Fact 9 *(Daubechies)* If $Q(z)$ is such that

$$\sup_{\omega \in \mathbb{R}} \prod_{j=0}^{l} \left| Q\left(\frac{\omega}{M^j}\right) \right| < M^{l(K-m-\frac{1}{2})}, \tag{4.51}$$

then $\psi_0(t)$ is m times continuously differentiable.

The wavelets, being finite linear combination of translates of the scaling function, are as regular as the scaling function. In particular if $\sup_\omega Q(\omega) < M^{K-1}$, then the scaling function and wavelets are continuous.

Remark: Regularity as defined here is a property of the scaling vector and *not* of the scaling function. It is easy to construct examples of of unitary scaling vectors with different orders of regularity such that the scaling function corresponding to the less regular scaling vector is smoother than the scaling function corresponding to the more regular scaling vector [36].

We now describe the construction of K-regular multiplicity M scaling vectors *of minimal length*. We have seen that K-regularity is equivalent to $(M-1)(K-1)$ linear constraints on h_0, and that an arbitrary multiplicity M scaling vector of length $N = MK$ is determined by $(M-1)(K-1)$ parameters. By imposing the regularity constraints on the general parametrization of unitary scaling vectors, one expects to obtain K-regular scaling vectors. However, there is no analytical method to solve the resultant set of $(M-1)(K-1)$ nonlinear equations (in the parameters) and until now numerical techniques have been the answer. Here we provide a numerical scheme by solving a systems of linear equations (these equations can be explicitly solved for also [42, 78]). We postulate the form of the scaling vector (Eqn. 4.48) and try to solve for the polynomial $Q(z)$ such that $H_0(z)$ is a unitary scaling vector. This approach is particularly simple because the unitariness conditions are linear in the autocorrelation of $Q(z)$.

With $N = MK$, $Q(z)$ is seen from Eqn. 4.48 to be a polynomial of degree $(K-1)$ in z^{-1}. With the definitions $H_0(z) = P(z)Q(z)$, $R(z) = P(z)P(z^{-1})$, and $A(z) =$

$Q(z)Q(z^{-1})$,

$$\begin{aligned} 1 &= [\downarrow M]\, H_0(z)H_0(z^{-1}) \\ &= [\downarrow M]\, P(z)P(z^{-1})Q(z)Q(z^{-1}) \\ &= [\downarrow M]\, R(z)A(z). \end{aligned} \tag{4.52}$$

$P(z)$ is explicitly known and is a polynomial of degree $(M-1)K$. Therefore, $R(z)$ can be precomputed and is a polynomial with $2K(M-1)+1$ terms.

$$R(z) = P(z)P(z^{-1}) = \sum_{i=-(M-1)K}^{(M-1)K} r(i)z^{-i}.$$

Then

$$A(z) = Q(z)Q(z^{-1}) = \sum_{i=-(K-1)}^{(K-1)} a(n)z^{-n}.$$

Define the $K \times K$ matrix, $S = [s_{i,j}]$ for $i, j \in \mathcal{R}(K)$,

$$s_{i,j} = \begin{cases} r(Mi) & \text{for } j = 0 \\ r(Mi+j-1) + r(Mi-j+1) & \text{for } 1 \le j \le K-1 \end{cases} \tag{4.53}$$

Then Eqn. 4.52 becomes,

$$S \begin{bmatrix} a(0) \\ a(1) \\ \vdots \\ a(K-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4.54}$$

For arbitrary M and K, S is invertible. However, for large M or K S may be badly scaled. By scaling all but the first row of L (for which the rhs is zero), this problem can be circumvented. Once $A(z)$ is obtained, if it is positive definite by spectral factorization $Q(z)$ can be obtained. Positive definiteness of $A(z)$ can be inferred from the general Lagrange interpolation arguments in [80]. There is a degree of freedom in the choice of $Q(z)$ depending on which spectral factors are chosen. One may choose a minimum phase, maximum phase, or mixed solution. For each such choice of $Q(z)$ one has a corresponding K-regular unitary scaling vector.

Independently of our effort several authors have obtained explicit formulae for the autocorrelation sequence $A(z)$ [42, 78].

4.4.4 Examples of K-regular Unitary h_0 and $\psi_0(t)$

The minimal phase solutions for K-regular unitary scaling vectors. for $M = 3$ and $M = 4$ are given in Table 4.1. For $K = 2$, the minimal phase and maximal phase solutions (there are only two solutions in this case) for arbitrary M is given by the following formula:

$$H_0(z) = \left[\frac{1 + z^{-1} + \ldots + z^{-(M-1)}}{M}\right]^2 (q(0) + q(1)z^{-1}),$$

where

$$q(0) = \frac{\sqrt{M}}{2}\left[1 \pm \sqrt{\frac{2M^2+1}{3}}\right] \quad \text{and} \quad q(1) = \frac{\sqrt{M}}{2}\left[1 \mp \sqrt{\frac{2M^2+1}{3}}\right].$$

Figs. 4.1-4.3 show the scaling functions, their Fourier transform, and the Fourier transform of the scaling vector, for 3-band, 4-band and 5-band case for $K = 2, 3, 4$ and 5. Notice that the shape of a multiplicity M, K-regular scaling function is largely determined by its regularity, K. Notice also that the Fourier transforms of the scaling functions vanish at multiples of 2π, and that $H_0(\omega)$ does not vanish for $|\omega| < \frac{\pi}{M}$ (in fact the first zero is at $\frac{2\pi}{M}$). This fact implies (from Corollary 3) that all K-regular minimal length scaling vectors give rise to orthonormal wavelet bases.

4.4.5 Construction of Regular Multiplicity M Wavelets

K-regular compactly supported WTFs are now parameterized in state-space. In the K-regular case since the scaling vectors are obtained independently of the Householder parameterization the state-space approach gives an elegant way to generate the wavelets. We have the following state-space characterization of compactly supported WTFs.

Table 4.1: K-Regular Minimal Length Unitary Scaling Vectors

$M=3$	n	$h_0(n)$	$M=3$	n	$h_0(n)$	$M=3$	n	$h_0(n)$
$K=2$	0	0.33838609728386	$K=3$	0	0.20313514584456	$K=6$	0	0.04641991275121
	1	0.53083618701374		1	0.42315033910807		1	0.16394657299264
	2	0.72328627674361		2	0.70731556228155		2	0.40667150052122
	3	0.23896417190576		3	0.44622537783130		3	0.56561987503637
	4	0.04651408217589		4	0.19864508103414		4	0.58223034773984
	5	-0.14593600755399		5	-0.17723527558292		5	0.24390438994869
$K=5$	0	0.07550761756143		6	-0.07201025448623		6	-0.03360979671399
	1	0.23086070821719		7	-0.04444515095259		7	-0.25350741685252
	2	0.51304535032014		8	0.04726998249100		8	-0.08274027041541
	3	0.59269796491023	$K=4$	0	0.12340698195349		9	-0.00156787261030
	4	0.50343156427108		1	0.31789563892953		10	0.11605073148585
	5	0.07274582768779		2	0.62131686335095		11	0.00346097586136
	6	-0.11559776131042		3	0.56142607070711		12	0.00040170813801
	7	-0.21804646388388		4	0.36890783202512		13	-0.03676774192987
	8	0.00692356260197		5	-0.08625807908307		14	0.00823961325941
	9	0.02913316570545		6	-0.12777980080646		15	0.00008644258833
	10	0.07286749987661		7	-0.13375920464072		16	0.00539777575368
	11	-0.02130382202714		8	0.05875903404127		17	-0.00218593998563
	12	-0.00439071767705		9	0.02029701733548			
	13	-0.01176303929137		10	0.02430600287569			
	14	0.00593935060686		11	-0.01646754911953			

$M=4$	n	$h_0(n)$	$M=4$	n	$h_0(n)$	$M=4$	n	$h_0(n)$
$K=2$	0	0.26978904939721	$K=4$	0	0.08571412050958	$K=5$	0	0.04916991424487
	1	0.39478904939721		1	0.19313899295294		1	0.12913015554835
	2	0.51978904939721		2	0.34917971394336		2	0.26140970524347
	3	0.64478904939721		3	0.56164878348085		3	0.46212341604513
	4	0.23021095060279		4	0.49550221952707		4	0.50348969444395
	5	0.10521095060279		5	0.41456599638527		5	0.49742757908607
	6	-0.01978904939721		6	0.21903222760227		6	0.35826639102137
	7	-0.14478904939721		7	-0.11453658682193		7	0.02935921939015
$K=3$	0	0.15083145463571		8	-0.09529322382982		8	-0.06205420421862
	1	0.28192600003506		9	-0.13069539487629		9	-0.17204166252712
	2	0.44427054543441		10	-0.08275002028156		10	-0.16539775306492
	3	0.63786509083375		11	0.07198039995437		11	0.03112914045751
	4	0.41021527232597		12	0.01407688379317		12	0.01081024188105
	5	0.27302618152727		13	0.02299040553808		13	0.05413053935774
	6	0.07333709072858		14	0.01453807873593		14	0.05420808584699
	7	-0.18885200007012		15	-0.01909259661330		15	-0.02997902951088
	8	-0.06104672696168					16	-0.00141564635125
	9	-0.05495218156233					17	-0.00864661146505
	10	-0.01760763616298					18	-0.00848642904691
	11	0.05098690923636					19	0.00736725361809

