Table 4.3: 2 Regular Modulated Scaling Vector : Type 1, $M = 2$, $N = 8$

| | Case 1 | | Case 2 | |
|---|---|---|---|---|
| $n$ | $h(n)$ | $h_0(n)$ | $h(n)$ | $h_0(n)$ |
| 0 | 0.33569552586546 | 0.31014222550271 | 0.03057338229891 | 0.02824612214561 |
| 1 | 0.67514166633581 | 0.62374956707322 | -0.06148835084290 | -0.05680782883163 |
| 2 | 0.29245823397072 | 0.11191892079935 | -0.44417178320799 | -0.16997718255776 |
| 3 | -0.58818400664583 | 0.22508827452548 | -0.89330615021238 | 0.34185346371612 |
| 4 | -0.58818400664583 | 0.54341116509056 | -0.89330615021238 | 0.82530726844767 |
| 5 | 0.29245823397072 | -0.27019617647995 | -0.44417178320799 | 0.41036121942490 |
| 6 | 0.67514166633581 | -0.25836553020608 | -0.06148835084290 | 0.02353057315103 |
| 7 | 0.33569552586546 | 0.12846511606780 | 0.03057338229891 | 0.01169992687716 |

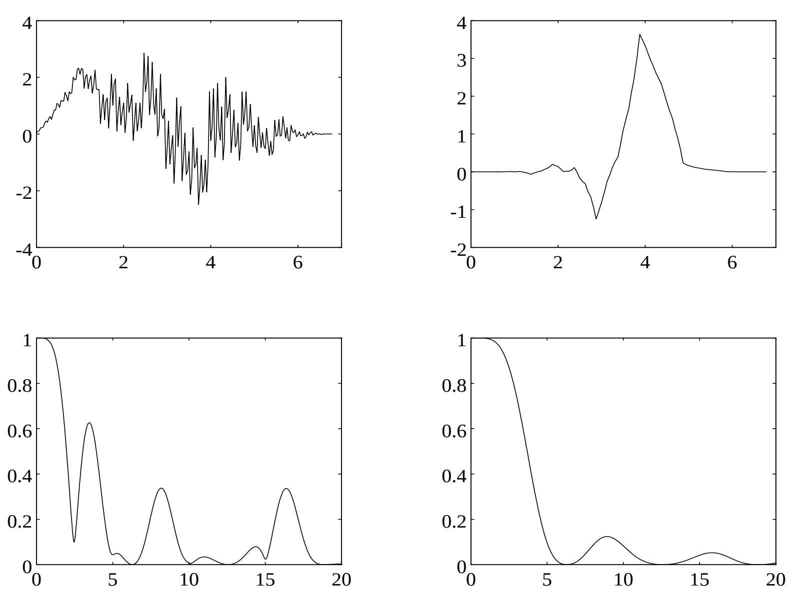


Figure 4.14: $\psi_0(t)$ and $\left|\hat{\psi}_0(\omega)\right|$: Case 1 and 2

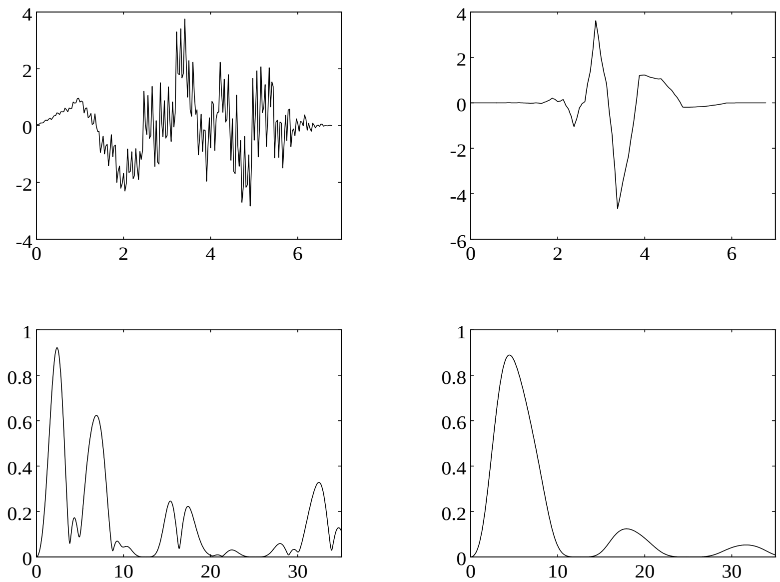


Figure 4.15: $\psi_1(t)$ and $\left|\hat{\psi}_1(\omega)\right|$: Case 1 and 2

solutions are given in Table 4.4 and the corresponding prototype filters are given in Table 4.5. The scaling functions in the three cases are given in Fig. 4.16. Only Case 2 corresponds to a smooth WTF and its Fourier transform is given in Fig. 4.17.

Since all our examples have been obtained by numerical techniques, there is no guarantee that for a given $M$ and $K$, there exists a multiplicity $M$ modulated WTF. However, we believe that this is so.

**Conjecture 1** For all $M$ and $K$, there exist multiplicity $M$ $K$-regular, modulated WTFs (of Types 1 and 2).

Table 4.4: Angle Parameters : Type 1, $M = 2$, $N = 12$, $K = 2$

| | | | |
|---|---|---|---|
| $\theta_{0,0}$ | -1.03346571960832 | 3.76296745429585 | 4.27075141557234 |
| $\theta_{0,1}$ | 0.55717636703648 | -1.51698381446403 | -0.75894250622404 |
| $\theta_{0,2}$ | 1.65438659766802 | -1.06788639473564 | -2.33371166425213 |

Table 4.5: 3 Regular Prototype Filter : Type 1, $M = 2$, $N = 12$, $K = 3$

| | Case 1 | Case 2 | Case 3 |
|---|---|---|---|
| $n$ | $h(n)$ | $h(n)$ | $h(n)$ |
| 0 | -0.03627175678668 | -0.02107816639314 | 0.21430340959772 |
| 1 | 0.06087838025582 | -0.01509168336228 | 0.45328110558723 |
| 2 | 0.43291206114071 | 0.03831788415099 | 0.22416303569681 |
| 3 | 0.72659797622866 | -0.02743508917869 | -0.47413556715302 |
| 4 | 0.30764278707689 | -0.43119668793692 | -0.64251558992040 |
| 5 | -0.43008909111475 | -0.90065333172258 | -0.24643539122724 |
| 6 | -0.43008909111475 | -0.90065333172258 | -0.24643539122724 |
| 7 | 0.30764278707689 | -0.43119668793692 | -0.64251558992040 |
| 8 | 0.72659797622866 | -0.02743508917869 | -0.47413556715302 |
| 9 | 0.43291206114071 | 0.03831788415099 | 0.22416303569681 |
| 10 | 0.06087838025582 | -0.01509168336228 | 0.45328110558723 |
| 11 | -0.03627175678668 | -0.02107816639314 | 0.21430340959772 |

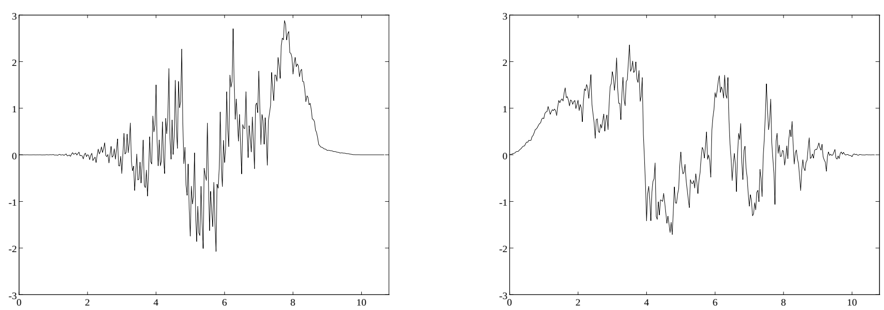

Figure 4.16: $\psi_0(t)$: Case 1 and Case 3

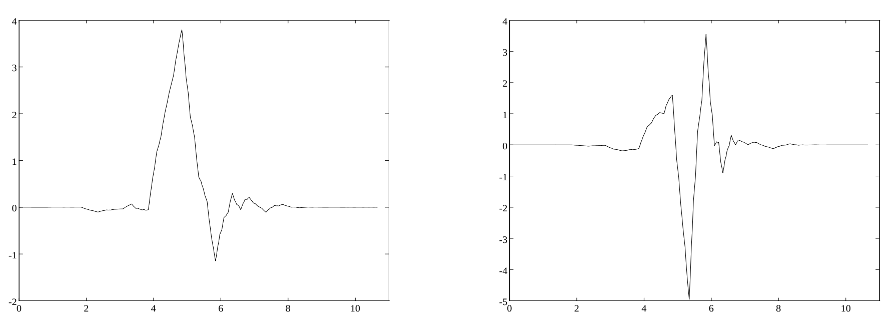

Figure 4.17: $\psi_0(t)$ and $\psi_1(t)$: Case 2

In most numerical analysis applications only low order of $K$ is required. For $M \in \{2, \ldots, 8\}$, 2 regular prototype filters for canonical Type 1 MWTFs are given in Table 4.6. Since the prototype filters are even symmetric, $h(n)$ is only given for $n \in \{0, 1, \ldots, N/2 - 1\}$.

