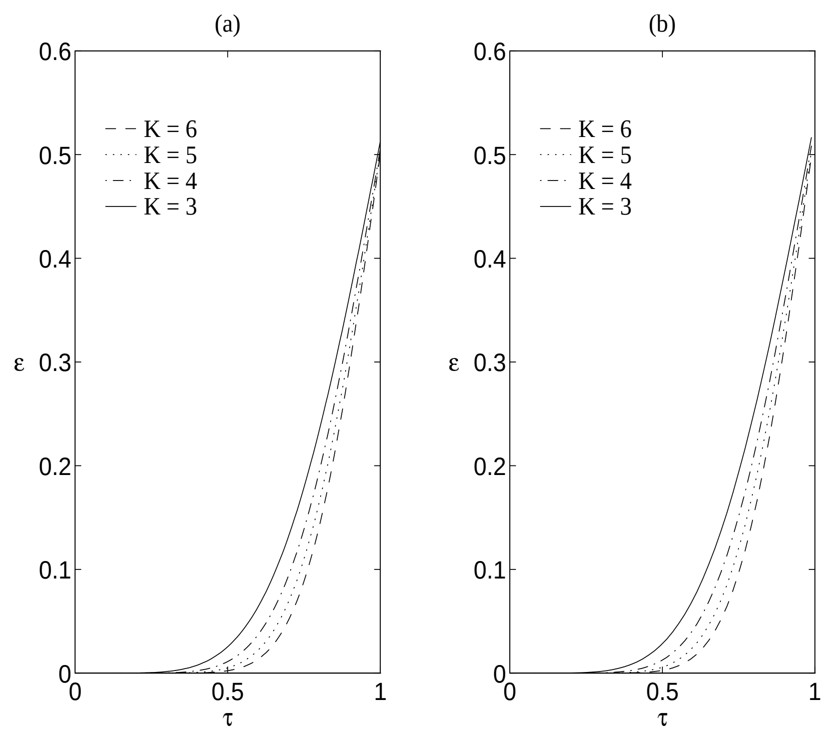

Figure 5.11: (a) Multiplicity 2 - ϵ_τ for K regular Wavelet Bases (b) Multiplicity 3 - ϵ_τ for K regular Wavelet Bases

i.e., $\frac{1}{p}+\frac{1}{q}=1$, and $1 \leq p, q \leq \infty$,

$$\int_{\mathbf{R}} dt f(t) g(t) \leq \|f\|_p \|g\|_q . \tag{5.43}$$

Notice that for $p = 2$, Hölder's inequality is Cauchy-Schwartz inequality. From Eqn. 5.26, for bandlimited $f(t)$ ($\hat{f}(\omega) = 0$ for $\omega \notin \Omega$), the L^1 norm of the error is given by

$$\begin{aligned} \left\|\widehat{Qf}\right\|_1 &= \int_{\mathbf{R}} d\omega \left|\widehat{Qf}(\omega)\right| \\ &= \int_{\Omega} d\omega \left|\hat{f}(\omega)\right| S_p(\omega) \\ &\leq \left\|\hat{f}\right\|_p \|S_p\|_q \qquad \text{From Eqn. 5.43.} \end{aligned} \tag{5.44}$$

Therefore, the design of the optimal robust multiresolution analysis for L^p classes of signals, with the L^1 error norm is equivalent to the problem

$$\min_{\Theta} \|S_p\|_q^q = \min_{\Theta} \left\{ \left|1 - \left|\hat{\psi}_0\left(\frac{\omega}{M^J}\right)\right|^2\right|^p + \left|\hat{\psi}_0^*\left(\frac{\omega}{M^J}\right)\right|^p \sum_{k\neq 0} \left|\hat{\psi}_0\left(\frac{\omega + 2\pi M^J k}{M^J}\right)\right|^p \right\}^q .$$

5.4.3 Numerical Design of Optimal Wavelets

This section describes the details of the design of the optimal multiresolution analysis for the specific case of $p = 2$ because of its importance. For $p = 2$, by Parseval's theorem, the frequency domain design technique actually minimizes the energy of the approximation error. From Eqn. 5.38, the design objective is

$$\max_{\Theta} \frac{1}{2\pi} \int_{\Omega} d\omega \left|\hat{f}(\omega)\right|^2 \left|\hat{\psi}_0\left(\frac{\omega}{M^J}\right)\right|^2 . \tag{5.45}$$

Since $f(t)$ is bandlimited to Ω, it is given by

$$f(t) = \sum_n f(M^{-J} n) \frac{\sin(2\pi(M^J t - k))}{2\pi(M^J t - k)} .$$

It is also true that for any $L \geq 1$,

$$f(t) = \sum_n f(M^{-(J+L)} n) \frac{\sin(2\pi(M^{J+L} t - k))}{2\pi(M^{J+L} t - k)} .$$