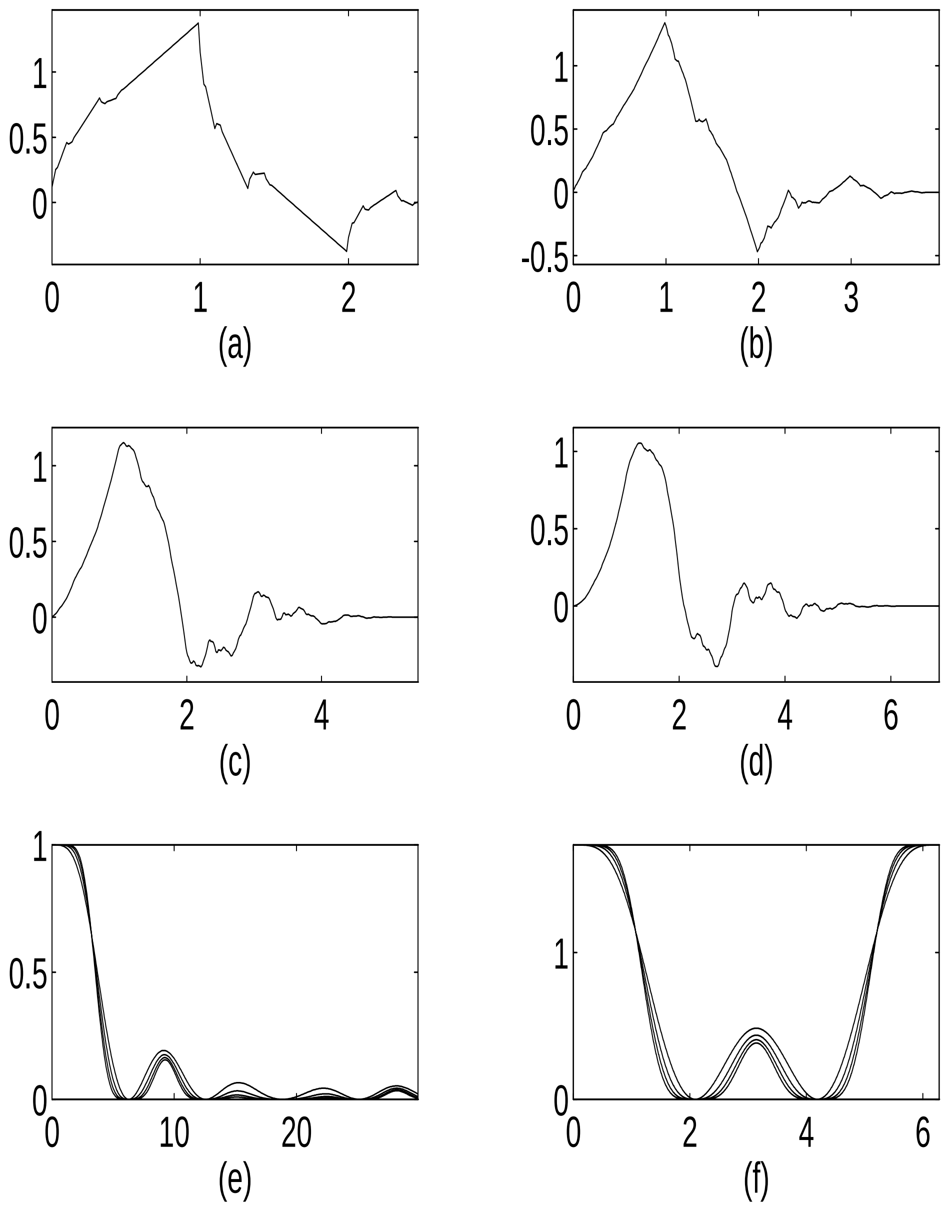

Figure 4.1: K-regular, multiplicity 3 Scaling Functions: (a) $\psi_0(t)$ for $K = 2$ (b) $\psi_0(t)$ for $K = 3$ (c) $\psi_0(t)$ for $K = 4$ (d) $\psi_0(t)$ for $K = 5$ (e) $\hat{\psi}_0(\omega)$ for $K = 2$ through $K = 5$ with maximal flatness of the Fourier transform increasing with K (f) $H_0(\omega)$ for $K = 2$ through $K = 5$

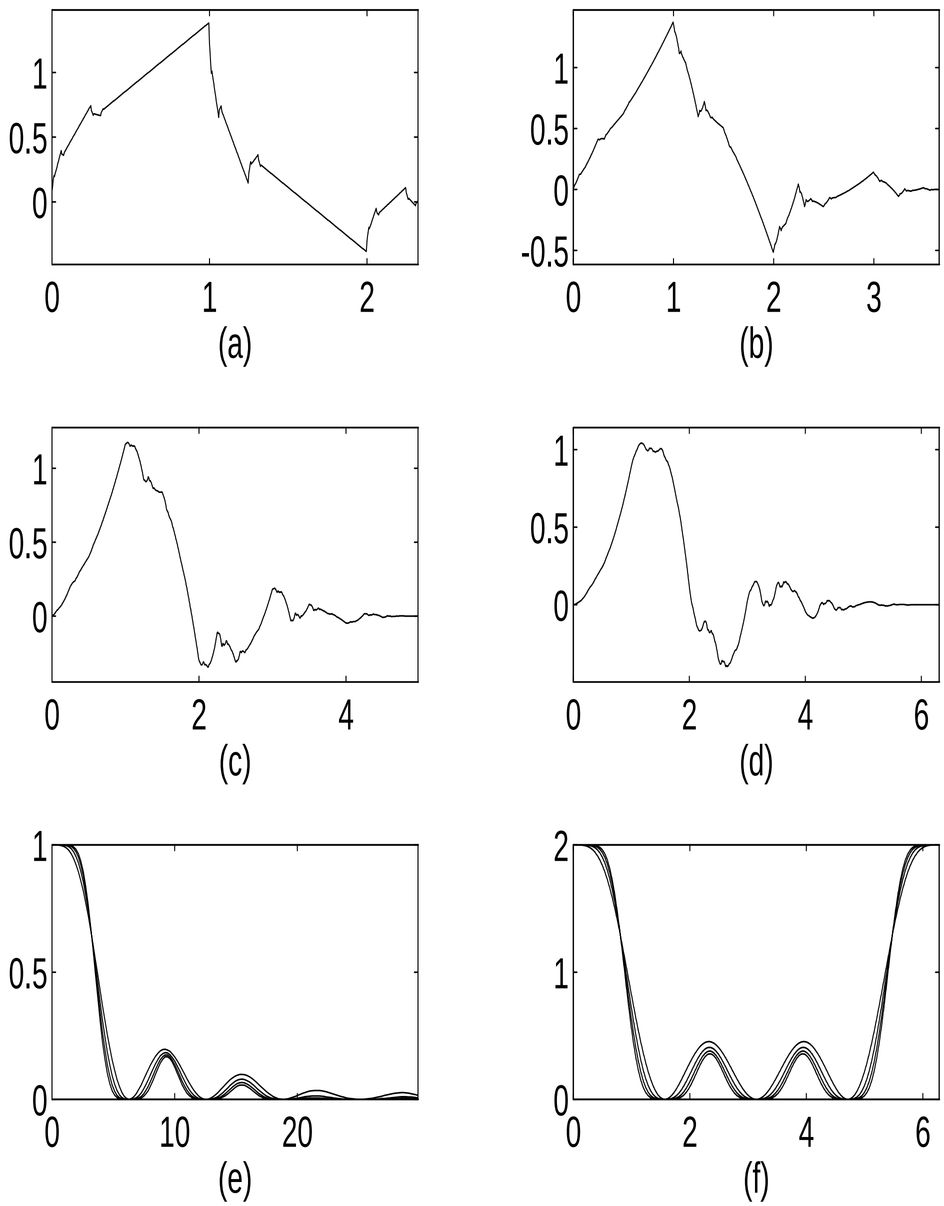

Figure 4.2: K-regular multiplicity 4 Scaling Functions: (a) $\psi_0(t)$ for $K = 2$ (b) $\psi_0(t)$ for $K = 3$ (c) $\psi_0(t)$ for $K = 4$ (d) $\psi_0(t)$ for $K = 5$ (e) $\hat{\psi}_0(\omega)$ for $K = 2$ through $K = 5$ with maximal flatness of the Fourier transform increasing with K (f) $H_0(\omega)$ for $K = 2$ through $K = 5$

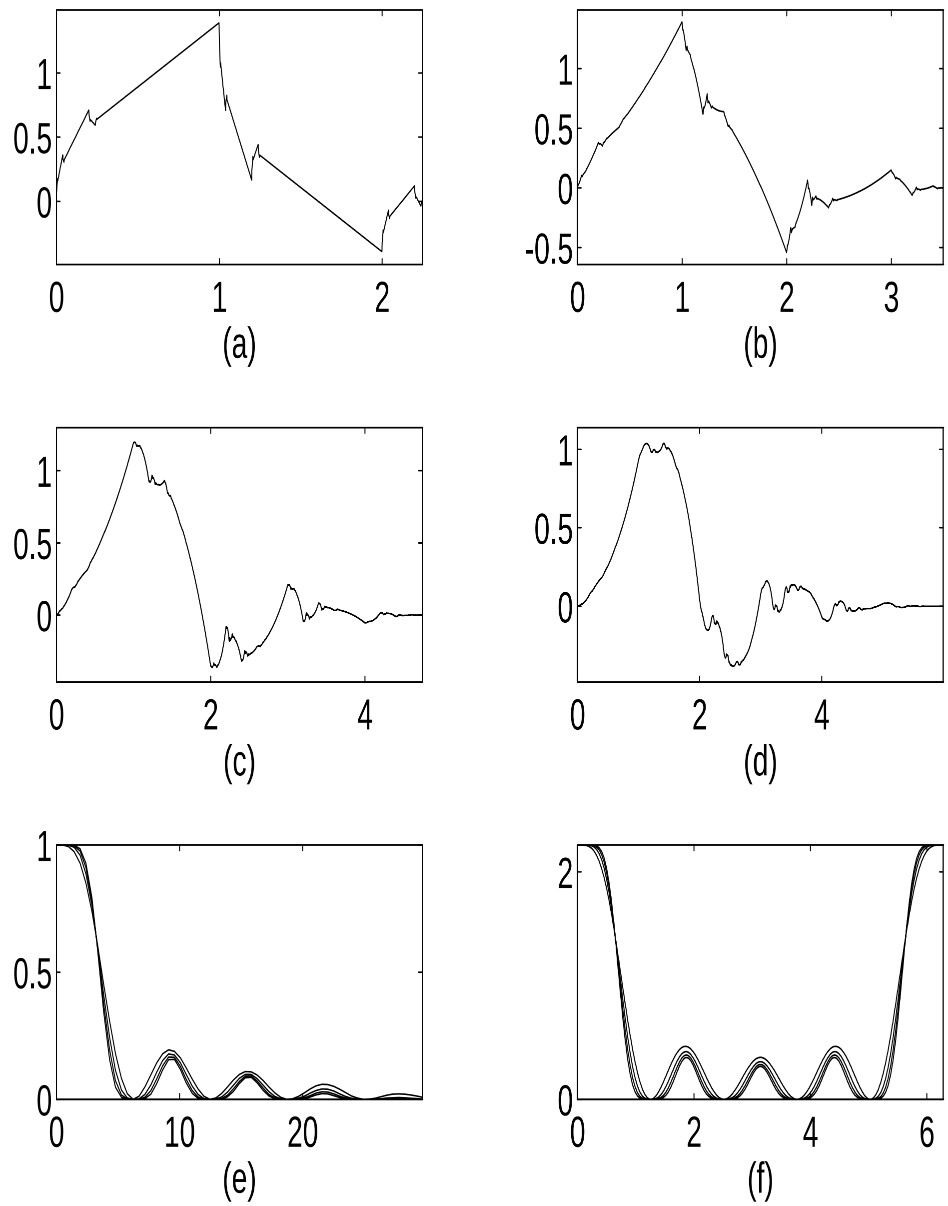

Figure 4.3: K-regular multiplicity 5 Scaling Functions: (a) $\psi_0(t)$ for $K = 2$ (b) $\psi_0(t)$ for $K = 3$ (c) $\psi_0(t)$ for $K = 4$ (d) $\psi_0(t)$ for $K = 5$ (e) $\hat{\psi}_0(\omega)$ for $K = 2$ through $K = 5$ with maximal flatness of the Fourier transform increasing with K (f) $H_0(\omega)$ for $K = 2$ through $K = 5$