Table 4.6: Prototype filters for $K$-regular WTFs

$K = 2$

| $M = 3,\ N = 18$ | $M = 5,\ N = 30$ | $M = 6,\ N = 36$ | $M = 7,\ N = 42$ | $M = 8,\ N = 48$ |
|---|---|---|---|---|
| -0.00388862514471 | -0.00301097038898 | -0.00054944468846 | -0.00246739083642 | -0.00123930643393 |
| 0.00000000000000 | -0.00201208359560 | -0.01910742365573 | -0.00237194122154 | -0.00779774978931 |
| -0.01478482519704 | 0.00000000000000 | -0.00577566692183 | -0.00107950489740 | -0.01487487447880 |
| -0.02767653677256 | -0.00448172450143 | 0.02844407733543 | 0.00000000000000 | -0.01932102524122 |
| 0.00000000000000 | -0.01512586189495 | 0.01910967329910 | -0.00191595817083 | 0.08973245101789 |
| 0.10522813154117 | -0.02162484892812 | -0.00265043105253 | -0.00743504813986 | 0.06138828332885 |
| 0.31266437180164 | -0.01722782422909 | -0.01338305956067 | -0.01402758013820 | 0.03296682908719 |
| 0.57735026918963 | 0.00000000000000 | -0.00910367899718 | -0.01803575190864 | 0.00750441807299 |
| 0.74621377262522 | 0.03837333703375 | -0.00579716969829 | -0.01735610151889 | -0.01402667671272 |
| | 0.10863423950739 | -0.02854997448027 | -0.01164038179004 | -0.02960269106783 |
| $M = 4,\ N = 24$ | 0.20456111193383 | -0.00910475083353 | 0.00000000000000 | -0.03627575519340 |
| 0.00877038984401 | 0.32349005648916 | 0.06455768411718 | 0.02065991979828 | -0.03352506774898 |
| -0.01075359886307 | 0.44721359549996 | 0.13936757164327 | 0.05440415181523 | -0.15570014852204 |
| 0.05457943758320 | 0.54181245657050 | 0.19273020820114 | 0.10253663567074 | -0.14970925240107 |
| 0.01059371915934 | 0.58791824411149 | 0.31714293350835 | 0.15991677470121 | -0.12515236873747 |
| -0.01813390625490 | | 0.48068967221677 | 0.22857379514531 | -0.08493625413770 |
| -0.01403215531070 | | 0.54340809109609 | 0.30396352191435 | -0.03716699040685 |
| -0.07121961258650 | | 0.55637746698764 | 0.37796447300923 | 0.01219222010045 |
| 0.02190387354986 | | | 0.43903678212524 | 0.07111424153313 |
| 0.16862395303534 | | | 0.47973571249855 | 0.14217546831856 |
| 0.31087426774399 | | | 0.49909818659237 | 0.44271260995028 |
| 0.62848508348218 | | | | 0.46607316030963 |
| 0.68597973556552 | | | | 0.48183430588546 |
| | | | | 0.49107010469637 |

$K = 3$

| $M = 2,\ N = 12$ | $M = 3,\ N = 30$ | |
|---|---|---|
| -0.01047051524186 | -0.00002420872650 | -0.01708533214152 |
| -0.04042851803229 | 0.00000000000000 | -0.02916098647507 |
| -0.03544505570212 | -0.00023932006300 | 0.00000000000000 |
| 0.13685965212863 | 0.00016871556489 | 0.10673613707477 |
| 0.50907256925186 | 0.00000000000000 | 0.32065980778007 |
| 0.84800595877688 | -0.00166787045212 | 0.57735026918963 |
| | 0.00090672612149 | 0.74249942047633 |
| | 0.00000000000000 | |

### 4.5.5 Orthonormality of Modulated Wavelet Tight Frames

Do modulated wavelet tight frames always form orthonormal basis? All the examples considered so far have been verified to be orthonormal wavelet bases. The classical Daubechies example of non-orthonormal tight frame corresponding to a scaling vector of $h_0 = [1, 0, 0, 1]^T$ is not a modulated WTF. We make the following conjecture

**Conjecture 2** Every multiplicity $M$, modulated WTF is necessarily an orthonormal wavelet basis.

Since (non-orthonormal) wavelet tight frames are far fewer than orthonormal wavelet bases numerical examples with random scaling vectors will not give any conclusive answer.

## 4.6 Linear Phase and Related WTFs

In this section WTFs associated with unitary FIR filter banks with symmetry (see Section 3.6) are parameterized. First consider the case of PS symmetry in which case $H_p(z)$ is parameterized in Eqn. 3.71. We have a WTF iff

$$\text{first row of } H_p(z)|_{z=1} = \left[\begin{array}{ccc} 1/\sqrt{M} & \ldots & 1/\sqrt{M} \end{array}\right]. \tag{4.73}$$

In Eqn. 3.71 since $P$ permutes the columns, the first row is unaffected. Hence Eqn. 4.73 is equivalent to the first rows of both $W_0'(z)$ and $W_1'(z)$ when $z = 1$ is given by

$$\left[\begin{array}{ccc} \sqrt{2/M} & \ldots & \sqrt{2/M} \end{array}\right].$$

This is precisely the condition to be satisfied by a WTF of multiplicity $M/2$. Therefore both $W_0'(z)$ and $W_1'(z)$ give rise to multiplicity $M/2$ compactly supported WTFs. If the McMillan degree of $W_0'(z)$ and $W_1'(z)$ are $L_0$ and $L_1$ respectively, then they are parameterized respectively by $\begin{pmatrix} M/2-1 \\ 2 \end{pmatrix} + (M/2-1)L_0$ and $\begin{pmatrix} M/2-1 \\ 2 \end{pmatrix} + (M/2-1)L_1$ parameters. In summary, a WTF with PS symmetry can be *explicitly* parameterized by $2\begin{pmatrix} M/2-1 \\ 2 \end{pmatrix} + (M/2-1)(L_0+L_1)$ parameters. Both $L_0$ and $L_1$ are greater than or equal to $K$.

PS symmetry does not reflect itself as any simple property of the scaling function $\psi_0(t)$ and wavelets $\psi_i(t)$, $i \in \{1, \ldots, M-1\}$ of the WTF. However, from design and implementation points of view PS symmetry is useful (because of the reduction in the number of parameters).

Next consider PCS symmetry. From Eqn. 3.72 one sees that Eqn. 4.73 is equivalent to the first rows of the matrices $A$ and $B$ defined by

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix} = \left\{ \prod_{i=K}^{0} \begin{bmatrix} A_i & B_i \\ -B_i & A_i \end{bmatrix} \right\}$$

are of the form $\begin{bmatrix} 1/\sqrt{M} & \ldots & 1/\sqrt{M} \end{bmatrix}$. Here we only have an implicit parameterization of WTFs, unlike the case of PS symmetry. As in the case of PS symmetry there is no simple symmetry relationships between the wavelets.

Now consider the case of linear-phase. In this case it can be seen [76] that the wavelets are also linear-phase. If we define

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} = \left\{ \prod_{i=K}^{0} \begin{bmatrix} A_i & B_i \\ B_i & A_i \end{bmatrix} \right\},$$

then it can be verified that one of the rows of the matrix $A + B$ has to be of the form $\begin{bmatrix} \sqrt{2/M} & \ldots & \sqrt{2/M} \end{bmatrix}$. This is an implicit parameterization of the WTF.

Finally consider the case of linear-phase with PCS symmetry. In this case also the wavelets are linear-phase. From Eqn. 3.74 it can be verified that we have a WTF iff the first row of $W_0'(z)$ for $z = 1$, evaluates to the vector $\begin{bmatrix} \sqrt{2/M} & \ldots & \sqrt{2/M} \end{bmatrix}$. Equivalently, $W_0'(z)$ gives rise to a multiplicity $M/2$ WTF. In this case the WTF is parameterized by precisely $\binom{M/2-1}{2} + (M/2-1)L$ parameters where $L \geq K$ is the McMillan degree of $W_0'(z)$.

## 4.7 Wavelet Frames

The theory of unitary filter banks gives rise to the theory of WTFs. The crucial property of unitary filter banks used in the construction is the PR property. In this section we show that PR filter banks in general give rise to wavelet frames for $L^2(\mathbb{R})$. The results obtained are an extension of the multiplicity 2 results of Cohen [19] and Vetterli et.al [90]. The relationship between PR filter bank theory and wavelet theory in its most general form is depicted in Table 4.7. For unitary filter banks the dual scaling functions and wavelets (in Table 4.7) are the same as the scaling functions and wavelets because of the fact that the synthesis filters are time-reverses of the analysis filters. The theory of wavelet frames is not as complete as the theory of wavelet tight frames because of the fact that $H_p(z)$ for an arbitrary FIR filter bank does not have as clean a characterization as $H_p(z)$ for a unitary filter bank.

**Theorem 34** *(Wavelet Frames Theorem)* Let $h_i$, and $g_i$ be the analysis and synthesis filters in an FIR FB such that $H_0(0) = G_0(0) = \sqrt{M}$ and for $i = 1, 2, \ldots, M-1$ $H_i(0) = G_i(0) = 0$. Define the functions

$$\psi_0(\omega) = \prod_{j\geq 1} H_0\left(\frac{\omega}{M^j}\right) \quad \text{and} \quad \tilde{\psi}_0(\omega) = \prod_{j\geq 1} G_0\left(\frac{-\omega}{M^j}\right).$$

Assume further that (the distributions) $\psi_0(t)$ and $\tilde{\psi}_0(t)$ are in $L^2(\mathbb{R})$ (notice that this implies restrictions on the $h_i$ and $g_i$). Now define

$$\psi_i(t) = \sqrt{M}\sum_k h_i(k)\psi_0(Mt-k), \quad \psi_{i,j,k}(t) = M^{j/2}\psi_i(M^j t - k),$$

$$\tilde{\psi}_i(t) = \sqrt{M}\sum_k g_i(k)\tilde{\psi}_0(Mt+k) \quad \text{and} \quad \tilde{\psi}_{i,j,k}(t) = M^{j/2}.\tilde{\psi}_i(M^j t - k)$$

Then $\{\psi_{i,j,k}\}$ and $\left\{\tilde{\psi}_{i,j,k}\right\}$ form dual frames for $L^2(\mathbb{R})$:

$$f(t) = \sum_{i=1}^{M-1}\sum_{j,k}\left\langle \tilde{\psi}_{i,j,k}(t), f\right\rangle \psi_{i,j,k} = \sum_{i=1}^{M-1}\sum_{j,k}\left\langle f, \psi_{i,j,k}(t)\right\rangle \tilde{\psi}_{i,j,k}.$$

Table 4.7: Relationship between Filter Bank Theory and Wavelet Theory

| Filter Bank Theory | Wavelet Theory |
|---|---|
| Lowpass Analysis filter $h_0$ | Scaling Vector/Function $(h_0, \psi_0)$ |
| Bandpass Analysis filter $h_l$ | Scaling Vector/Function $(h_l, \psi_l)$ |
| Lowpass Synthesis filter $g_0$ | Dual Scaling Vector/Function $(g_0, \tilde{\psi}_0)$ |
| Bandpass Synthesis filter $g_l$ | Dual Scaling Vector/Function $(g_l, \tilde{\psi}_l)$ |

$$f(t) = \sum_k \left\langle \tilde{\psi}_{0,0,k}(t), f \right\rangle \psi_{0,0,k}(t) + \sum_{i=1}^{M-1} \sum_{j=1}^{\infty} \sum_k \left\langle \tilde{\psi}_{i,j,k}(t), f \right\rangle \psi_{i,j,k}(t).$$

$$f(t) = \sum_k \langle f, \psi_{0,0,k}(t) \rangle \tilde{\psi}_{0,0,k}(t) + \sum_{i=1}^{M-1} \sum_{j=1}^{\infty} \sum_k \langle f, \psi_{i,j,k}(t) \rangle \tilde{\psi}_{i,j,k}(t).$$