Because of the flatness of $\widehat{\psi}_0(\omega)$ close to the origin, for large enough L,

$$\widehat{\psi}_0\left(\frac{\omega}{M^{L+J}}\right) \approx 1 \text{ on } \Omega.$$

Therefore, from Eqn. 4.78, we have

$$\widehat{\psi}_0\left(\frac{\omega}{M^J}\right) \approx \left|\frac{1}{\sqrt{M}}H_0\left(\frac{\omega}{M^{J+1}}\right)\right|^2 \cdots \left|\frac{1}{\sqrt{M}}H_0\left(\frac{\omega}{M^{J+L}}\right)\right|^2 \text{ for } \omega \in \Omega$$

and Eqn. 5.45 can be approximated by

$$\max_{\Theta} \frac{1}{2\pi}\int_{\Omega} d\omega \left|\hat{f}(\omega)\right|^2 \left|\frac{1}{\sqrt{M}}H_0\left(\frac{\omega}{M^{J+1}}\right)\right|^2 \cdots \left|\frac{1}{\sqrt{M}}H_0\left(\frac{\omega}{M^{J+L}}\right)\right|^2$$

or equivalently,

$$\max_{\Theta} \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \left|\hat{f}\left(\frac{\omega}{M^{J+L}}\right)\right|^2 \left|H_0(M^{L-1}\omega)\right|^2 \cdots \left|H_0(\omega)\right|^2.$$

If we define

$$a(n) = f(M^{-(J+L)}n) * f(-M^{-(J+L)}n),$$

and let $r(n)$ be the sequence whose (discrete time) Fourier transform is

$$\left|H_0(M^{L-1}\omega)\right|^2 \cdots \left|H_0(\omega)\right|^2,$$

then the optimal design problem reduces to

$$\max_{\Theta} \sum_n a(n)r(n).$$

Since $a(n)$ and $r(n)$ (being related to autocorrelation of sequences) are symmetric this is equivalent to

$$\max_{\Theta}\left\{\frac{1}{2}a(0)r(0) + \sum_{n=1}^{\infty} a(n)r(n)\right\}.$$

Furthermore, since $h_0(n)$ is a finite length sequence $r(n)$ is finite, and therefore the above sum is also finite. Thus, the objective function is a simple (discrete) inner product between the autocorrelation of the samples of the function and the autocorrelation of the samples of the scaling function.

We now give two examples of the design of the optimal scaling function for a few bandlimited signals. The first example designs the optimal multiplicity $M = 2$, length $N = 8$ (i.e., $K = 4$ regular), scaling function that represents a smooth sinusoidal signal. The second example illustrates the optimal design of the scaling function, ψ_0, for $M = 3, N = 8$ for two segments of speech (voiced and unvoiced).

Example 19 The desired signal given by

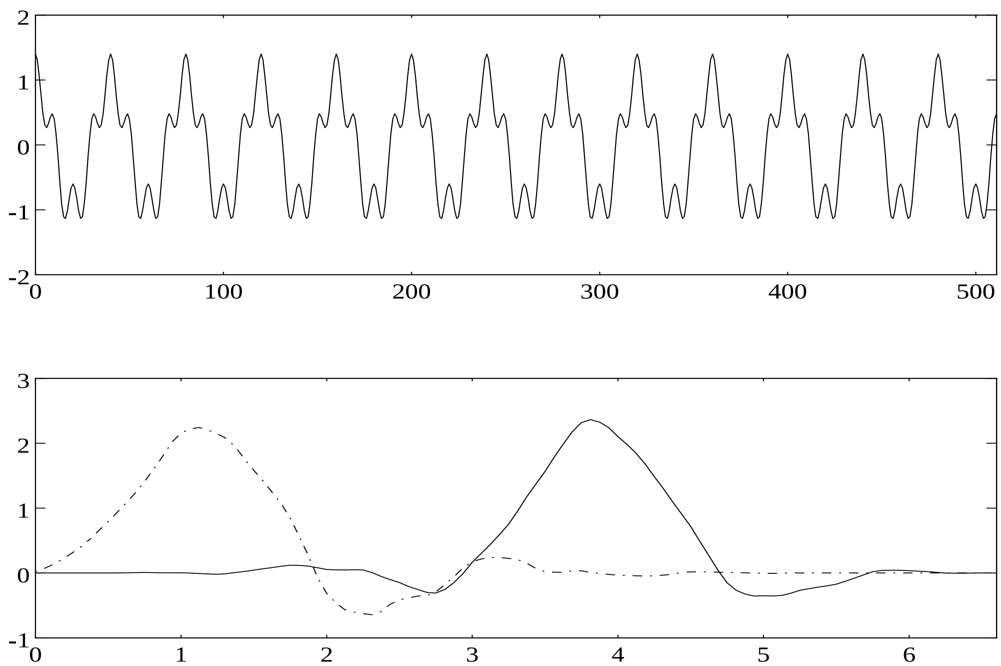

Figure 5.12: Optimal Design: (a) $f(t)$ and (b) $\psi_0(t)$ with $M = 2$, $N = 8$ (where the solid line shows the optimal design and the dashed line shows the 4-regular, multiplicity 2 scaling function).