Theorem 30 Let Y be an $(K+M)\times(K+M)$ orthogonal matrix with an $N \times N$ nilpotent submatrix. Without loss of generality (by permutation) assume Y is of the form

$$Y = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A is nilpotent. Furthermore for some row $\begin{bmatrix} c & d \end{bmatrix}$ of $\begin{bmatrix} C & D \end{bmatrix}$

$$c(I-A)^{-1}B + d = [\frac{1}{\sqrt{M}}, \ldots, \frac{1}{\sqrt{M}}] = \mathbf{e} \tag{4.55}$$

Then there exists a compactly supported WTF, with the functions $\psi_i(t)$ supported in $[0, \frac{MK-1}{M-1}]$, such that $[A, B, c, d]$ is a realization of the (dual first-orthant) polyphase representation of the scaling vector:

$$H_0(z) = c(zI - A)^{-1}B + d.$$

Conversely, given an arbitrary WTF with $[0, \frac{MK-1}{M-1})$, there exists an orthogonal Y with the above properties.

Proof: Given a wavelet tight frame, let $[A, B, C, D]$ be a balanced realization of the polyphase matrix $H_p(z)$. Since $H_p(z)$ is unitary, Y is unitary (Fact 8). Also, from Eqn. 4.13, if c and d are the first rows of C and D respectively, then, Eqn. 4.55 is satisfied. On the other hand, given Y, we can unitarily *dilate* it (i.e add rows to make it a unitary matrix) by a Gram-Schmidt process. We can also permute rows so that $\begin{bmatrix} c & b \end{bmatrix}$ is the first row of $\begin{bmatrix} C & B \end{bmatrix}$. Also, the fact that $\hat{H}_p(1) = \mathbf{e}$ implies that

$$c(zI - A)^{-1}B + d = \mathbf{e}$$

Conversely given Y define $H_p(z) = C(zI - A)^{-1}B + D$. The nilpotency of A ensures that $H_p(z)$ is a polynomial in z^{-1}. It is easy to check that $[A, B]$ is controllable and $[C, A]$ is observable. Hence $[A, B, C, D]$ is a minimal realization of $H_p(z)$. The fact that $c(zI - A)^{-1}B + d = \mathbf{e}$ ensures that $H_0(0) = \sqrt{M}$. Therefore, since Eqn. 4.47 is satisfied we have a wavelet tight frame. □

Given h_0 Theorem 30 implies the existence of an orthogonal matrix (the *state-space wavelet matrix*) Y. From Y, one can read off the associated wavelets. A complete parameterization of all possible Y associated with a given h_0 (such that $H_p(z)$ has the same McMillan degree as the polyphase vector of h_0) gives a corresponding parameterization of wavelet vectors and wavelets. Let $[\hat{A}, \hat{B}, \hat{C}, \hat{D}]$ be a minimal realization of $\hat{H}_p(z) = [H_{0,0}(z), H_{0,1}(z), \ldots, H_{0,M-1}(z)]$. From this information we now explicitly construct Y.

Lemma 27 If $[\hat{A}, \hat{B}, \hat{C}, \hat{D}]$ is a minimal realization of $\hat{H}_p(z)$, there exists a non-singular matrix T and an associated partition of any *state-space wavelet* matrix Y (corresponding to h_0),

$$Y = \left[\begin{array}{c} Y_1 \\ \hline Y_2 \end{array}\right] = \left[\begin{array}{cc} A & B \\ C_1 & D_1 \\ \hline C_2 & D_2 \end{array}\right], \tag{4.56}$$

such that

$$\begin{bmatrix} A & B \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} T\hat{A}T^{-1} & T\hat{B} \\ \hat{C}T^{-1} & \hat{D} \end{bmatrix}. \tag{4.57}$$

Proof: Theorem 30 implies the existence of Y which can be partitioned so that $H_1(z) = C_1(zI-A)^{-1}B + D_1$ and $H_2(z) = C_2(zI-A)^{-1}B + D_2$ where $H_1(z)$ is $1 \times M$ and $H_2(z)$ is $M-1 \times M$ and $H_p(z) = \begin{bmatrix} H_1(z) \\ H_2(z) \end{bmatrix}$. Since $H_1(z) = \hat{H}(z)$, $[A, B, C_1, D_1]$ and $[\hat{A}, \hat{B}, \hat{C}, \hat{D}]$ are similar realizations and hence there exists a non-singular matrix T such that the result holds. □

Given a minimal realization $[\hat{A}, \hat{B}, \hat{C}, \hat{D}]$ of $\hat{H}(z)$, if by constructing T one obtains Y_1. Y_2 is then constructed by a Gram-Schmidt process (using the SVD algorithm). Since Y is orthogonal T must be chosen so that $Y_1 Y_1^T = I$. This implies that

$$AA^T + BB^T = I, \quad AC_1^T + BD_1^T = 0, \quad \text{and} \quad C_1C_1^T + D_1D_1^T = 1. \tag{4.58}$$

Theorem 31 Let $[\hat{A}, \hat{B}, \hat{C}, \hat{D}]$ be a minimal realization of $\hat{H}(z)$. Let $\hat{W}_c$ be the controllability matrix of $\hat{H}(z)$. Then $T = W_c^{-\frac{1}{2}}$ is the balancing transformation for $\hat{H}(z)$.

Proof: Eqn. 4.58 follows from Eqn. 4.57 and Eqn. 4.44.

$$\begin{aligned} AA^T + BB^T &= T\hat{A}T^{-1}(T^{-1})^T\hat{A}^T T + T\hat{B}\hat{B}^T T^T \\ &= W_c^{-\frac{1}{2}}\hat{A}W_c\hat{A}^T W_c^{-\frac{1}{2}} + W_c^{-\frac{1}{2}}\hat{B}\hat{B}^T W_c^{-\frac{1}{2}} \\ &= W_c^{-\frac{1}{2}}\left[W_c - \hat{B}\hat{B}^T\right]W_c^{-\frac{1}{2}} + W_c^{-\frac{1}{2}}\hat{B}\hat{B}^T W_c^{-\frac{1}{2}} \\ &= I \end{aligned} \tag{4.59}$$

$$\begin{aligned} AC_1^T + BD_1^T &= W_c^{-\frac{1}{2}}\hat{A}(W_c^{-\frac{1}{2}})^{-1}(W_c^{-\frac{1}{2}})^{-1}\hat{C}^T + W_c^{-\frac{1}{2}}\hat{B}\hat{D}^T \\ &= W_c^{-\frac{1}{2}}\left[\hat{A}W_c\hat{C}^T + \hat{B}\hat{D}^T\right] = 0 \end{aligned} \tag{4.60}$$

$$\begin{aligned} C_1C_1^T + D_1D_1^T &= \hat{C}(W_c^{-\frac{1}{2}})^{-1}(\hat{C}(W_c^{-\frac{1}{2}})^{-1})^T + \hat{D}\hat{D}^T \\ &= \hat{C}W_c\hat{C}^T + \hat{D}\hat{D}^T = I. \end{aligned} \tag{4.61}$$

□

It is interesting to note that though in principle T could have depended on any of the four state-space matrices that describe the scaling vector, it really depended only upon the state and input matrices (A and B respectively).

Parametrization Of Multiplicity M Wavelets

SVD completion of Y_1

We now describe *one way* to construct the wavelet filters in this state-space setting by using the singular-value-decomposition (SVD). Given right unitary Y_1, one has to find Y_2 such that

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}^T = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{4.62}$$

Define the symmetric positive definite matrix $X = Y_2 Y_2^T$ so that $X = I - Y_1 Y_1^T$. Let $X = U\Sigma U^T$ be the SVD of X. Σ is a diagonal matrix of positive entries. Then a general solution for Y_2 is given by $Y_2 = U(\Sigma)^{\frac{1}{2}}\Theta$ where Θ is an arbitrary $M-1 \times M-1$ constant unitary matrix. This follows since $Y_2 Y_2^T = U(\Sigma)^{\frac{1}{2}}\Theta\Theta^T(\Sigma)^{\frac{1}{2}}U^T = X$. The number of degrees of freedom in the choice of the wavelet filters is $\begin{pmatrix} M-1 \\ 2 \end{pmatrix}$.

The state-space parameterization gives a method for the numerical design of wavelets with respect to any objective function (of Θ) as an unconstrained minimization problem. It may also be used for the design of unitary filter banks. Since the scaling vector could be considered as the lowpass filter of a unitary filter bank, the above parameterization could be used to design the rest of the filters in a unitary filter bank.

4.4.6 Regularity of the Wavelets: Is it Important?

There are many applications where K regular wavelet tight frames have been found useful. Usually this is attributed to the smoothness (high order differentiability) of the scaling functions and wavelets. This section argues that the smoothness of the functions is usually irrelevant. It is the regularity of the scaling vector (and the equivalent vanishing moments property of the wavelet vectors and wavelets), that is critical.