**Proof:** The first part of the proof of this theorem parallels the wavelets tight frames theorem. The functions $\psi_i(t)$ and $\tilde{\psi}_i(t)$ are all compactly supported in an interval of length $\frac{N-1}{M-1}$ as in the tight frames case. Define two families of operators

$$\tilde{I}_{i,j} = \sum_k \tilde{\psi}_{i,j,k} >< \psi_{i,j,k} \quad \text{and} \quad I_{i,j} = \sum_k \psi_{i,j,k} >< \tilde{\psi}_{i,j,k}.$$

These operators makes sense for all $i$ for an arbitrary $f \in L^2(\mathbb{R})$. In fact it suffices to show that for $f$ given

$$\sum_k |\langle f, \psi_{i,j,k} \rangle|^2 \leq C_i \|f\|^2 \quad \text{and} \quad \sum_k \left| \left\langle f, \tilde{\psi}_{i,j,k} \right\rangle \right|^2 \leq \tilde{C}_i \|f\|^2$$

for suitable non-negative constants $C_i$. This follows from Bessel's inequality by noting that the functions $\left\{ \psi_{i,j,(N-1)m+k} / \|\psi_i\| \right\}$ for fixed $i$, $j$ and $k$, with $k = 0, 1, \ldots, N-2$ form an orthonormal family (since they don't overlap) and hence

$$\sum_k |\langle f, \psi_{i,j,k} \rangle|^2 \leq (N-1) \|\psi_i\| \|f\|^2 .$$

Similarly for the case of $\tilde{\psi}_{i,j,k}$. Notice that for all $j$

$$\psi_{i,j-1,k}(t) = \sum_l h_i(l) \psi_{0,j,Mk+l} \quad \text{and} \quad \tilde{\psi}_{i,j-1,k}(t) = \sum_l g_i(l) \tilde{\psi}_{0,j,Mk-l}. \tag{4.74}$$

For any fixed $j$ from the fact that $h_i(-n)$ and $g_i(-n)$ form a PR filter bank we can invoke Theorem 24 to get

$$I_{0,j} = \sum_{i=0}^{M-1} I_{i,j-1} \quad \text{and} \quad \tilde{I}_{0,j} = \sum_{i=0}^{M-1} \tilde{I}_{i,j-1}. \tag{4.75}$$

The filter bank PR property (Eqn. 3.8) is crucial to obtain this result. By repeatedly substituting for $I_{0,j-1}$, $I_{0,j-2}$ etc., for fixed $J$

$$I_{0,J} = \sum_{i=1}^{M-1} \sum_{j=-\infty}^{J-1} I_{i,j}.$$

Notice that $\psi_{i,j,k}(t)$ approaches the zero function for all $i$ and $k$ as, $j \to \infty$ and hence the infinite sum over $j$ makes sense. We need to show that $\lim_{j\to\infty} I_{0,j} f = f$ for all $f$ in $L^2(\mathbb{R})$. Since $\psi_{0,0,0}(t)$ and $\tilde{\psi}_{0,0,0}(t)$ may *not* form a partition of unity, we cannot use the approach taken in the tight frames case.

As in [19] we first show that there is weak convergence and then strengthen it to strong convergence in $L^2(\mathbb{R})$. For weak convergence we must have for any $f$, $g$ in $L^2(\mathbb{R})$

$$\sum_{i=1}^{M-1} \sum_{j=-\infty}^{\infty} \langle g, I_{i,j} f\rangle = \lim_{j\to\infty} \langle g, I_{0,j} f\rangle = \langle g, f\rangle .$$

Since both $\psi_{0,j,k}$ and $\tilde{\psi}_{0,j,k}$ approach the Dirac measure at $2^{-j}k$ for large $j$ (recall that $H_0(0) = G_0(0) = \sqrt{M}$ and hence $\hat{\psi}_0(0) = \hat{\tilde{\psi}}_0(0) = 1$) this is expected and can be proved rigorously by noticing that

$$\begin{aligned}
&\langle g, I_{0,j} f\rangle \qquad (4.76)\\
&= \sum_k \langle g, \psi_{0,j,k}\rangle \left\langle \tilde{\psi}_{0,j,k}, f\right\rangle \\
&= M^j \sum_k \left\langle g, \psi_0(M^j t - k)\right\rangle \left\langle \tilde{\psi}_0(M^j t - k, f\right\rangle \\
&= M^{-j} \sum_k \left[\frac{1}{2\pi}\int \hat{g}(\omega)\hat{\psi}_0^*\left(\frac{\omega}{M^j}\right) e^{i\frac{\omega}{M^j}k} d\omega\right] \left[\frac{1}{2\pi}\int \hat{f}^*(\lambda)\hat{\tilde{\psi}}_0(\frac{\lambda}{M^j}) e^{-i\frac{\lambda}{M^j}k} d\lambda\right] \\
&= \frac{1}{2\pi}\int \hat{g}(\omega)\hat{\psi}_0\left(\frac{\omega}{M^j}\right)\left[\hat{f}^*(\lambda)\hat{\psi}_0\left(\frac{\omega}{M^j}\right)\frac{1}{2\pi M^j}\sum_k e^{i\frac{\omega-\lambda}{M^j}k} d\lambda\right] d\omega \\
&= \frac{1}{2\pi}\int \hat{g}(\omega)\hat{\psi}_0\left(\frac{\omega}{M^j}\right)\left[\hat{f}^*(\lambda)\hat{\psi}_0(\frac{\lambda}{M^j})\sum_l \delta\left(\frac{\omega-\lambda}{M^j} + 2\pi l\right) d\lambda\right] d\omega \\
&= \sum_l \frac{1}{2\pi}\int \hat{g}(\omega)\hat{\psi}_0\left(\frac{\omega}{M^j}\right)\hat{f}^*(\omega + 2\pi M^j l)\hat{\psi}_0\left(\frac{\omega}{M^j} + 2\pi l\right) d\omega \qquad (4.77)
\end{aligned}$$

and all terms in the right hand side are small for $l \neq 0$ and large $j$ and the $l = 0$ term approaches the desired inner product. To strengthen this weak convergence to strong convergence we need the following result:

$$\sum_{i,j,k} |\langle f, \psi_{i,j,k} \rangle|^2 \leq C \|f\|^2$$

for some $C \geq 0$. Since the sum over $i$ is finite it suffices to show that

$$\sum_{j,k} |\langle f, \psi_{i,j,k} \rangle|^2 \leq C_i \|f\|^2$$

for some $C_i \geq 0$. This can be proved by elementary means using epsilon-delta arguments. Two important facts are required for the proof: firstly that for $i \neq 0$, $\hat{\psi}_i(0) = 0$ and secondly that $\hat{\psi}_i(\omega)$ decays faster than $|\omega|^{-\frac{1}{2}}$ for large $\omega$. The first fact follows trivially from the hypothesis that $H_i(0) = G_i(0) = 0$, and the second from the fact that $\psi_i(t)$ is in $L^2(\mathbb{R})$. These two facts can be used to show that for $j$ and $k$ tending to $\infty$, the numbers $|\langle f, \psi_{i,j,k} \rangle|^2$ decay rapidly enough for the double sum to converge. Now we have strong convergence because

$$\begin{aligned}
&\left\| f - \sum_{|j| \leq L, |k| \leq L} \sum_i I_{i,j} f \right\| \\
&= \sup_{\|g\|=1} \left| \langle g, f \rangle - \sum_{|j| \leq L, |k| \leq L} \sum_i \langle g, I_{i,j} f \rangle \right| \\
&\leq \sup_{\|g\|=1} \sum_{i=1}^{M-1} \sum_{|j|>L, |K|>L} \left| \sum_{|j|>L, |k|>L} \sum_i \langle g, I_{i,j} f \rangle \right| \\
&\leq \sup_{\|g\|=1} \left[ \sum_{i=1}^{M-1} \sum_{j,k} |\langle g, \psi_{i,j,k} \rangle| \right]^{\frac{1}{2}} \left[ \sum_{j>L, k>L} \left| \left\langle \tilde{\psi}_{i,j,k}, f \right\rangle \right| \right]^{\frac{1}{2}} \\
&\leq C \left[ \sum_{i=1}^{M-1} \sum_{j>L, k>L} \left| \left\langle \tilde{\psi}_{i,j,k}, f \right\rangle \right| \right]^{\frac{1}{2}}
\end{aligned}$$

□

Wavelet frames also have an associated multiresolution analysis the structure of which is relatively more complex (compared to the tight frames case). Define the spaces

$$W_{i,j} = Span\left\{\psi_{i,j,k}(t)\right\} \quad \text{and} \quad \tilde{W}_{i,j} = Span\left\{\tilde{\psi}_{i,j,k}(t)\right\}.$$

For $f \in L^2(\mathbb{R})$ $I_{i,j}f \in \tilde{W}_{i,j}$. From Eqn. 4.75

$$I_{0,j}f = \sum_{i=0}^{M-1} I_{i,j-1}f \quad \text{and} \quad \tilde{I}_{0,j}f = \sum_{i=0}^{M-1} \tilde{I}_{i,j-1}f$$

and hence

$$W_{0,j} = W_{0,j-1} \oplus W_{1,j-1} \ldots \oplus W_{M-1,j-1}$$

$$\tilde{W}_{0,j} = \tilde{W}_{0,j-1} \oplus \tilde{W}_{1,j-1} \ldots \oplus \tilde{W}_{M-1,j-1}.$$

Also $\lim_{j\to\infty} W_{0,j} = L^2(\mathbb{R}) = \lim_{j\to\infty} \tilde{W}_{0,j}$ and $\lim_{j\to-\infty} W_{0,j} = \{0\} = \lim_{j\to-\infty} \tilde{W}_{0,j}$. We have a double multiresolution analysis of $L^2(\mathbb{R})$ with a chain of closed subspaces:

$$\{0\} \subset \ldots W_{0,-1} \subset W_{0,0} \subset W_{0,1} \ldots \subset L^2(\mathbb{R})$$

and

$$\{0\} \subset \ldots \tilde{W}_{0,-1} \subset \tilde{W}_{0,0} \subset \tilde{W}_{0,1} \ldots \subset L^2(\mathbb{R})$$

What is the relationship between, the spaces $W_{i,j}$ and $\tilde{W}_{i,j}$? In general there is no relationship. However if the frame is a Riesz basis for all $j$ and $i \neq l$, $W_{i,j} \perp \tilde{W}_{l,j}$.

