac{1}{B} \leq \left\|T^{-1}\right\| \leq \frac{1}{A}. \tag{4.4}$$

The upper bound follows from

$$\begin{aligned} \|Tf\|^2 &= \langle Tf, Tf \rangle = \sum_n \langle f, \phi_n \rangle \langle \phi_n, Tf \rangle \\ &\leq \left\{ \sum_n |\langle f, \phi_n \rangle|^2 \right\}^{\frac{1}{2}} \left\{ \sum_n |\langle \phi_n, Tf \rangle|^2 \right\}^{\frac{1}{2}} \\ &= B^{\frac{1}{2}} \|f\| B^{\frac{1}{2}} \|Tf\| = B\|f\|\|Tf\| \end{aligned} \tag{4.5}$$

and the lower bound from (see Eqn. 4.3)

$$A\|f\|^2 \leq \langle Tf, f \rangle = \sum_n |\langle f, \phi_n \rangle|^2 \leq \|Tf\|\|f\| \leq \|T\|\|f\|^2. \tag{4.6}$$

Therefore $A \leq \|T\| \leq B$ and T is invertible. Eqn 4.5 can be applied to the vector $T^{-1}f$ to give $\|f\| \leq B\|T^{-1}f\| \leq B\|T^{-1}\|\|f\|$ and Eqn. 4.6 can be applied to the vector $T^{-1}f$ to give $\|T^{-1}f\| \leq \frac{1}{A}\|f\|$. Also $\forall f \in \mathcal{H}$

$$f = TT^{-1}f = \sum_j \left\langle T^{-1}f, \phi_j \right\rangle \phi_j = \sum_j \left\langle f, T^{-1}\phi_j \right\rangle \phi_j \tag{4.7}$$

If $\tilde{\phi}_j \stackrel{\text{def}}{=} T^{-1}\phi_j$, Eqn. 4.7 looks like the expansion of f in a Riesz basis. Hence $\left\{\tilde{\phi}_j\right\}$ is called the *dual* frame. Indeed $\left\{\tilde{\phi}_j\right\}$ is a frame since (from Eqn. 4.4) for all $f \in \mathcal{H}$,

$$\frac{1}{B}\|f\|^2 \leq \sum_j \left|\left\langle f, \tilde{\phi}_j \right\rangle\right| \leq \frac{1}{A}\|f\|^2$$

The *frame operator* for the dual frame is T^{-1}.

$$\begin{aligned}\sum_j \left\langle f, \tilde{\phi}_j \right\rangle \tilde{\phi}_j &= \sum_j \left\langle f, T^{-1}\phi_j \right\rangle T^{-1}\phi_j \\ &= T^{-1}\left[\sum_j \left\langle T^{-1}f, \phi_j \right\rangle \phi_j\right] \\ &= T^{-1}T(T^{-1}f) = T^{-1}f\end{aligned}$$

If $\tilde{\phi}_j = \phi_j$ for all j the frame is said to be a tight frame. A tight frame is to an orthonormal basis what a frame is to a Riesz basis (recall that a Riesz basis with self-dual vectors, i.e., $\tilde{\phi}_j = \phi_j$ is an orthonormal basis).

In signal analysis one is interested in the *representation* of signals. Hence Eqn. 4.7, which gives an expansion of a signal, is more natural than Eqn. 4.3 (which precisely defines a frame). One could take Eqn. 4.7 as the *definition* of a frame, dual frame pair for $\mathcal{H}$. The disadvantage with this approach is that for a fixed frame, the dual pair may not be unique (see Example 8). However, the construction of the dual frame from Eqn. 4.3 by inversion of the frame operator *is unique*. The particular dual frame constructed by inversion of the frame operator has an *extremal* property. If $\left\{\breve{\phi}_j\right\}$ is any dual frame, $\left\{\tilde{\phi}_j\right\}$ is the dual frame obtained by inversion of the frame operator and $\breve{\phi}_j = \tilde{\phi}_j + e_j$, then, $\sum_j e_j \rangle\langle\, \phi_j$ is the zero operator. We now introduce the notion of a *generalized frame pair*, which is useful in the study of the action of PR filter banks on separable Hilbert spaces. Generalized frame pairs have two sets of vectors spanning different Hilbert spaces, with the additional property that from the projection of a vector in one space onto the other, one can recover the vector. In a sense, the pair of Hilbert spaces are *at an angle* with respect to each other (i.e not perpendicular).