Wavelet theory has been developed from a filter banks point of view. This is the most general approach to the wavelet theory. In special cases, one can construct WTFs independently of filter bank theory [17, 14]. We are not aware of any construction of multiplicity M wavelet bases, that do not start from a filter bank perspective. Filter bank theory is more fundamental than wavelet theory. It is popularly believed that wavelet theory gives an underlying continuous time interpretation of sequences in a filter bank. Be that as it may, it must be realized that digital signal processing had its roots in the Shannon sampling theorem, allowing one to go from continuous time to discrete time. Therefore, an underlying continuous time interpretation of filter banks is classical.

Consider the practical analysis of a signal with respect to an orthonormal wavelet basis $\{\psi_{i,j,k}(t)\}$. Assume that a finite set of samples of the signal $f(t)$ is given. In the wavelet analysis of $f(t)$ (see Section 5.2) usually, the samples of $f(t)$ are assumed to be the expansion coefficient of $f(t)$ at the finest scale of interest, say J_f. That is,

$$f(t) \approx Wf(0, J_f, k)\psi_{0,J_f,k}(t) \tag{4.63}$$

From this information using a filter bank, $Wf(i,j,k)$ is computed for $j < J_f$. These coefficients (computed approximately) are the referred to as the DWT (discrete wavelet transform) of $f(t)$. We now to give an *exact* interpretation of this process, that does not require knowledge of wavelet theory.

The idea is to interpret the samples $f(\frac{n}{T})$ as the the Nyquist rate samples of a bandlimited signal. These turn out to be the expansion coefficients with respect to the *sinc* basis. Notice that the sinc function is infinitely differentiable. There is no problem with the fact that there are only finitely many samples of f, since bandlimited signals could have finitely many Nyquist samples. Let $\mathcal{H}$ denote the space of bandlimited signals in $[0, \pi/T)$. Then $f \in \mathcal{H}$. From Theorem 24, we are essentially decomposing $\mathcal{H}$, recursively using the filter bank, and the numbers $W(i,j,k)$ are precisely the expansion coefficients with respect to the bases in the decomposition subspaces. Since all the basis function at any level are linear combinations (finite if the filter bank is FIR) of the sinc function, they are infinitely smooth, regardless, of whether the filters in the filter bank give rise to K *regular* wavelets. Hence, even assuming we have a filter bank with tight frame wavelets, there is no reason to impose constraints on the sequences h_0 so that they give smooth wavelets, because, all practical computations, can be interpreted as being, done with some other smooth basis. Thus smoothness of the underlying functions, is not a valid argument for using

K regular h_0 and associated WTFs. However, K regularity of h_0 is important only we have an infinite cascade of decompositions.

For best frequency localization, therefore, the filters $h_i(n)$, need not even give rise to a WTF, let alone be regular. However, in numerical analysis applications (like the approximation of differential operators), the vanishing moments property of the scaling vector (which causes smoothness of the corresponding $\psi_0(t)$) is important [39]. However, in signal processing applications, usually regularity is unimportant (unless filter banks are used in tree structures).

4.5 Modulated Wavelet Tight Frames

Unitary filter banks give rise to WTFs. We now show that modulated unitary filter banks gives rise to modulated WTFs. In MFBs the filter h_0 determines the rest of the filter bank (since it determines the prototype filter). Therefore for modulated WTFs the scaling vector uniquely fixes the wavelet vectors (and hence the wavelets). Since the scaling function determines the multiresolution analysis, the multiresolution analysis would also determine the wavelets. For large M modulated WTFs are possibly the only WTFs that can be *designed* to meet desired specifications.

4.5.1 Parameterization of Modulated Wavelet Tight Frames

A unitary filter bank is associated with a WTF iff it satisfies Eqn. 4.13, i.e., $H_0(1) = \sqrt{M}$. The filters in an MFB are given by $h_i(n) = c_{i,n-\frac{\alpha}{2}} h(n)$ where α is equal to $(M-1)$ or $(M-2)$ depending on whether the MFB is Type 1 or Type 2. The modulation vector satisfies $c_{i,n+2lM} = (-1)^l c_{i,n}$ and $c_{i,M} = -(-1)^i s_{i,0}$. This periodicity can be exploited to express the filters $h_i(n)$ in terms of the polyphase components of the prototype filter $h(n)$. For $l \in \mathbf{Z}$, let $p_l(n) = h(Mn + l)$, and for $j \in \{0, 1\}$, let $p_{l,j}(n) = h(2Mn + Mj + l)$. The analysis filters can be expressed as

$$H_i(z) = \sum_n z^{-n} c_{i,n-\frac{\alpha}{2}} h(n)$$

$$
\begin{aligned}
&= \sum_{l=0}^{M-1} c_{i,l-\frac{\alpha}{2}} z^{-l} \left[\sum_n (-1)^l h(2Mn+l) z^{-2Mn} \right] \ldots \\
&\quad \ldots - (-1)^i s_{i,l-\frac{\alpha}{2}} z^{-M-l} \left[\sum_n (-1)^l h(2Mn+M+l) z^{-2Mn} \right] \\
&= \sum_{l=0}^{M-1} c_{i,l-\frac{\alpha}{2}} z^{-l} P_{l,0}(-z^{2M}) - (-1)^i s_{i,l-\frac{\alpha}{2}} z^{-M-l} P_{l,1}(-z^{2M}). \qquad (4.64)
\end{aligned}
$$

In particular

$$
H_0(1) = \sum_{l=0}^{M-1} \cos\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})\right) P_{l,0}(-1) - \sin\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})\right) P_{l,1}(-1). \qquad (4.65)
$$

For each of the J lattices of the unitary MFB (from Eqn. 3.47 substituting $z=-1$)

$$
\begin{bmatrix} P_{l,0}(-1) & P_{\alpha-l,0}(-1) \\ P_{l,1}(-1) & -P_{\alpha-l,1}(-1) \end{bmatrix} = \sqrt{\frac{2}{M}} \begin{bmatrix} \cos(\Theta_l) & \sin(\Theta_l) \\ \sin(\Theta_l) & -\cos(\Theta_l) \end{bmatrix} \qquad (4.66)
$$

where $\Theta_l = \sum_{k=0}^{k_l-1} \theta_{l,k}$. Even if the MFB is not FIR we may define Θ_l as in Eqn. 4.66 and hence the results that follow hold in the IIR case also.

Type 1: M even

For Type 1 MFBs $c_{i,M-1-n-\frac{\alpha}{2}} = c_{i,n-\frac{\alpha}{2}}$ and $s_{i,M-1-n-\frac{\alpha}{2}} = -s_{i,n-\frac{\alpha}{2}}$. The MFB forms a WTF iff

$$
\begin{aligned}
\sqrt{M} &= H_0(1) \\
&= \sqrt{\frac{2}{M}} \sum_{l=0}^{M-1} \cos\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})\right) P_{l,0}(-1) - \sin\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})\right) P_{l,1}(-1) \\
&= \sqrt{\frac{2}{M}} \sum_{l=0}^{J-1} \cos\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})\right) P_{l,0}(-1) \\
&\quad + \cos\left(\frac{\pi}{2M}(M-1-l-\frac{\alpha}{2})\right) P_{M-1-l,0}(-1) \\
&\quad - \sin\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})\right) P_{l,1}(-1) - \sin\left(\frac{\pi}{2M}(M-1-l-\frac{\alpha}{2})\right) P_{M-1-l,1}(-1) \\
&= \sqrt{\frac{2}{M}} \sum_{l=0}^{J-1} \cos\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})\right) [\cos(\Theta_l) + \sin(\Theta_l)]
\end{aligned}
$$

$$
\begin{aligned}
& -\sin\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})\right)[\sin(\Theta_l)-\cos(\Theta_l)] \\
= & \sqrt{\frac{2}{M}}\sum_{l=0}^{J-1}\cos\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})+\Theta_l\right)+\sin\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})+\Theta_l\right) \\
= & \frac{2}{\sqrt{M}}\sum_{l=0}^{J-1}\sin\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})+\Theta_l+\frac{\pi}{4}\right)
\end{aligned}
$$

Equivalently

$$
\sum_{l=0}^{J-1}\sin\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})+\Theta_l+\frac{\pi}{4}\right)=\frac{M}{2}=J.
$$

Since a sum of J sinusoids is J iff each individual sinusoid is 1

$$
\sin\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})+\Theta_l+\frac{\pi}{4}\right)=1 \quad \text{and} \quad \Theta_l=\frac{\pi}{4}+\frac{\pi}{2M}(\frac{\alpha}{2}-l). \tag{4.67}
$$

Type 1: M odd

Now $J=\frac{M-1}{2}$ and the situation is similar to when M is even except that the expression for $H_0(1)$ in Eqn. 4.65 has an extra term corresponding to $l=J=\frac{M-1}{2}$. Also $P_{J,0}(-1)=\pm\frac{1}{\sqrt{M}}$, $c_{0,J-\frac{\alpha}{2}}=c_{0,0}=1$ and $s_{0,J-\frac{\alpha}{2}}=s_{0,0}=0$. Hence

$$
\begin{aligned}
H_0(1) &= \frac{2}{\sqrt{M}}\sum_{l=0}^{J-1}\sin\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})+\Theta_l-\frac{\pi}{4}\right)+c_{0,J}P_{J,0}(-1)-s_{0,J}P_{J,1}(-1) \\
&= \frac{2}{\sqrt{M}}\sum_{l=0}^{J-1}\sin\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})+\Theta_l-\frac{\pi}{4}\right)\pm\frac{1}{\sqrt{M}}.
\end{aligned}
$$

The MFB forms a tight frame iff

$$
\sum_{l=0}^{J-1}\sin\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})+\Theta_l+\frac{\pi}{4}\right)\pm\frac{1}{\sqrt{M}}\pm\frac{1}{2}=\frac{M}{2}=J+\frac{1}{2}
$$

Since a sum of J sinusoids is at most J, the above equation is satisfied only when $P_J(-1)=\frac{1}{\sqrt{M}}$ whence Θ_l satisfies Eqn. 4.67.