Necessary and sufficient conditions so that the frames become Riesz basis can be derived exactly as in the multiplicity 2 case in [19]. Most wavelet frames will be Riesz bases. Regularity conditions can also be imposed on the wavelets and scaling functions. $\tilde{\psi}_0(t)$ is called the dual scaling function and the functions $\tilde{\psi}_i(t)$ are called dual wavelets. Multiplicity 2 biorthogonal wavelet bases have been constructed by Vetterli [90], Chui [14] and Cohen [19]. But for Chui, the others have used filter bank theory as the starting point. Chui starts from a multiresolution analysis with spline functions and constructs spline wavelets. Since spline functions are as smooth as we require these are smooth wavelet bases.

## 4.8 Oversampling Invariance of Wavelet Frames

Frames and tight frames being redundant systems are robust in the presence of coefficient quantization (and other uncertainties) [23]. Given the importance of redundancy in these representations, it is natural to enquire into whether oversampling a wavelet frame gives another wavelet frame. Recently Chui and Shi [16] showed that under certain conditions on the oversampling factor a wavelet frame continues to be a wavelet frame. In this section, we strengthen their results by showing that in general if the conditions in their paper are not met then oversampling does not preserve the frame property. We also show that when the frame property is preserved the dual frame is obtained by oversampling the original frame. Our proof is elementary (compared to the one in [16]). Since the theory of filter banks is used in our proof our result is only valid for wavelet frames that are derived from filter banks (i.e., the frames considered in this thesis).

Section 4.3 gives a general technique to construct wavelet frames (with compact support) from FIR filter banks. Such *FB wavelet frames* have a scaling function, $\psi_0(t)$, associated with them. We show that if $\psi_{i,j,k}(t)$ forms a multiplicity $M$ wavelet frame then $\psi_{i,j,k/N}$ forms a wavelet frame with dual frame $\tilde{\psi}_{i,j,k/N}$ iff the $N$ is relatively prime to $M$. The result follows from a sequence of lemmas.

**Lemma 28** Given FIR $h(n)$ let

$$\hat{\psi}(\omega) = \prod_{j=1}^{\infty} H(e^{i\omega/M^j}) \tag{4.78}$$

exist as the Fourier transform of a function $\psi(t) \in L^2(\mathbb{R})$. For any positive integer $N$ if $H'(z) = H(z^N)$ we have

$$\hat{\psi}'(\omega) \stackrel{\text{def}}{=} \prod_{j=1}^{\infty} H'(e^{i\omega/M^j}) = \psi(N\omega)$$

and therefore $\psi'(t) = \frac{1}{N}\psi(\frac{t}{N})$

**Proof:** Follows directly by substituting $H'(e^{i\omega}) = H(e^{iN\omega})$. □

The result says that if a function is generated from a sequence by the infinite product in Eqn. 4.78, interpolating the sequence by a factor of $N$ is equivalent to stretching the function by a factor of $N$ and (and scaling by $1/N$). A direct consequence of the above Lemma is the following result for filter banks and frames generated from them.

**Lemma 29** Let the filters $H_i(z)$ and $G_i(z)$ generate a multiplicity $M$ wavelet frame (with functions $\psi_i(t)$ and $\tilde{\psi}_i(t)$). If the filters are upsampled by a factor of $N$, the associated functions $\psi_i(t)$ and $\tilde{\psi}_i(t)$ are given by $\frac{1}{N}\psi_i\left(\frac{t}{N}\right)$ and $\frac{1}{N}\tilde{\psi}_i\left(\frac{t}{N}\right)$ respectively. Conversely, given an FB frame if the scaling function and wavelets are all stretched by a factor of $N$ they can be obtained by upsampling each analysis and synthesis filter by $N$.

**Proof:** Apply Lemma 28 for all the functions involved. □

**Lemma 30** Let $H_i(z)$ and $G_i(z)$ be the filters in a perfect reconstruction filter bank. Then the filter bank obtained by interpolating each of the analysis and synthesis filters by a factor of $N$ is perfect reconstruction iff (generically) $(N, M) = 1$.

**Proof:** Since $G_p^T(z)H_p(z) = H_p(z)G_p^T(z) = I$ we have in particular

$$\sum_k H_{i,k}(z)G_{j,k}(z) = \delta(i-j) \tag{4.79}$$

Let $h_i'(n)$ and $g_i'(n)$ denote the new filters. Define $\alpha(l) = \sum_k h_i'(k)g_j'(Ml-k)$. Then from Eqn. 3.1 and Eqn. 3.2,

$$\begin{aligned}
\mathcal{Z}(a(l)) &= [\downarrow M]H_i'(z)G_j'(z) \\
&= [\downarrow M]\sum_{k=0}^{M-1}\sum_{l=0}^{M-1} z^{-Nk}H_{i,k}(z^{NM})z^{Nl}G_{j,l}(z^{NM}) \\
&= [\downarrow M]\sum_{k=0}^{M-1}\sum_{l=0}^{M-1} z^{N(l-k)}H_{i,k}(z^{NM})G_{j,l}(z^{NM})
\end{aligned}$$

Now let $(N,M)=1$. As $l$ and $k$ range from 0 through $(M-1)$ the only value of $N(k-l)$ that is a multiple of $M$ is zero. Hence from Eqn. 4.79

$$\mathcal{Z}(a(l)) = \sum_{k=0}^{M-1} H_{i,k}(z^N)G_{j,k}(z^N) = \delta(i-j)$$

Or equivalently in terms of sequences

$$\sum_k h_i'(k)g_j'(Ml-k) = \delta(i-j)\delta(l)$$

Therefore the new filters form a perfect reconstruction filter bank. If $(N,M)\neq 1$ the above relationships do not hold and hence (generically) with the new filters we *do not* have a PR filter bank. □

**Theorem 35** Let $\{\psi_{i,j,k}\}$ and $\left\{\tilde{\psi}_{i,j,k}\right\}$ form a frame, dual frame pair. Also let

$$\psi_i'(t) = \frac{1}{\sqrt{N}}\psi_i\left(\frac{t}{N}\right), \quad \tilde{\psi}_i'(t) = \frac{1}{\sqrt{N}}\tilde{\psi}_i\left(\frac{t}{N}\right),$$
$$\psi_{i,j,k}' = M^{j/2}\psi_i'(M^j t-k), \quad \text{and} \quad \tilde{\psi}_{i,j,k}' = M^{j/2}\tilde{\psi}_i'(M^j t-k).$$

Then $\{\psi_{i,j,k}'\}$ forms a frame with dual frame $\left\{\tilde{\psi}_{i,j,k}'\right\}$ iff $(N,M)=1$.

**Proof:** Let $H_i(z)$ and $G_i(z)$ denote the analysis and synthesis filters of the filter bank associated with the FB wavelet frame. Then from Lemma 29, the functions $\psi_i'(t)$ and $\tilde{\psi}_i'(t)$ are generated by replacing the filters $H_i(z)$ by $H'(z)=H_i(z^N)$ and $G_i(z)$ by $G_i'(z)=G_i(z^N)$ respectively. Now these functions generate a frame iff the corresponding filters form a perfect reconstruction filter bank. From Lemma 30, this occurs iff $(N,M)=1$. Hence the result follows. □

**Theorem 36** Let $\{\psi_{i,j,k}\}$ form an FB wavelet frame for $L^2(\mathbb{R})$ with dual frame $\left\{\psi_{i,j,k}^{\sim}\right\}$. Then $\left\{\frac{1}{\sqrt{N}}\psi_{i,j,\frac{k}{N}}\right\}$ and $\left\{\frac{1}{\sqrt{N}}\tilde{\psi}_{i,j,\frac{k}{N}}\right\}$ form dual frames for $L^2(\mathbb{R})$ iff $(N,M)=1$.

**Proof:** Notice that

$$\begin{aligned}
\frac{1}{\sqrt{N}}\psi_{i,j,\frac{k}{\sqrt{N}}}(t) &= \frac{1}{\sqrt{N}}\left[M^{j/2}\psi_i\left(M^j t - \frac{k}{N}\right)\right] \\
&= \frac{1}{\sqrt{N}}\left[M^{j/2}\psi_i\left(\frac{M^j}{N}(Nt - M^{-j}k)\right)\right] \\
&= M^{j/2}\psi_i'\left(M^j(Nt - M^{-j}k)\right) \\
&= M^{j/2}\psi_i'\left(M^j Nt - k\right)\right) \\
&= \psi_{i,j,k}'(Nt) \qquad (4.80)
\end{aligned}$$

Therefore $\left\{\frac{1}{\sqrt{N}}\psi_{i,j,\frac{k}{N}}\right\}$ forms a frame for $L^2(\mathbb{R})$ with dual frame $\left\{\frac{1}{\sqrt{N}}\tilde{\psi}_{i,j,\frac{k}{N}}\right\}$, iff $\left\{\psi_{i,j,k}'(Nt)\right\}$ and $\left\{\tilde{\psi}_{i,j,k}'(Nt)\right\}$ form dual frames for $L^2(\mathbb{R})$ and from Theorem 35 the result follows. □

In summary, all wavelet frames, that have filter banks associated with them, exhibit the following property: oversampling by a factor or $N$ preserves the frame property with the *same dual frame* (oversampled by $N$) iff $(N, M) = 1$. In particular, tight frames are preserved by oversampling by $N$, with $(N, M) = 1$.

# Chapter 5

# Computational Aspects and Applications

## 5.1 Implementation of FIR Filter Banks

Most results in filter bank and wavelet theory have been developed in a general setting. However, this section discusses the implementation of FIR filter banks only. There do exist classes of IIR filter banks (based on allpass structures) with low computational cost [84]. The question we address is how to most efficiently implement the analysis and synthesis banks in a filter bank or transmultiplexer. Section 5.2 relates the computations in FIR filter banks to computations in compactly supported wavelet frames. Throughout this section, by the polyphase representation we refer to the first orthant polyphase representation. Moreover, in the final implementation we will always assume that $H_p(z)$ and $G_p(z)$ will be multiplied by appropriate delays to make them causal.

### 5.1.1 General FIR Filter Banks

The polyphase representation is the most efficient way to implement both the analysis and synthesis banks in the general FIR case [70]. The essential idea is that only one out of $M$ outputs of a filter in any branch of the analysis bank must be computed. Similarly in the synthesis bank, the inputs to the filter along each branch has $M-1$ interlaced zeros which can be exploited by the polyphase representation.