Definition 11 The (bounded) sets of vectors $\left(\{\phi_j\}, \left\{\tilde{\phi}_j\right\}\right)$ is said to be a *generalized frame pair* in a separable Hilbert space $\mathcal{X}$ if $\forall f \in \mathcal{H}$,

$f = \sum_j \left\langle f, \tilde{\phi}_j \right\rangle \phi_j$ and $\forall f \in \tilde{\mathcal{H}}$ $f = \sum_j \langle f, \phi_j \rangle \tilde{\phi}_j$, where $\mathcal{H} = Span\{\phi_j\}$ and $\tilde{\mathcal{H}} = Span\left\{\tilde{\phi}_j\right\}$.

In particular if $\mathcal{H} = \tilde{\mathcal{H}}$, $\{\phi_j\}$ forms a frame for $\mathcal{H}$, and hence the terminology *generalized* frame pair. It turns out that PR filter banks give a natural change of coordinates for separable Hilbert spaces with generalized frame pairs. A generalized frame pair is said to span the pair of spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$. A generalized frame pair is said to be a *generalized Riesz basis pair* if $\left\langle \phi_i, \tilde{\phi}_j \right\rangle = \delta(i-j)$.

Example 8 (Frame) Let $\mathcal{H} = \mathbb{R}^2$. The vectors $\{\phi_1, \phi_2, \phi_3\}$ given by

$$\begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

form a frame with frame operator

$$T = \sum_j \phi_j \rangle\langle \phi_j = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

A dual frame is given by

$$\begin{bmatrix} \tilde{\phi}_1 & \tilde{\phi}_2 & \tilde{\phi}_3 \end{bmatrix} = T^{-1} \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}.$$

The frame bounds (extremal eigenvalues T) are $A = 1$ and $B = 3$. Also

$$\sum_j \tilde{\phi}_j \rangle\langle \phi_j = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Another dual frame (there are infinitely many of them) is given by

$$\begin{bmatrix} \tilde{\phi}_1 & \tilde{\phi}_2 & \tilde{\phi}_3 \end{bmatrix} = \begin{bmatrix} 5/6 & -1/6 & -1/6 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}.$$

In general if A is an $m \times n$ matrix ($m \leq n$) of rank m with no trivial columns, the columns of A form a frame. The columns of the transpose of any *right* inverse of A (with no trivial columns) form a dual frame. The frame operator is given by AA^*, and the *unique* dual frame obtained by inversion of the frame operator is given by the columns of $(AA^*)^{-1}A$. This dual is related to the minimum norm solution of the linear equation $A^*x = y$, which is $x = (AA^*)^{-1}Ay$. This extremal property is always exhibited by the particular dual frame $\{T^{-1}\phi_j\}$.

Every right-invertible matrix (with non-trivial columns) gives rise to a frame. In particular right-unitary matrices are right invertible and they give rise to all possible tight frames.

Example 9 (Tight Frame) For $\mathbb{R}^3$ the set of vectors $\{\phi_j\}$ given by

$$\begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

form a tight frame with frame bounds $A = B = 1$ since the frame operator

$$T = \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ -1/2 & -1/2 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $\{\phi_j\}$ forms a tight frame and $\|\phi_j\|^2 = \frac{1}{\alpha}$ then $\alpha > 1$ measures the *redundancy* of the tight frame relative to an orthonormal basis. If $\varphi_j = \sqrt{\alpha}\phi_j$, $\|\varphi_j\| = 1$ and

$$f = \frac{1}{\alpha} \sum_j \langle f, \varphi_j \rangle \varphi_j.$$

Consider the construction of such tight frames in $\mathbb{R}^m$ with n vectors φ_j. In some coordinate system they are described by an $m \times n$ right unitary matrix. Right unitariness