Type 2: M even

Since $\alpha = (M-2)$, $c_{i,M-2-n-\frac{\alpha}{2}} = c_{i,n-\frac{\alpha}{2}}$ and $s_{i,M-2-n-\frac{\alpha}{2}} = -s_{i,n-\frac{\alpha}{2}}$. When M is even $J = \frac{M-1}{2}$ and from Eqn. 4.66

$$\begin{bmatrix} P_{l,0}(-1) & P_{M-2-l,0}(-1) \\ P_{l,1}(-1) & -P_{M-2-l,1}(-1) \end{bmatrix} = \sqrt{\frac{2}{M}} \begin{bmatrix} \cos(\Theta_l) & \sin(\Theta_l) \\ \sin(\Theta_l) & -\cos(\Theta_l) \end{bmatrix}.$$

Also from Theorem 12 $P_{M-1}(z) = \pm\sqrt{\frac{2}{M}} z^n$ for some n. Hence either $P_{M-1,0}(z)$ is zero or $P_{M-1,1}(z)$ is zero. The other evaluated at $z=-1$ is $\sqrt{\frac{2}{M}}$. Moreover

$$\cos\left(\frac{\pi}{2M}(M-1-\frac{\alpha}{2})\right) = \sin\left(\frac{\pi}{2M}(M-1-\frac{\alpha}{2})\right) = \frac{1}{\sqrt{2}}.$$

The term corresponding to $(M-1)$ in Eqn. 4.65 is $\pm\sqrt{\frac{1}{M}}$. Hence the MFB is a WTF iff

$$\begin{aligned} \sqrt{M} &= H_0(1) \\ &= \sqrt{\frac{2}{M}} \sum_{l=0}^{M-1} \cos\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})\right) P_{l,0}(-1) - \sin\left(\frac{\pi}{2M}(l-\frac{\alpha}{2})\right) P_{l,1}(-1) \\ &= \frac{2}{\sqrt{M}} \sum_{l=0}^{J-1} \sin\left(\frac{\pi}{2M}(l-\frac{\alpha}{2}) + \Theta_l + \frac{\pi}{4}\right) \pm \sqrt{\frac{1}{M}}. \end{aligned}$$

Equivalently

$$\frac{M \pm 1}{2} = \sum_{l=0}^{J-1} \sin\left(\frac{\pi}{2M}(l-\frac{\alpha}{2}) + \Theta_l + \frac{\pi}{4}\right).$$

Since $J = \frac{M-1}{2}$, there exists a solution only if $P_{M-1}(z)$ is such that the left hand side is $\frac{M-1}{2}$. This is always possible and once again Θ_l must satisfy Eqn. 4.67.

Type 2: M odd

As before, $P_{M-1}(z)$ contributes $\sqrt{\frac{1}{M}}$ to $H_0(1)$. From Theorem 12 $P_{\frac{M-2}{2},0}(z) = \frac{1}{\sqrt{M}} z^n$ for some n. Hence the MFB forms a WTF iff

$$\sqrt{M} = H_0(1) \frac{2}{\sqrt{M}} \sum_{l=0}^{J-1} \sin\left(\frac{\pi}{2M}(l-\frac{\alpha}{2}) + \Theta_l + \frac{\pi}{4}\right) \pm \sqrt{\frac{1}{M}} \pm \sqrt{\frac{1}{M}}.$$

The two terms $\sqrt{\frac{1}{M}}$ are contributed by $P_{M-1}(z)$ and $P_{\frac{M-2}{2}}(z)$. Therefore for a WTF

$$\frac{M \pm 1 \pm 1}{2} = \sin\left(\frac{\pi}{2M}(l - \frac{\alpha}{2}) + \Theta_l + \frac{\pi}{4}\right).$$

The maximum value of the right hand side is $\frac{M-2}{2}$. There exists a solution only if the left hand side is $\frac{M-2}{2}$. This is always possible and Θ_l satisfies Eqn. 4.67.

> **Theorem 32** *(Modulated Wavelet Tight Frames Theorem)* For all M there exist multiplicity M modulated WTFs. Every compactly supported modulated WTF is associated with an FIR unitary MFB and is parameterized by J unitary lattices such that the sum of the angles in the lattices satisfy (for $l \in \mathcal{R}(J)$) Eqn. 4.67. If a canonical MFB has Jk parameters, the corresponding WTF has $J(k-1)$ parameters.

Remark: Recall going from a unitary filter bank to a WTF requires the imposition of Eqn. 4.13, i.e., one constraint. However, this one constraint, in the modulated WTF case, becomes, J constraints (a constraint each on each of the lattices), and some additional assumptions on the delays (the non-lattice $p_l(n)$).

4.5.2 Some Examples of Modulated Wavelet Tight Frames

Example 12 (Type 1: M Even) For $M = 2$ the number of lattices $J = \frac{M}{2} = 1$. If the length of the scaling vector is $N = 4$ (and the MFB is canonical) this lattice has one parameter $\theta_{0,0}$ since $N_p = k_0 = 1$. Therefore the WTF has no free parameters ($J(k_0 - 1) = 0$). From Eqn. 4.67 $\Theta_0 = \theta_{0,0} = \frac{\pi}{8} + \frac{\pi}{4} = \frac{3\pi}{8}$. The prototype filter and the scaling and wavelet vectors are given by

$$h = \begin{bmatrix} \cos(\frac{\pi}{8}) \\ \cos(\frac{\pi}{8}) \\ \cos(\frac{3\pi}{8}) \\ \cos(\frac{5\pi}{8}) \end{bmatrix}, \quad h_0 = \begin{bmatrix} \cos(\frac{\pi}{8})\sin(\frac{\pi}{8}) \\ \cos^2(\frac{\pi}{8}) \\ \cos(\frac{\pi}{8})\sin(\frac{\pi}{8}) \\ -\sin^2(\frac{\pi}{8}) \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{\sqrt{2}+1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ -\frac{\sqrt{2}-1}{2\sqrt{2}} \end{bmatrix} \quad \text{and} \quad h_1 = \begin{bmatrix} -\frac{\sqrt{2}-1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ -\frac{\sqrt{2}+1}{2\sqrt{2}} \end{bmatrix}.$$

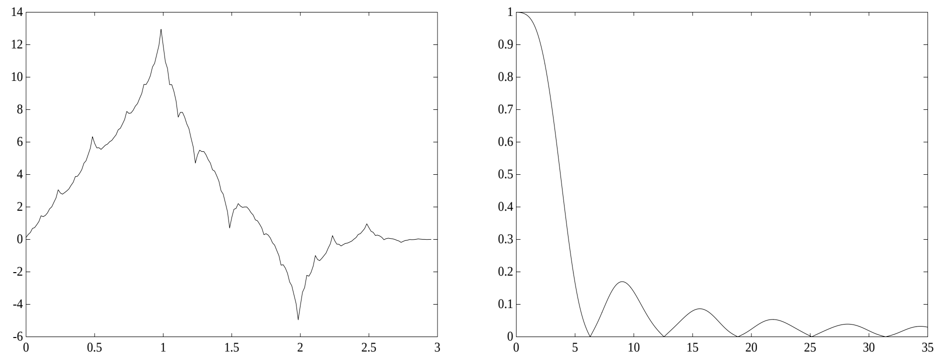

Figure 4.4: $\psi_0(t)$ and $\left|\hat{\psi}_0(\omega)\right|$: Type 1, $M = 2$, $N = 4$.

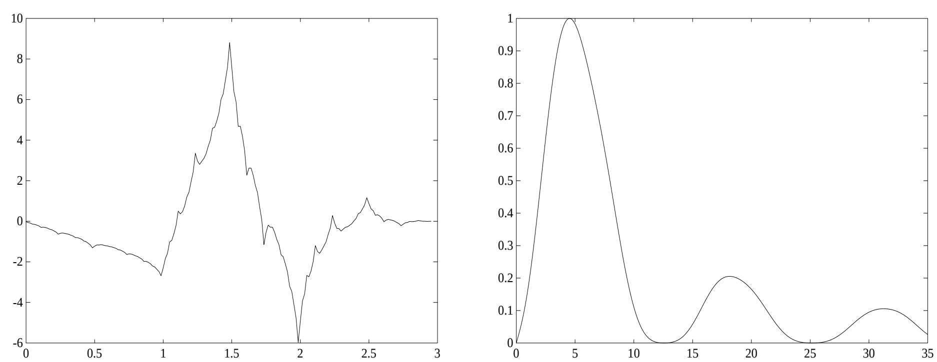

Figure 4.5: $\psi_1(t)$ and $\left|\hat{\psi}_1(\omega)\right|$: Type 1, $M = 2$, $N = 4$.

The scaling function and wavelet associated with this WTF (along with their Fourier transforms) is shown in Fig. 4.4-4.5. The autocorrelation of h_0 and hence the Lawton matrix corresponding to this scaling vector are given by

$$r = \frac{1}{8}\begin{bmatrix} 1-\sqrt{2} \\ 0 \\ 3+\sqrt{2} \\ 8 \\ 3+\sqrt{2} \\ 0 \\ 1-\sqrt{2} \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 1 & \frac{3+\sqrt{2}}{4} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & \frac{1-\sqrt{2}}{8} & 0 \end{bmatrix}.$$

The characteristic polynomial of L has a unique eigenvector of eigenvalue one:

$$\begin{aligned} \det(\lambda I - L) &= \frac{1}{8}\left(8\lambda^3 - 12\lambda^2 + (3+\sqrt{2})\lambda + (1-\sqrt{2})\right) \\ &= \frac{1}{8}(\lambda - 1)\left(8\,\lambda^2 - 4\,\lambda - 1 + \sqrt{2}\right). \end{aligned}$$

Hence the WTF is an orthonormal basis.

Example 13 (Type 1: M odd) If $M = 3$ and $N = 6$ (and the MFB is canonical) $J = 1$ and $N_p = k_0 = 1$. Once again the WTF has no free parameters $((k_0 - 1) = 0)$ and $\Theta_0 = \theta_{0,0} = \dfrac{5\,\pi}{12}$. The prototype filter $h(n)$ and h_0 are given by

$$h = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos(\theta_{0,0}) \\ \frac{1}{\sqrt{2}} \\ \sin(\theta_{0,0}) \\ \sin(\theta_{0,0}) \\ \frac{1}{\sqrt{2}} \\ \cos(\theta_{0,0}) \end{bmatrix} \quad h_0 = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos(\theta_{0,0})\cos(\frac{\pi}{6}) \\ \frac{1}{\sqrt{2}} \\ \sin(\theta_{0,0})\cos(\frac{\pi}{6}) \\ \sin(\theta_{0,0})\cos(\frac{\pi}{3}) \\ 0 \\ \cos(\theta_{0,0})\cos(\frac{2\pi}{3}) \end{bmatrix}.$$

The scaling function and its Fourier transform is shown in Fig. 4.6 while the wavelets and their Fourier transforms are given in Fig. 4.7.