Consider the analysis bank. From Eqn. 3.5

$$D_p(z) = H_p(z)X_p(z). \tag{5.1}$$

The above equation is represented in Fig. 5.1. One readily checks that there is a factor of $M$ savings in computation over direct implementation. If each filter $h_i(n)$ is of length $MK$, a direct implementation requires $M^3K$ multiplications and $M^2(MK-1)$ additions per output vector $d_p(\cdot)$. The polyphase implementation requires $M^2$ convolutions with filters of length $K$, and therefore requires $M^2K$ multiplications and $M^2(K-1)$ additions. Further savings are possible by implementing each of the $M^2$ convolutions in Eqn. 5.1 using fast-convolution algorithms or the FFT [6, 7].

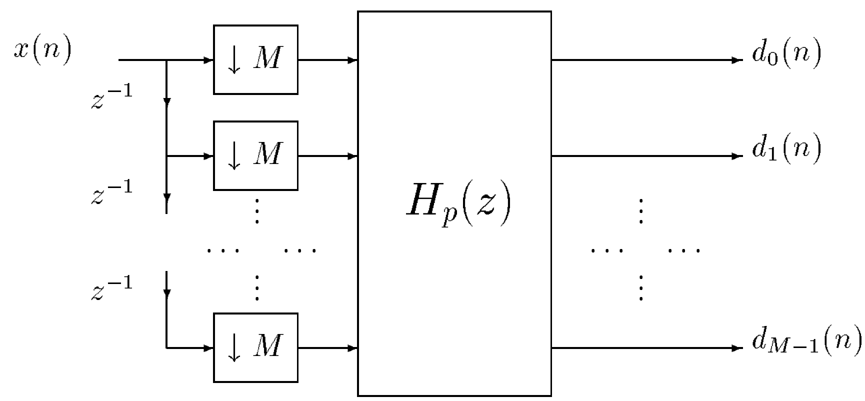


Figure 5.1: Polyphase Structure for the Analysis Bank

As for the synthesis bank, from Eqn. 3.6

$$Y_p(z) = G_p^T(z)D_p(z). \tag{5.2}$$

This is represented in Fig. 5.2. Clearly, the savings from going from the direct to the polyphase representation is a factor of $M$ as in the case of the analysis bank. Notice the non-causal delays in Fig. 5.2 exhibiting the structural finite non-causality of $M-1$ of the filter bank.

**General Unitary Filter Banks**

If $H_p(z)$ is unitary the Householder parameterization described earlier provides the most efficient technique for implementing general unitary filter banks. Recall that

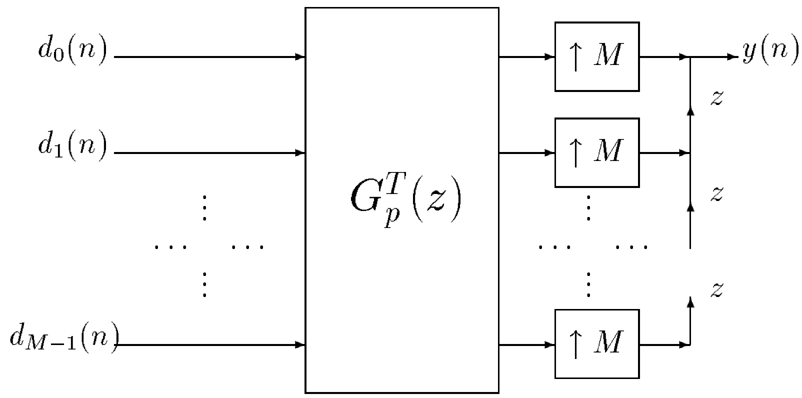


Figure 5.2: Polyphase Structure for the Synthesis Bank

in Section 3.4, Householder type factorization of $H_p(z)$ was discussed for general $M$. For $M = 2$, a Givens' rotation type factorization was also discussed, (which played an important role in the theory of unitary modulated filter banks). The Givens' type factorization is most efficient for $M = 2$, and the Householder type factorization is most efficient otherwise.

Let $H_p(z)$ be unitary of degree $K - 1$. Then the filters are of length $MK$. Now from Theorem 9 (Eqn. 3.22) $H_p(z)$ can be parameterized by $K - 1$ unitary vectors $v_i, i = 1, 2, \ldots, K - 1$ and a unitary matrix $V_0$.

$$H_p(z) = [I - v_{K-1}v_{K-1}^T + z^{-1}v_{K-1}v_{K-1}^T] \ldots [I - v_1 v_1^T + z^{-1}v_1 v_1^T]V_0. \tag{5.3}$$

Therefore, the analysis FB can be implemented as shown in Fig. 5.3. The structure is

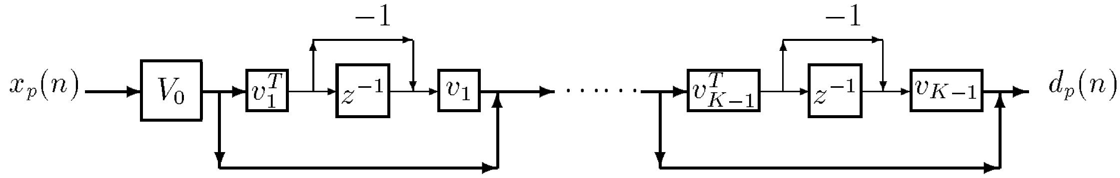


Figure 5.3: Implementation of Unitary FBs : Analysis Bank

regular and easily coded. The thick lines in Fig. 5.3 correspond to vector data, and the thin lines to scalar data. The the number of storage elements required to implement the entire filter bank is $K-1$, the McMillan degree of $H_p(z)$. Since each of the filters is of length $MK$, the usual polyphase implementation requires $M^2K$ multiplications and $M^2(K-1)$ additions. As for the cascade implementation above, there are $K-1$ blocks, followed by a unitary matrix multiplication. Since each block requires $2M$ multiplies and $M$ adds, the entire cascade requires $[(K-1)2M+M^2]$ multiplies and $[(K-1)M+(M-1)M]$ additions. These counts could be moderately improved by taking advantage of the fact that the unitary matrix multiplication can be done more efficiently than a general matrix multiplication. As for storage requirements in a direct form implementation of $H_p(z)$, the obvious convolution implementation requires $(MK-1)M$ scalar storages (delays), the polyphase implementation $(K-1)M$ delays, and the cascade implementation requires just $K-1$ delays (the minimal possible since the McMillan degree of $H_p(z)$ is $K-1$).

The polyphase synthesis bank matrix is given by

$$G_p^T(z) = H_p^T(z^{-1}) = V_0^T[I - v_1v_1^T + zv_1v_1^T]\ldots[I - v_{K-1}v_{K-1}^T + zv_{K-1}v_{K-1}^T]. \quad (5.4)$$

The corresponding structure for the synthesis bank is shown in Fig. 5.4 The computa-

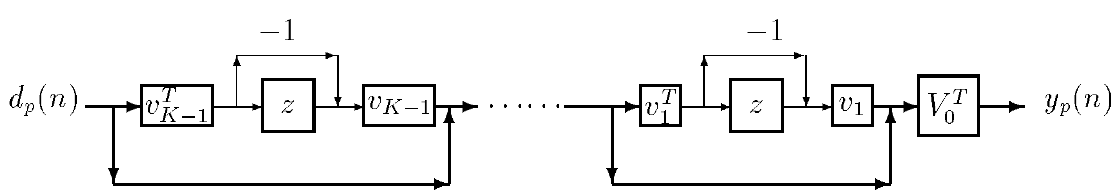


Figure 5.4: Implementation of Unitary FBs : Synthesis Bank

tional complexity of this structure is identical to that of the analysis bank. However because $z$-blocks are involved the implementation is non-causal. A causal reorganization of the computation is given in Fig. 5.5. The storage complexity of this bank is $(M-1)(K-1)$ (which in turn is the McMillan degree of $z^{K-1}H_p(z^{-1})$).

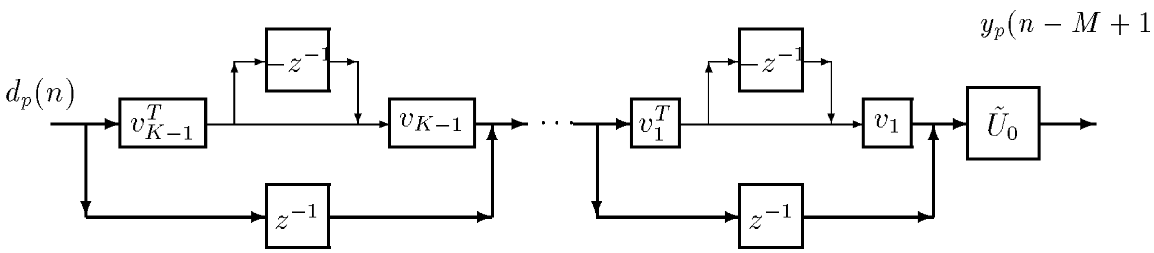


Figure 5.5: Cascade Implementation of the Synthesis Filter Bank - Causal

### Two Channel Unitary Filter Banks

For two-channel unitary filter banks, the *denormalized* orthogonal lattice (based on Givens' rotations) is the most efficient [86]. Recall that a degree $(K-1)$ unitary matrix $H_p(z)$ is parameterized by $(K-1)$ angle parameters $\theta_i$. Define $\gamma_i = \tan\theta_i$ and $\beta_i = \prod_{i=0}^{K-1} \cos\theta_i$. Then (from Eqn. 3.23)

$$H_p(z) = \beta \left\{ \prod_{i=K-1}^{1} \begin{bmatrix} 1 & z^{-1}\gamma_i \\ -\gamma_i & z^{-1} \end{bmatrix} \right\} \begin{bmatrix} 1 & \gamma_0 \\ -\gamma_0 & 1 \end{bmatrix} \tag{5.5}$$

and since each factor in the product above requires two multiplies (as opposed to four for a planar rotation matrix), the multiplication cost of the above implementation is one half of a rotation-based implementation. Furthermore, $H_p(z)$ above is structurally unitary: independent of the quantization of $\gamma_i$, $H_p(z)$ is unitary [86].