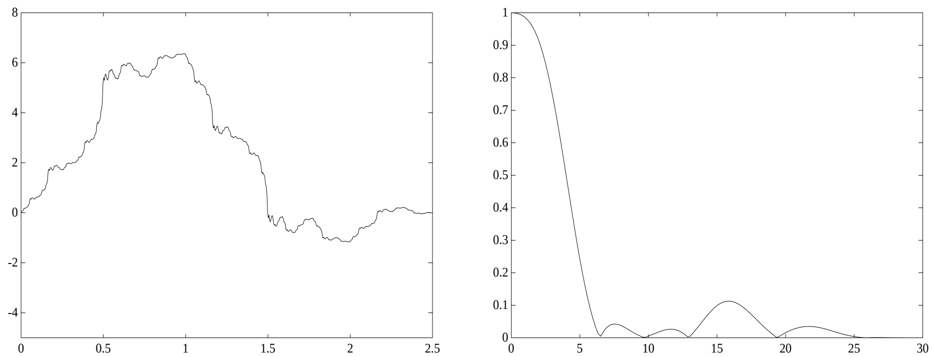

Figure 4.6: $\psi_0(t)$ and $\left|\hat{\psi}_0(\omega)\right|$: Type 1, $M = 3$, $N = 6$.

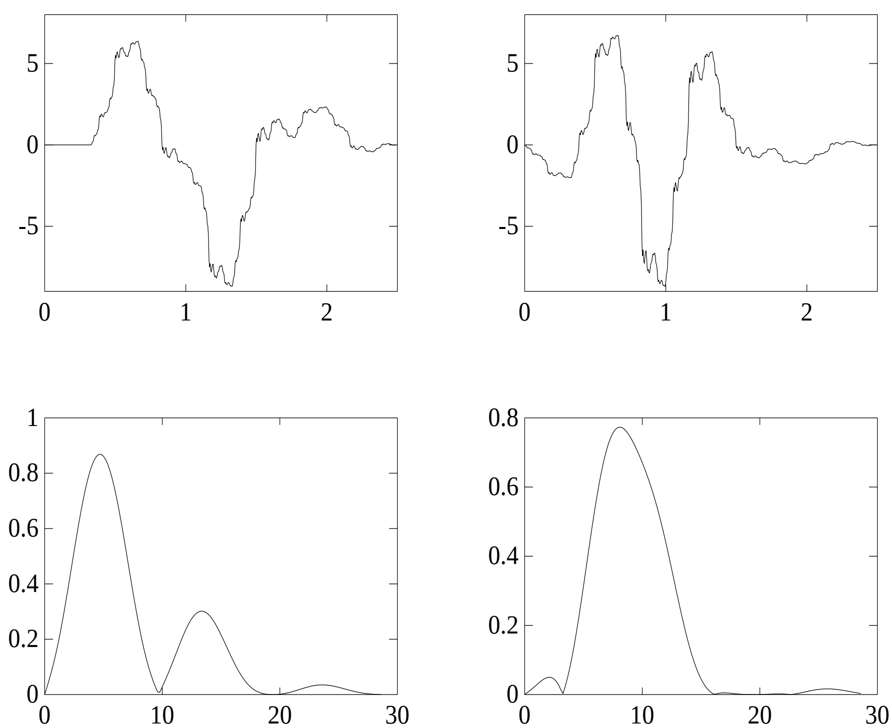

Figure 4.7: $\psi_i(t)$ and $\left|\hat{\psi}_i(\omega)\right|$: Type 1, $M = 3$, $N = 6$.

Example 14 (Type 2: M even) In this example $M = 4$ and $N = 7$. Since $J = 1$ and $k_0 = 1$ $N_p = 2$. The only parameter is $\theta_{0,0} = \frac{3\pi}{8}$. Moreover we have to choose $h(3) = \frac{1}{\sqrt{2}}$. Hence $h(n)$ and $h_0(n)$ are given by

$$h = \sqrt{\frac{1}{2}} \begin{bmatrix} \cos(\theta_{0,0}) \\ \frac{1}{\sqrt{2}} \\ \sin(\theta_{0,0}) \\ 1 \\ \sin(\theta_{0,0}) \\ \frac{1}{\sqrt{2}} \\ \cos(\theta_{0,0}) \end{bmatrix} \quad \text{and } h_0 = \sqrt{\frac{1}{2}} \begin{bmatrix} \cos(\theta_{0,0})\cos(\frac{\pi}{8}) \\ \frac{1}{\sqrt{2}} \\ \sin(\theta_{0,0})\cos(\frac{\pi}{8}) \\ \frac{1}{\sqrt{2}} \\ \sin(\theta_{0,0})\cos(\frac{3\pi}{8}) \\ 0 \\ -\cos(\theta_{0,0})\cos(\frac{3\pi}{8}) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1+\sqrt{2}}{4} \\ \frac{1}{2} \\ \frac{1}{4} \\ 0 \\ \frac{1-\sqrt{2}}{4} \end{bmatrix} \tag{4.68}$$

The scaling function and wavelets are shown in Fig 4.8.

Example 15 (Type 2: M odd) Let $M = 5$ and $N = 9$. Here $J = 2$ and h and h_0 are given by

$$h = \sqrt{\frac{2}{5}} \begin{bmatrix} \cos(\theta_{0,0}) \\ \cos(\theta_{1,0}) \\ \sin(\theta_{1,0}) \\ \sin(\theta_{0,0}) \\ 1 \\ \sin(\theta_{0,0}) \\ \sin(\theta_{1,0}) \\ \cos(\theta_{1,0}) \\ \cos(\theta_{0,0}) \end{bmatrix} \quad \text{and} \quad h_0 = \sqrt{\frac{2}{5}} \begin{bmatrix} \cos(\theta_{0,0})\cos(\frac{3\pi}{20}) \\ \cos(\theta_{1,0})\cos(\frac{\pi}{20}) \\ \sin(\theta_{1,0})\cos(\frac{\pi}{20}) \\ \sin(\theta_{0,0})\cos(\frac{3\pi}{20}) \\ \frac{1}{\sqrt{2}} \\ \sin(\theta_{0,0})\cos(\frac{7\pi}{20}) \\ \sin(\theta_{1,0})\cos(\frac{9\pi}{20}) \\ \cos(\theta_{1,0})\cos(\frac{11\pi}{20}) \\ \cos(\theta_{0,0})\cos(\frac{13\pi}{20}) \end{bmatrix}.$$

Fig 4.9 and Fig 4.10 show the functions and their Fourier transforms.

Example 16 (Type 1: M odd - Non-canonical Example) In this example $M = 5$ and $N = 19$. There are two lattices the first with one parameter and the second with two

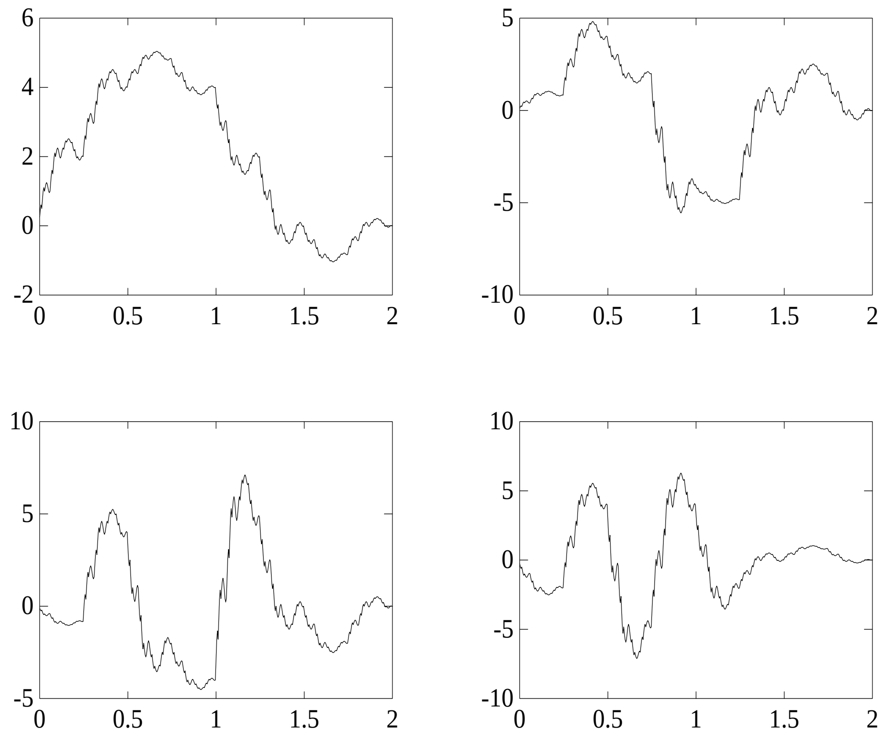

Figure 4.8: $\psi_i(t)$: Type 2, $M = 4$, $N = 7$.

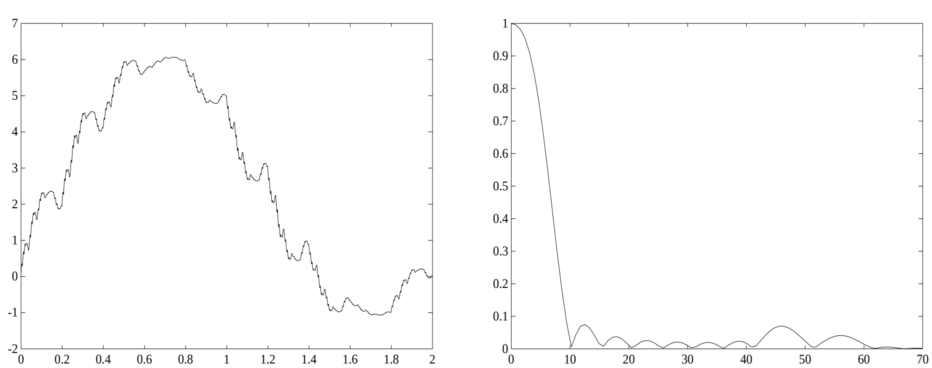

Figure 4.9: $\psi_0(t)$ and $\left|\hat{\psi}_0(\omega)\right|$: Type 1, $M = 5$, $N = 9$.