### 5.1.2 Modulated Filter Banks

A unitary PR MFB reduces to a set of $J$ two channel PR filter banks. This section shows that the implementation of an MFB is equivalent to the implementation of $J$ two channel PR filter banks, a DCT and an inverse DCT. The type of DCT depends on the parity of the *modulation phase* $\alpha$. If $\alpha$ a Type III DCT is required and if $\alpha$ is odd a Type IV DCT is required. Recall that the four types of DCT are defined as follows [72]:

**Type I DCT:** For $k \in \{0, 1, \ldots, M\}$,

$$X(k) = \sqrt{\frac{2}{M}} \sum_{n=0}^{M} x(n) \cos\left(\frac{\pi}{M} kn\right). \tag{5.6}$$

**Type II DCT:** For $k \in \{1, \ldots, M-1\}$

$$X(k) = \sqrt{\frac{2}{M}} \sum_{n=0}^{M-1} x(n) \cos\left(\frac{\pi}{M} k(n + \frac{1}{2})\right) \tag{5.7}$$

and

$$X(0) = \sqrt{\frac{1}{M}} \sum_{n=0}^{M-1} x(n). \tag{5.8}$$

**Type III DCT:** For $k \in \{0, 1, \ldots, M-1\}$

$$X(k) = \sqrt{\frac{2}{M}} \sum_{n=1}^{M-1} x(n) \cos\left(\frac{\pi}{M}(k + \frac{1}{2})n\right) + \sqrt{\frac{1}{M}} x(0). \tag{5.9}$$

**Type IV DCT:** For $k \in \{0, 1, \ldots, M-1\}$

$$X(k) = \sqrt{\frac{2}{M}} \sum_{n=0}^{M-1} x(n) \cos\left(\frac{\pi}{M}(k + \frac{1}{2})(n + \frac{1}{2})\right). \tag{5.10}$$

DCT I and DCT IV are involutary (i.e., the inverse transform is the forward transform) while the inverse of DCT III is DCT II and vice versa [72].

### MFB Analysis Bank

**Modulation Phase $\alpha$ is even:** This covers Type 1 MFBs with $M$ odd and Type 2 MFBs with $M$ even. In this case $J = \frac{\alpha}{2}$ (from Eqn. 3.42) and we have

$$\begin{aligned} d_i(n) &= \sum_k x(Mn - k) h_i(k) \\ &= \sum_k x(Mn - k) h(k) \cos\left(\frac{\pi}{2M}(2i + 1)(k - \frac{\alpha}{2})\right) \\ &= \sum_k x(Mn - k - J) h(k + J) \cos\left(\frac{\pi}{2M}(2i + 1)k\right). \end{aligned}$$

Since $c_{i,k+2Ml} = (-1)^l c_{i,k}$, $c_{i,k} = c_{i,-k}$ and $c_{i,M} = 0$ (where $c_{i,\tau}$ is defined in Eqn. 3.33),

$$\begin{aligned} d_i(n) &= \sum_l (-1)^l x(Mn - 2Ml - J) h(2Ml + J) c_{i,0} \\ &\quad + \sum_{k=1}^{M-1} \left\{ \sum_l (-1)^l x(Mn - 2Ml - k - J) h(2Ml + k + J) \right\} c_{i,k} \\ &\quad + \sum_{k=1}^{M-1} \left\{ \sum_l (-1)^l x(Mn - 2Ml + k - J) h(2Ml - k + J) \right\} c_{i,k}. \end{aligned}$$

If $x_k(n) = x(Mn - k)$, $p_{k,0}(n) = h(2Mn + k)$ and $p_{k,1}(n) = h(2Mn + M + k)$, then

$$D_i(z) = X_J(z) P_{J,0}(-z^2) c_{i,0} + \sum_{k=1}^{M-1} \left[ X_{J+k}(z) P_{J+k,0}(-z^2) + X_{J-k}(z) P_{J-k,0}(-z^2) \right] c_{i,k}.$$

Define

$$T_k(z) = \begin{cases} \sqrt{M} X_J(z) P_{J,0}(-z^2) & \text{for } k = 0 \\ \sqrt{\frac{M}{2}} \left( X_{J+k}(z) P_{J+k,0}(-z^2) + X_{J-k}(z) \right) P_{J-k,0}(-z^2) & \text{for } k \in \mathcal{R}(M) \backslash \{0\}. \end{cases}$$

Then

$$D_i(z) = \sqrt{\frac{1}{M}} T_0(z) + \sqrt{\frac{2}{M}} \sum_{k=1}^{M-1} T_k(z) \cos\left( \frac{\pi}{M} \left( i + \frac{1}{2} \right) k \right).$$

which shows that the output at the instant $n$ is the Type III DCT of the sequence $t_k(n)$.

Let the MFB be Type 1 ($M$ is odd). Notice that $X_{Mn+k}(z) = z^{-n} X_k(z)$ and that

$$\begin{bmatrix} P_{2Mn+k,0}(-z^2) \\ P_{2Mn+k,1}(-z^2) \\ P_{M+k,0}(-z^2) \\ P_{M+k,1}(-z^2) \end{bmatrix} = \begin{bmatrix} (-1)^n z^{2n} P_{k,0}(-z^2) \\ (-1)^n z^{2n} P_{k,1}(-z^2) \\ P_{k,1}(-z^2) \\ -z^2 P_{k,0}(-z^2) \end{bmatrix}.$$

Hence for $k \in \{0, 1, \ldots, J-1\}$

$$\begin{aligned} T_{J+1+k}(z) &= \sqrt{\frac{M}{2}} \left( X_{M+k}(z) P_{M+k,0}(-z^2) + X_{-M+M-1-k}(z) P_{-M+M-1-k,0}(-z^2) \right) \\ &= \sqrt{\frac{M}{2}} \left( z^{-1} X_k(z) P_{k,1}(-z^2) - z^1 X_{M-1-k}(z) z^{-2} P_{M-1-k,1}(-z^2) \right) \\ &= \sqrt{\frac{M}{2}} \left( z^{-1} X_k(z) P_{k,1}(-z^2) - z^{-1} X_{M-1-k}(z) P_{M-1-k,1}(-z^2) \right). \end{aligned}$$

Fig 5.6 shows the implementation of the analysis filter bank. Notice that (as expected) $P_{J,1}(z^2)$ does not enter the computation and could be arbitrary. If the MFB is Type

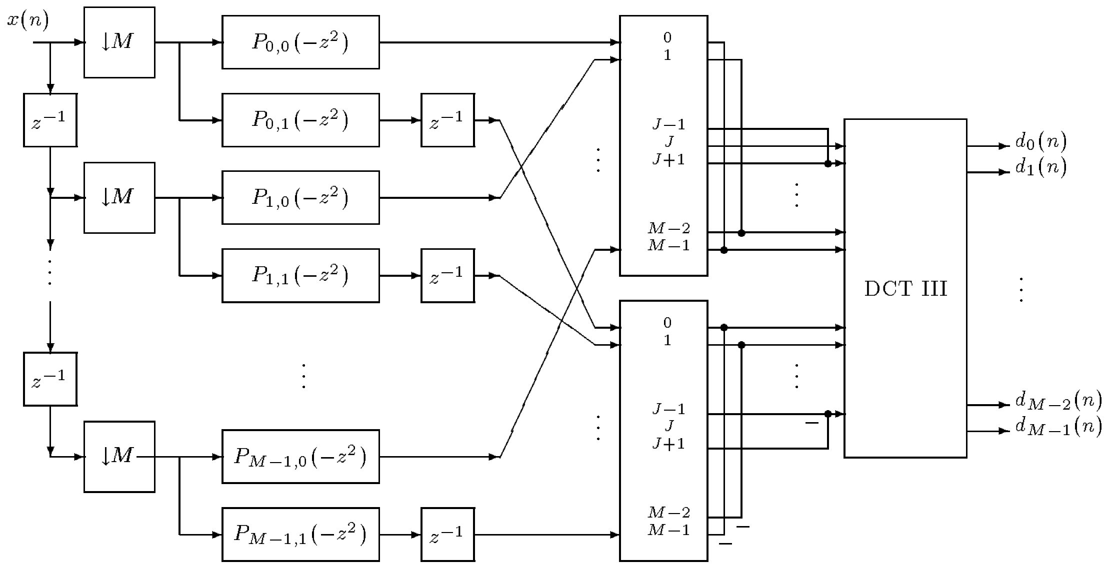


Figure 5.6: MFB Analysis Bank Implementation: Type 1, odd $M$

2 and $M$ even then

$$\begin{aligned} T_{J+1}(z) &= \sqrt{\frac{M}{2}}\left(X_{M-1}(z)P_{M-1,0}(-z^2) + X_{-M+M-1}(z)P_{-M+M-1,0}(-z^2)\right) \\ &= \sqrt{\frac{M}{2}}\left(X_{M-1}(z)P_{M-1,0}(-z^2) - z^{-1}X_{M-1}(z)P_{M-1,1}(-z^2)\right). \end{aligned}$$

For $k \in \{0, 1, \ldots, J-1\}$

$$\begin{aligned} T_{J+2+k}(z) &= \sqrt{\frac{M}{2}}\left(X_{M+k}(z)P_{M+k,0}(-z^2) + X_{-M+M-2-k}(z)P_{-M+M-2-k,0}(-z^2)\right) \\ &= \sqrt{\frac{M}{2}}\left(z^{-1}X_k(z)P_{k,1}(-z^2) - z^{-1}X_{M-2-k}(z)P_{M-2-k,1}(-z^2)\right). \end{aligned}$$

The implementation of this analysis bank is shown in Fig 5.7.

**Modulation Phase $\alpha$ is odd:** This covers Type 1 MFBs with $M$ even and Type 2 MFBs with $M$ odd. In this case $J = \frac{\alpha+1}{2}$ and

$$d_i(n) = \sum_k x(Mn-k)h(k)\cos\left(\frac{\pi}{2M}(2i+1)(k-\frac{\alpha}{2})\right)$$

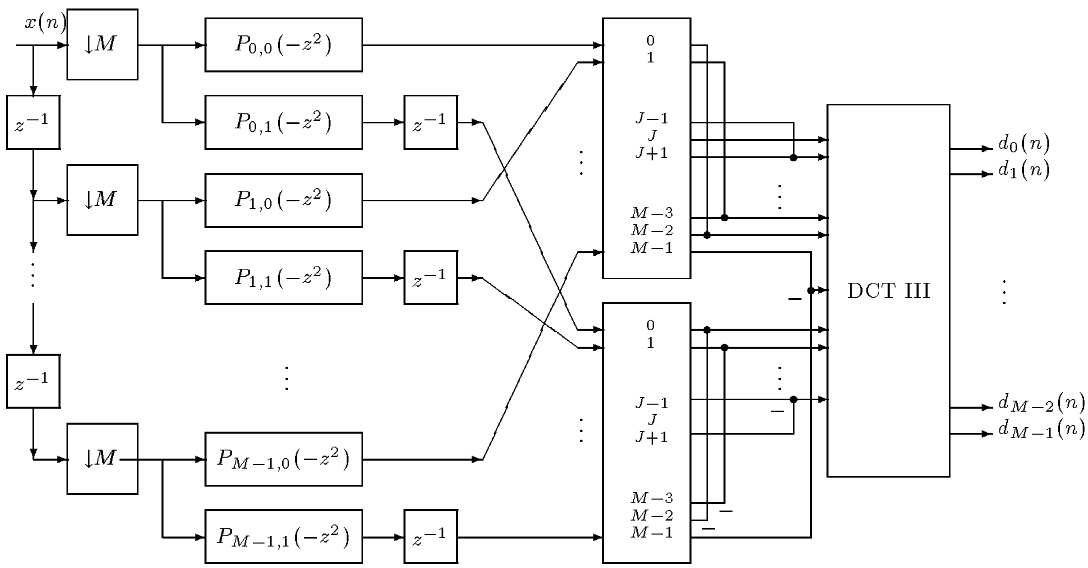


Figure 5.7: MFB Analysis Bank Implementation: Type 2, even $M$

$$
\begin{aligned}
&= \sum_k x(Mn-k-J)h(k+J)\cos\left(\frac{\pi}{2M}(2i+1)(k+\frac{1}{2})\right) \\
&= \sum_{k=0}^{M-1}\left\{\sum_l (-1)^l x(Mn-2Ml-k-J)h(2Ml+k+J)\right\} c_{i,k+\frac{1}{2}} \\
&\quad + \sum_{k=0}^{M-1}\left\{\sum_l (-1)^l x(Mn-2Ml+k+1-J)h(2Ml-k-1+J)\right\} c_{i,-k-1+\frac{1}{2}} \\
&= \sum_{k=0}^{M-1}\left\{\sum_l (-1)^l x(Mn-2Ml-k-J)h(2Ml+k+J)\right\} c_{i,k+\frac{1}{2}} \\
&\quad + \sum_{k=0}^{M-1}\left\{\sum_l (-1)^l x(Mn-2Ml+k+1-J)h(2Ml-k+J-1)\right\} c_{i,k+\frac{1}{2}}.
\end{aligned}
$$

Equivalently

$$D_i(z) = \sum_{k=0}^{M-1}\left[X_{J+k}(z)P_{J+k,0}(-z^2) + X_{J-1-k}(z)P_{J-1-k,0}(-z^2)\right] c_{i,k+\frac{1}{2}}.$$

For $k \in \mathcal{R}(M)$ let

$$T_k(z) \stackrel{\text{def}}{=} \sqrt{\frac{M}{2}}\left(X_{J+k}(z)P_{J+k,0}(-z^2) + X_{J-1-k}(z)P_{J-1-k,0}(-z^2)\right).$$

Then $D_i(z)$ is the Type IV DCT of $T_k(z)$.

$$D_i(z) = \sqrt{\frac{2}{M}} \sum_{k=0}^{M-1} T_k(z) \cos\left(\frac{\pi}{M}(i+\frac{1}{2})(k+\frac{1}{2})\right).$$

If the MFB is of Type 1 ($M$ is even) for $k \in \{0, 1, \ldots, J-1\}$

$$\begin{aligned} T_{J+k}(z) &= \sqrt{\frac{M}{2}} \left(X_{M+k}(z)P_{M+k,0}(-z^2) + X_{-M+M-1-k}(z)P_{-M+M-1-k,0}(-z^2)\right) \\ &= \sqrt{\frac{M}{2}} \left(z^{-1}X_k(z)P_{k,1}(-z^2) - z^{-1}X_{M-1-k}(z)P_{M-1-k,1}(-z^2)\right) \end{aligned}$$

The implementation of this analysis bank is shown in Fig 5.8. If the MFB is of Type

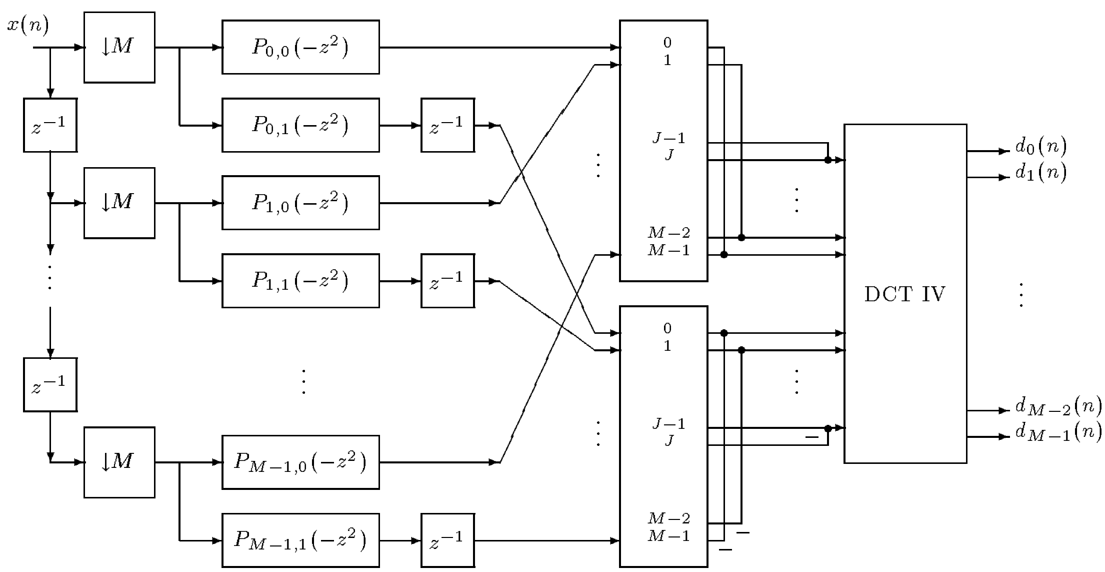


Figure 5.8: MFB Analysis Bank Implementation: Type 1, even $M$

2 ($M$ is odd). In this case

$$\begin{aligned} T_J(z) &= \sqrt{\frac{M}{2}} \left(X_{M-1}(z)P_{M-1,0}(-z^2) + X_{-M+M-1}(z)P_{-M+M-1,0}(-z^2)\right) \\ &= \sqrt{\frac{M}{2}} \left(X_{M-1}(z)P_{M-1,0}(-z^2) - z^{-1}X_{M-1}(z)P_{M-1,1}(-z^2)\right). \end{aligned}$$

For $k \in \{0, 1, \ldots, J-1\}$

$$T_{J+1+k}(z) = \sqrt{\frac{M}{2}} \left(X_{M+k}(z)P_{M+k,0}(-z^2) + X_{-M+M-2-k}(z)P_{-M+M-2-k,0}(-z^2)\right)$$

$$= \sqrt{\frac{M}{2}} \left( z^{-1} X_k(z) P_{k,1}(-z^2) - z^{-1} X_{M-2-k}(z) P_{M-2-k,1}(-z^2) \right).$$

The implementation of this analysis bank is shown in Fig 5.9.

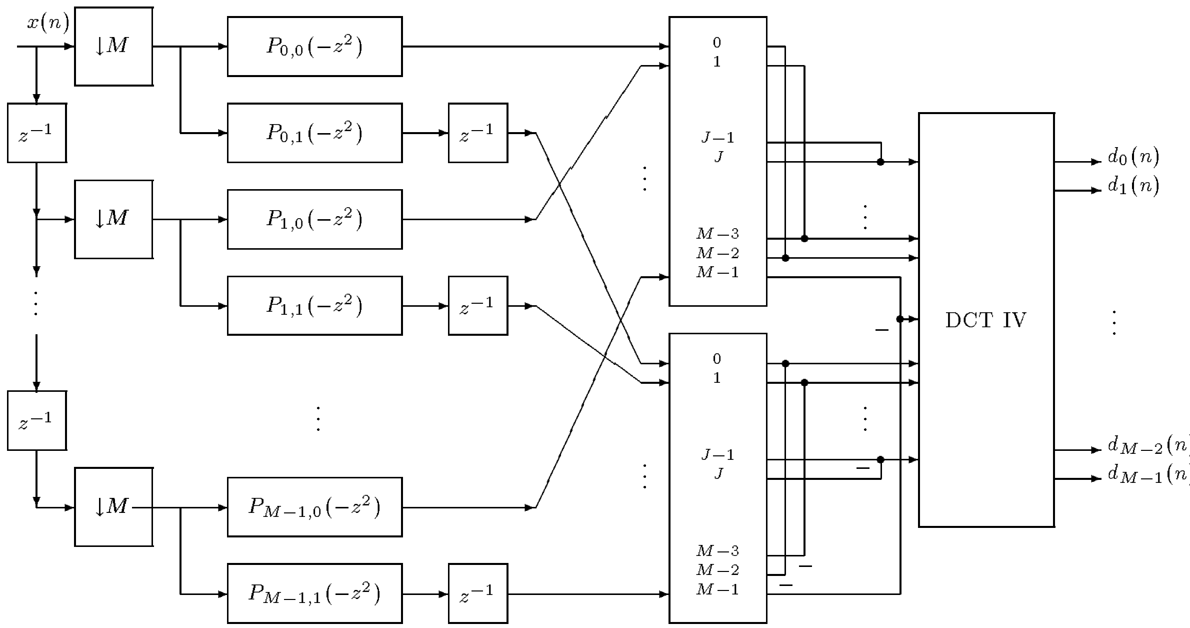


Figure 5.9: MFB Analysis Bank Implementation: Type 2, odd $M$

### MFB Synthesis Bank

**Modulation Phase $\alpha$ is even:** This covers Type 1 MFBs with $M$ odd and Type 2 MFBs with $M$ even. In this case $J = \frac{\alpha}{2}$ (from Eqn. 3.42) and

$$\begin{aligned} G_i(z) &= \sum_n g(n) c_{i,n+J} z^{-n} = \sum_n g(n-J) c_{i,n} z^{-n+J} \\ &= z^J c_{i,0} \left[ \sum_l z^{-2Ml} (-1)^l g(2Ml - J) \right] \\ &\quad + \sum_{k=1}^{M-1} z^{J-k} c_{i,k} \left[ \sum_l z^{-2Ml} (-1)^l g(2Ml + k - J) \right] \\ &\quad + \sum_{k=1}^{M-1} z^{J+k} c_{i,k} \left[ \sum_l z^{-2Ml} (-1)^l g(2Ml - k - J) \right] \end{aligned}$$

$$= z^J c_{i,0} G_{J,0}(-z^{2M}) + \sum_{k=1}^{M-1} \left[ z^{J+k} G_{J+k,0}(-z^{2M}) + z^{J-k} c_{i,k} G_{J-k,0}(-z^{2M}) \right] c_{i,k}$$