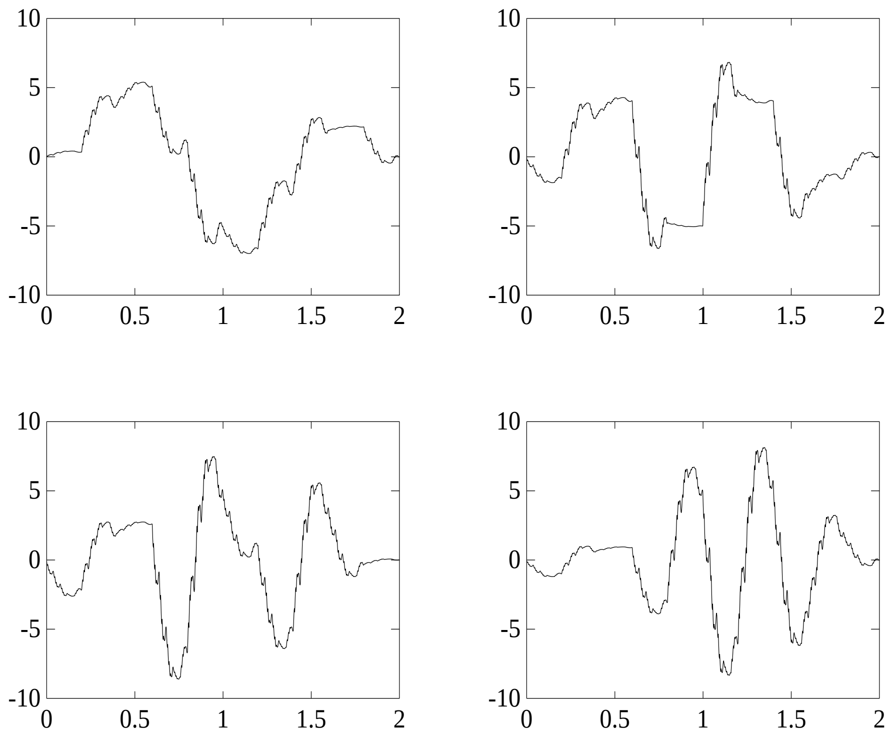

Figure 4.10: $\psi_i(t)$: Type 1, $M = 5$, $N = 19$.

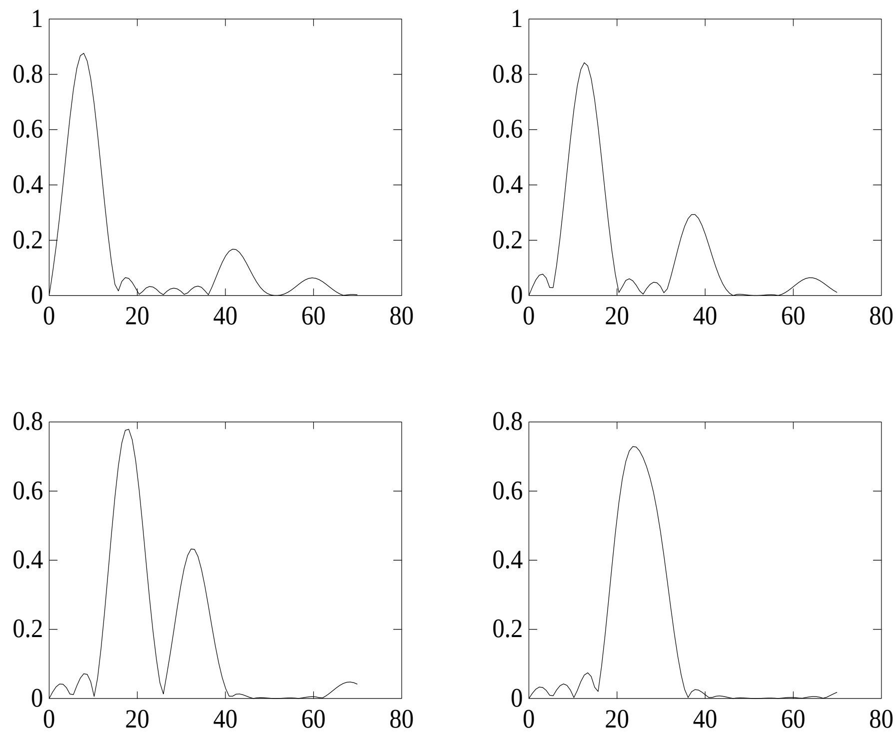

Figure 4.11: $\left|\hat{\psi}_i(\omega)\right|$: Type 1, $M = 5$, $N = 19$.

(i.e., $k_0 = 1$, $k_1 = 2$, and $N_p = 3$). Because Eqn. 4.67 has to be satisfied there is one free parameter.

$$\Theta_0 = \theta_{0,0} = \frac{\pi}{5} + \frac{\pi}{4} = \frac{9\,\pi}{20}.$$

$$\Theta_1 = \theta_{0,0} + \theta_{0,1} = \frac{\pi}{10} - \frac{\pi}{4} = -\frac{3\,\pi}{20}.$$

The polyphase component $p_{\frac{M-1}{2}}(n) = h(Mn+2)$ is not determined by the lattice. The numerical values of the prototype filter and the scaling vector with the *arbitrary* choice of $\theta_{1,0} = -\frac{3\pi}{10}$ (and hence $\theta_{1,1} = 3\frac{\pi}{20}$) are given in Table 4.2. The scaling function and wavelets are given n Fig. 4.12 and 4.13 respectively. Notice the *irregularity* of the scaling functions and wavelets. One verifies that the Lawton matrix for this example has a unique eigenvector of eigenvalue one. Hence this WTF is an orthonormal basis!

Table 4.2: Prototype filter and Scaling Vector : Type 1, $M = 5$, $N = 19$.

n	$h(n)$	$h_0(n)$	n	$h(n)$	$h_0(n)$
0	0.09893784281542	0.08004239622447	10	0.00000000000000	0.00000000000000
1	0.33122992405823	0.31501837766753	11	-0.23229208124280	0.22092289754973
2	0.00000000000000	0.00000000000000	12	-0.44721359549996	0.44721359549996
3	0.45589887899278	0.43358559963774	13	0.16877007594177	-0.16050988048005
4	0.62466895493456	0.50536780040049	14	0.00000000000000	0.00000000000000
5	0.62466895493456	0.36717119927548	15	0.00000000000000	0.00000000000000
6	0.16877007594177	0.05215282160796	16	0.45589887899278	-0.14088050132526
7	-0.00000000000000	-0.00000000000000	17	0.00000000000000	0.00000000000000
8	-0.23229208124280	0.07178220076275	18	0.33122992405823	0.10235567557952
9	0.09893784281542	-0.05815420490054			

4.5.3 Explicit Formula for Canonical 1-regular MWTFs

For Type 1 canonical WTFs with $N = 2M$ or canonical Type 2 MFBs with $N = 2M - 1$ (as in all the examples in Section. 4.5.2) there are no free parameters. It is possible to find an explicit formula for the scaling and wavelet vectors in such WTFs. These formulae have important consequences in transform coding applications

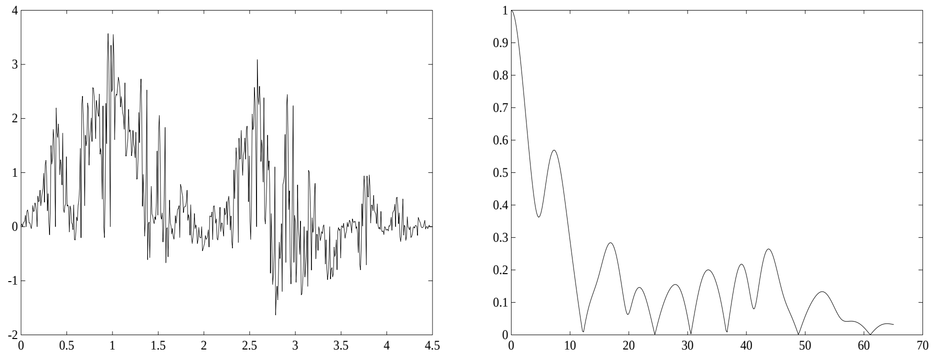

Figure 4.12: $\psi_0(t)$ and $\left|\hat{\psi}_0(\omega)\right|$: Type 1, $M = 5$, $N = 19$.

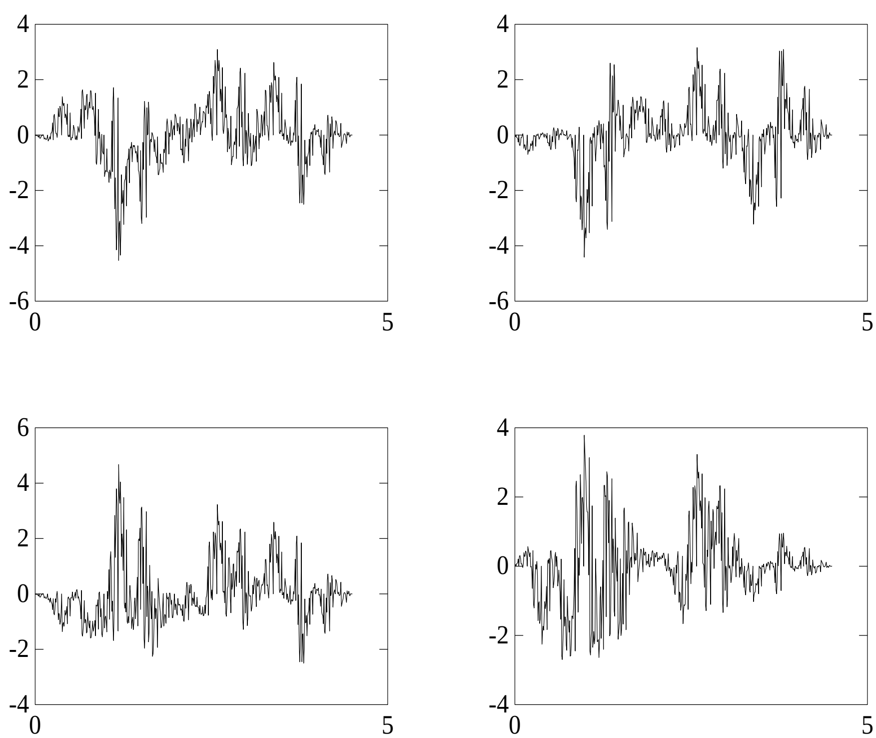

Figure 4.13: $\psi_i(t)$: Type 1, $M = 5$, $N = 19$.

(especially image coding), where very long filters would exhibit ringing artifacts and one has to keep N as small as possible while simultaneously having M large [62].

First consider the Type 1 case where $N = 2M$ (and for simplicity M even). It is easy to see that the prototype filter is of the form

$$\begin{bmatrix} h(0) \\ \cdots \\ h(J-1) \\ h(J) \\ \cdots \\ h(M-1) \\ h(M) \\ \cdots \\ h(M+J) \\ \cdots \\ h(2M-1) \end{bmatrix} = \sqrt{\frac{2}{M}} \begin{bmatrix} \cos(\Theta_0) \\ \cdots \\ \cos(\Theta_{J-1}) \\ \sin(\Theta_{J-1}) \\ \cdots \\ \sin(\Theta_0) \\ \sin(\Theta_0) \\ \cdots \\ \sin(\Theta_{J-1}) \\ \cdots \\ \cos(\Theta_0) \end{bmatrix}. \tag{4.69}$$

Therefore, for $n \in \{0, \ldots, J-1\}$, $h(n) = \cos\left(\frac{\pi}{2} - (2n+1)\frac{\pi}{4M}\right)$. Also for $n \in \{0, 1, \ldots, J-1\}$, $g(J+n) = \sin(\frac{\pi}{2} - (2(J-1-n)+1)\frac{\pi}{4M}) = \cos\left(\frac{\pi}{2} - (2(J+n)+1))\frac{\pi}{4M}\right)$. This implies that for $n \in \{0, \ldots, M-1\}$, $h(n) = \cos\left(\frac{\pi}{2} - (2n+1)\frac{\pi}{4M}\right)$. Similarly it can be shown that for $n \in \{0, \ldots, M-1\}$, $g(M+n) = \sin\left(\frac{\pi}{2} - (2n+1)\frac{\pi}{4M}\right)$. Equivalently, $g(M+n) = \cos\left(\frac{\pi}{2} - (2(M+n)+1)\frac{\pi}{4M}\right)$. Putting it all together we have the following simple formula for the prototype filter:

$$h(n) = \sin\left(\frac{\pi}{4M}(2n+1)\right)$$

One can similarly do an analysis for the case when M is odd and show that the prototype filter has the same form. Therefore the scaling and wavelet vectors are given by

$$h_i(n) \;=\; h(n)c_{i,n} = h(n)\cos\left(\frac{\pi}{2M}(2i+1)(n - \frac{M-1}{2})\right)$$

$$
\begin{aligned}
&= h(n)\cos\left(\frac{\pi}{2M}(2i+1)(n+\frac{1}{2}) - (2i+1)\frac{\pi}{4}\right) \\
&= \sqrt{\frac{1}{2M}}\left[\sin\left(\frac{\pi(i+1)(n+.5)}{M} - (2i+1)\frac{\pi}{4}\right)\right. \\
&\qquad \left. - \sin\left(\frac{\pi i(n+.5)}{M} - (2i+1)\frac{\pi}{4}\right)\right].
\end{aligned}
\tag{4.70}
$$

In the Type 2 case also (just as above by analyzing the cases of odd and even M independently) it can be shown that the prototype filter of length $N = 2M-1$ is given by

$$
h(n) = \sin\left(\frac{\pi}{2M}(n+1)\right)
$$

Therefore the scaling and wavelet vectors are given by

$$
\begin{aligned}
h_i(n) &= h(n)c_{i,n} = h(n)\cos\left(\frac{\pi}{2M}(2i+1)(n-\frac{M-2}{2})\right) \\
&= h(n)\cos\left(\frac{\pi}{2M}(2i+1)(n+1) - (2i+1)\frac{\pi}{4}\right) \\
&= \sqrt{\frac{1}{2M}}\left[\sin\left(\frac{\pi(i+1)(n+1)}{M} - (2i+1)\frac{\pi}{4}\right)\right. \\
&\qquad \left. - \sin\left(\frac{\pi i(n+1)}{M} - (2i+1)\frac{\pi}{4}\right)\right].
\end{aligned}
\tag{4.71}
$$

Theorem 33 Scaling/wavelet vectors in canonical modulated MFBs of Type 1 and Type 2 with $N = 2M$ and $N = 2M-1$ respectively are given by Eqn. 4.70 and Eqn. 4.71 respectively.

4.5.4 K Regular Modulated WTFs

Wavelets in a modulated WTF need not be smooth (see Example 16) This section discusses the construction of K regular modulated WTFs. The construction is based on the parameterization of compactly supported modulated WTFs. The regularity conditions on the scaling vector h_0 become a set of non-linear constraints on the parameters of the WTF. Therefore, if the WTF has a sufficient number of free parameters, by solving a set of non-linear equations, we can obtain regular modulated WTFs. In some cases, one can analytically solve these equations, while in general K regular modulated WTFs have to be designed numerically.

A multiplicity M, K-regular wavelet orthonormal basis satisfies $(K-1)(M-1)$ additional linear constraints (compared to a general WTF). Therefore to achieve K regularity a modulated WTF should have $(K-1)(M-1)$ free parameters to impose these constraints. In general, there seems to be no analytical procedure to obtain K-regular minimal length modulated WTFs (for moderate M and K, however, this is possible). In this section for the special case when $M = 2$ we obtain example K regular modulated WTFs. Our examples show that for a given regularity there may be more than one minimal length modulated WTF. The smoothness properties of the different WTFs could be vastly different.

For design purposes it is convenient to use the fact that a scaling vector is K-regular iff its partial moments up to order $K-1$ are equal (Theorem 29). Indeed for for $l \in \{1, \ldots, M-1\}$ and $k \in \{1, \ldots, K-1\}$

$$\eta_{k,0} \stackrel{\text{def}}{=} \sum_n (Mn)^k h_0(Mn) = \sum_n (Mn+l)^k h_0(Mn+l) \stackrel{\text{def}}{=} \eta_{k,l}.$$

Example 17 Consider a Type 1 MFB, with $M = 2$ and $N = 8$. The associated WTF has precisely one degree of freedom, since $J = 1$, and $k_0 = 2$. This degree of freedom is exploited make the WTF 2-regular. From Eqn. 4.67, $\Theta_0 = \theta_{0,0} + \theta_{0,1} = \frac{3\pi}{8}$. For this example

$$h = \begin{bmatrix} \cos(\theta_{0,1})\cos(\theta_{0,0}) \\ -\cos(\theta_{0,1})\sin(\theta_{0,0}) \\ \sin(\theta_{0,1})\cos(\theta_{0,0}) \\ \sin(\theta_{0,1})\sin(\theta_{0,0}) \\ \sin(\theta_{0,1})\sin(\theta_{0,0}) \\ \sin(\theta_{0,1})\cos(\theta_{0,0}) \\ -\cos(\theta_{0,1})\sin(\theta_{0,0}) \\ \cos(\theta_{0,1})\cos(\theta_{0,0}) \end{bmatrix} \quad \text{and} \quad h_0 = \begin{bmatrix} \cos(\theta_{0,1})\cos(\theta_{0,0})\cos(\frac{\pi}{8}) \\ -\cos(\theta_{0,1})\sin(\theta_{0,0})\cos(\frac{\pi}{8}) \\ \sin(\theta_{0,1})\cos(\theta_{0,0})\cos(\frac{3\pi}{8}) \\ -\sin(\theta_{0,1})\sin(\theta_{0,0})\cos(\frac{3\pi}{8}) \\ -\sin(\theta_{0,1})\sin(\theta_{0,0})\cos(\frac{\pi}{8}) \\ -\sin(\theta_{0,1})\cos(\theta_{0,0})\cos(\frac{\pi}{8}) \\ \cos(\theta_{0,1})\sin(\theta_{0,0})\cos(\frac{3\pi}{8}) \\ \cos(\theta_{0,1})\cos(\theta_{0,0})\cos(\frac{3\pi}{8}) \end{bmatrix} \tag{4.72}$$

This scaling vector is 2 regular iff $\sum_n (-1)^n k h_0(n) = 0$. Therefore

$$\theta_{0,1} = \frac{1}{2}\left[\frac{\pi}{4} + \arcsin(\frac{\sqrt{2}+1}{4})\right] \quad \text{or} \quad \frac{1}{2}\left[\frac{\pi}{4} + \pi - \arcsin(\frac{\sqrt{2}+1}{4})\right].$$

where arcsin is assumed to return a value in $[-\pi/2, \pi/2]$. There are two choices of h and hence there are two such 2-regular WTFs. For these two case h and h_0 are given in Table 4.3. The scaling and wavelet functions and their Fourier transforms the two choices of WTFs are given in Fig. 4.14 and Fig. 4.15. In Case 2 the scaling function and wavelets are relatively smooth (compared to Case 1) even though both generate 2-regular WTFs. Regularity of the scaling vector does no imply smoothness of the corresponding WTF. The scaling function has a piecewise linear characteristic which is tied to its being 2-regular.

Example 18 In this example $M = 2$ and $N = 12$. Therefore there are three angle parameters, and hence the WTF has two degrees of freedom. We therefore construct a WTF that is three regular. For it to be a WTF

$$\Theta_0 = \theta_{0,0} + \theta_{0,1} + \theta_{0,2} = \frac{3\,\pi}{8}.$$

For h_0 to be 3-regular $\sum_n (-1)^n n^k h_0(n) = 0$, for $k \in \{1, 2\}$. These nonlinear equations in the angle parameters can be solved numerically (there are several solutions). Three