Let $S_k(z)$ denote the DCT II of $D_i(z)$. Then

$$\begin{aligned} Y(z) &= \sum_i D_i(z^M) G_i(z) \stackrel{\text{def}}{=} \sum_{l=0}^{M-1} z^l Y_l(z^M) \\ &= \sqrt{M} z^J G_{J,0}(-z^{2M}) S_0(z^M) \\ &\quad + \sqrt{\frac{M}{2}} \sum_{k=1}^{M-1} \left[ z^{J+k} G_{J+k,0}(-z^{2M}) + z^{J-k} G_{J-k,0}(-z^{2M}) \right] S_k(z^M) \end{aligned}$$

Therefore the polyphase components of $Y(z)$ are given as follows.

$$Y_J(z) = \sqrt{M} z^J G_{J,0}(-z^{2M}) S_0(z^M).$$

When $M$ is odd, for $k \in \{0, 1, \ldots, J-1\}$

$$\begin{aligned} Y_k(z) &= \sqrt{\frac{M}{2}} \left[ G_{k,0}(-z^2) S_{J-k}(z) + z G_{k,1}(-z^2) S_{J+1+k}(z) \right] \\ Y_{M-1-k}(z) &= \sqrt{\frac{M}{2}} z \left[ G_{M-1-k,0}(-z^2) S_{J-k}(z) - z G_{M-1-k,1}(-z^2) S_{J+1+k}(z) \right] \end{aligned}$$

and when $M$ is even

$$\begin{aligned} Y_k(z) &= \sqrt{\frac{M}{2}} \left[ G_{k,0}(-z^2) S_{J-k}(z) + z G_{k,1}(-z^2) S_{J+2+k}(z) \right] \\ Y_{M-2-k}(z) &= \sqrt{\frac{M}{2}} z \left[ G_{M-2-k,0}(-z^2) S_{J-k}(z) - z G_{M-2-k,1}(-z^2) S_{J+2+k}(z) \right]. \end{aligned}$$

Additionally when $M$ is even

$$Y_{M-1}(z) = \sqrt{\frac{M}{2}} \left( G_{M-1,0}(-z^2) - z G_{M-1,1}(-z^2) \right) S_{J+1}(z).$$

**Modulation Phase $\alpha$ is even:** This covers Type 2 MFBs with $M$ even and Type 1 MFBs with $M$ odd. In this case $J = \frac{\alpha+1}{2}$ and

$$G_i(z) = \sum_n g(n) c_{i,n+J-\frac{1}{2}} z^{-n} = \sum_n g(n-J) c_{i,n-\frac{1}{2}} z^{-n+J}$$

$$
\begin{aligned}
&= \sum_{k=0}^{M-1} z^{J-1-k} c_{i,k+\frac{1}{2}} \left[ \sum_l z^{-2Ml} (-1)^l g(2Ml+k+1-J) \right] \\
&\qquad + \sum_{k=0}^{M-1} z^{J+k} c_{i,k+\frac{1}{2}} \left[ \sum_l z^{-2Ml} (-1)^l g(2Ml-k-J) \right] \\
&= \sum_{k=0}^{M-1} \left[ z^{J+k} G_{J+k,0}(-z^{2M}) + z^{J-1-k} G_{J-1-k,0}(-z^{2M}) \right] c_{i,k+\frac{1}{2}}
\end{aligned}
$$

Let $S_k(z)$ denote the DCT IV of $D_i(z)$. Then

$$
\begin{aligned}
Y(z) &= \sum_i D_i(z^M) G_i(z) \stackrel{\text{def}}{=} \sum_{k=0}^{M-1} z^k Y_k(z^M) \\
&= \sqrt{\frac{M}{2}} \sum_{k=0}^{M-1} \left[ z^{J+k} G_{J+k,0}(-z^{2M}) + z^{J-1-k} G_{J-1-k,0}(-z^{2M}) \right] S_k(z^M)
\end{aligned}
$$

For $k \in \{0, 1, \ldots, J-1\}$ we have when $M$ is even

$$
\begin{aligned}
Y_k(z) &= \sqrt{\frac{M}{2}} \left[ G_{k,0}(-z^2) S_{J-1-k}(z) + z G_{k,1}(-z^2) S_{J+k}(z) \right] \\
Y_{M-1-k}(z) &= \sqrt{\frac{M}{2}} z \left[ G_{M-1-k,0}(-z^2) S_{J-1-k}(z) - z G_{M-1-k,1}(-z^2) S_{J+k}(z) \right]
\end{aligned}
$$

and when $M$ is odd

$$
\begin{aligned}
Y_k(z) &= \sqrt{\frac{M}{2}} \left[ G_{k,0}(-z^2) S_{J-1-k}(z) + z G_{k,1}(-z^2) S_{J+1+k}(z) \right] \\
Y_{M-2-k}(z) &= \sqrt{\frac{M}{2}} z \left[ G_{M-2-k,0}(-z^2) S_{J-1-k}(z) - z G_{M-2-k,1}(-z^2) S_{J+1+k}(z) \right].
\end{aligned}
$$

Additionally when $M$ is even

$$
Y_{M-1}(z) = \sqrt{\frac{M}{2}} \left( G_{M-1,0}(-z^2) - z G_{M-1,1}(-z^2) \right) S_{J+1}(z).
$$

### 5.1.3 Modulated FIR Unitary Filter Banks

If an MFB is unitary FIR each of the pairs $P_{l,0}(-z^2)$ and $P_{l,1}(-z^2)$ can be implemented using the denormalized two-channel orthogonal lattice. Indeed

$$
\begin{bmatrix} P_{l,0}(-z^2) \\ P_{l,1}(-z^2) \end{bmatrix} = \prod_{k=k_l-1}^{1} \left\{ \begin{bmatrix} \cos(\theta_{l,k}) & -z^2 \sin(\theta_{l,k}) \\ \sin(\theta_{l,k}) & z^2 \cos(\theta_{l,k}) \end{bmatrix} \right\} \begin{bmatrix} \cos(\theta_{l,0}) \\ \sin(\theta_{l,0}) \end{bmatrix}
$$

$$= \beta_l \prod_{k=k_l-1}^{1} \left\{ \begin{bmatrix} 1 & -z^{-2}\gamma_{l,k} \\ \gamma_{l,k} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \gamma_{l,0}) \end{bmatrix} \right\},$$

where $\beta_l = \prod_{k=k_l-1}^{0} \cos(\theta_{l,k})$, and $\gamma_{l,k} = \tan(\theta_{l,k})$. Similarly the synthesis bank also can be implemented using the two-channel orthogonal lattice. Also notice that the PR pairs can be implemented simultaneously.

## 5.2 Computations in FB Wavelet Frames

Given $h_i(n)$ and $g_i(n)$ of a compactly supported multiplicity $M$ wavelet frame, how does one compute the samples of $\psi_i(t)$ and $\tilde{\psi}_i(t)$? Furthermore, given the samples of a signal how does one (*efficiently*) compute its Discrete Wavelet Transform (DWT)?

### 5.2.1 Samples of the Scaling Functions and Wavelets

Since between samples a function could behave badly, some form of a "continuity" assumption is implicit in the sampled representation of functions. To compute the samples of $\psi_i(t)$ and $\tilde{\psi}_i(t)$ there are two possible approaches: *the interpolation method* and *the infinite product method.* The latter is relatively more efficient while the former is relatively more accurate.

In the interpolation method $\{\psi_0(n)\}$ is first obtained by solving the linear equations

$$\psi_0(n) = \sqrt{M} \sum_{k=0}^{N-1} h_0(Mn-k)\psi_0(k).$$

Samples at the $M$-adic rationals are then computed recursively.

$$\psi_0\left(\frac{n}{M^J}\right) = \sqrt{M} \sum_{k=0}^{N-1} h_0(k)\psi_0\left(\frac{n-kM^{J-1}}{M^{J-1}}\right).$$

Similarly the samples of the wavelets at $M$-adics rationals are computed.

$$\psi_i\left(\frac{n}{M^J}\right) = \sqrt{M} \sum_{k=0}^{N-1} h_i(k)\psi_0\left(\frac{n-kM^{J-1}}{M^{J-1}}\right).$$

This method gives computes the exact values of $\psi_i(t)$ at the $M$-adic rationals.

The infinite product method is based on the following formula:

$$\widehat{\psi}_i(\omega) = \left[\frac{1}{\sqrt{M}} H_i\left(\frac{\omega}{M}\right)\right] \prod_{j=2}^{\infty} \left[\frac{1}{\sqrt{M}} H_0\left(\frac{\omega}{M^j}\right)\right].$$

If the infinite product above is truncated to $J$ terms the resulting function is periodic with fundamental period $[-\frac{\pi}{M^J}, \frac{\pi}{M^J}]$. This can be interpreted as the (discrete-time) Fourier transform of the samples $\psi_i\left(\frac{n}{M^J}\right)$.

$$\mathcal{Z}(\psi_i(M^{-J}n))(z) \approx M^{J/2} H_i\left(z^{M^{J-1}}\right) \prod_{i=0}^{J-2} H_0\left(z^{M^i}\right). \tag{5.11}$$

The above equation can be implemented as a recursive algorithm.

### 5.2.2 Analysis/Synthesis in Wavelet Bases

The DWT of a signal $f(t)$ is given by $\left\langle f, \tilde{\psi}_{i,j,k}\right\rangle$. Assuming only the samples of $f(t)$ are available the DWT can only be computed approximately. The inverse DWT is given by

$$f(t) = \sum_{i=1}^{M-1} \sum_{j,k} \langle f, \psi_{i,j,k}(t)\rangle \, \tilde{\psi}_{i,j,k}(t).$$

Let $Wf(i,j,k) = \langle f, \psi_{i,j,k}\rangle$, $i \in \{1, 2, \ldots, M-1\}$, denote the DWT (with respect to this frame). From computational considerations one defines the Discrete Scaling Transform (DST) at scale $j$ by $Wf(0,j,k) = \langle f, \psi_{0,j,k}\rangle$. The DST contains all the information in scale $j$. From a practical viewpoint the indices $j$ and $k$ run only over a finite set.

Any signal $f(t)$ can be approximated to any desired accuracy in $W_{0,j}$ for sufficiently large $j$. Let $j = J_f$ denote the finest scale of interest. Similarly a coarsest scale of interest $J_c$ can also be determined (if necessary). Information in coarser scales can be retained in the DST at that scale.

$$f(t) = \sum_k Wf(0, J_c, k)\tilde{\psi}_{0,J_c,k}(t) + \sum_{i=1}^{M-1} \sum_{j=J_c}^{J_f} \sum_k Wf(i,j,k)\tilde{\psi}_{i,j,k}.$$