$$f(t) = \cos\left(\frac{\pi t}{20}\right) + \frac{2}{5}\sin\left(\frac{\pi t}{5}\right),$$

and the corresponding optimal scaling function $\psi_0(t)$ are shown in Fig. 5.12. For comparison the 4-regular, multiplicity 2 scaling function is also shown. The Fourier magnitude of both functions in Fig. 5.12b is approximately identical. In fact, the optimal solution found is only a different spectral factor compared to the 4-regular multiplicity 2 scaling function. With different values of initial guesses in the numerical optimization scheme and different choices of $J(\geq 0)$, a number of optimal solutions were obtained (among which one is identical to the 4 regular multiplicity 2 scaling function shown in Fig. 5.12b). The scaling vector corresponding to the particular optimal design shown in Fig. 5.12 is given in Table 5.4.

Table 5.4: Optimal Scaling Vector for $M = 2$ and $N = 8$

h_0
0.03834055240569
-0.01386799694174
-0.09651950788871
0.30597938054698
0.79675871228695
0.50200805529179
-0.03147297561738
-0.08701265771049

Example 20 Fig 5.13 shows a speech signal (16000 samples at $8kHz$), and two segments of data taken from the speech signal (512 samples each representing voiced and unvoiced speech respectively). Using multiplicity $M = 3$ we have designed an optimal scaling function for representing each of the speech segments (voiced/unvoiced). The optimal scaling functions are shown in Fig. 5.14 for different fixed J. Once again the optimal scaling function is similar to a 3 regular multiplicity 3 wavelet as can be seen from Fig. 5.15.

5.5 Correlation Structure of ψ_i in ON Wavelet Bases

This section investigates the correlation structure of $\psi_i(t)$ in *compactly supported orthonormal wavelet bases.* This structure plays an important role in wavelet-based interpolation [38, 68].

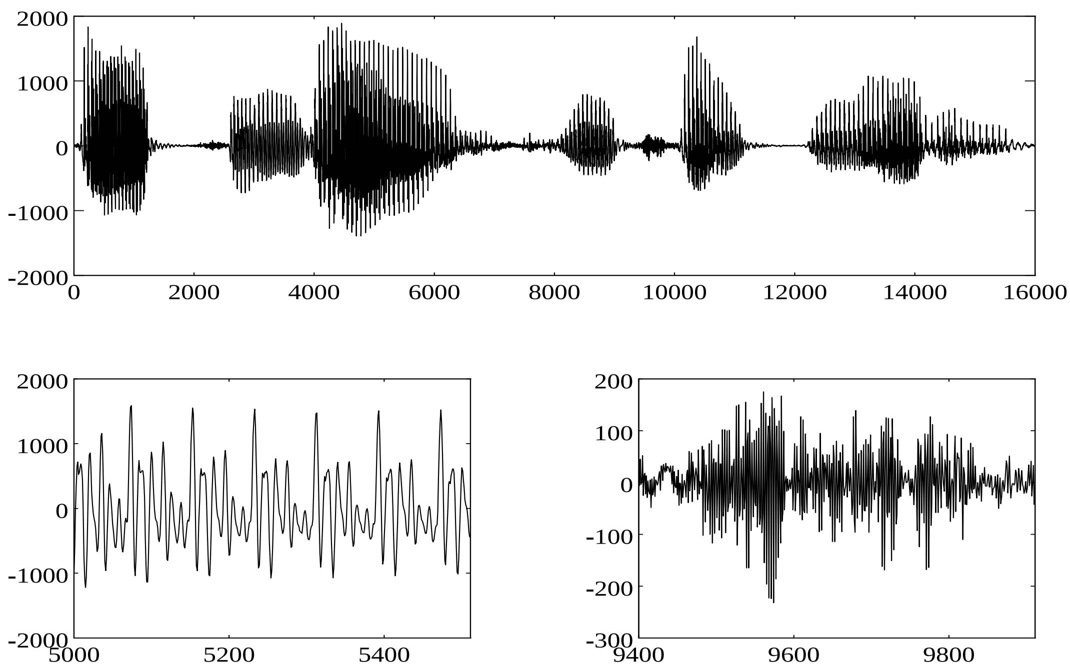

Figure 5.13: Speech signal (a) "Cats and dogs each hate the other" sampled at $8kHz$. (b) 512 samples of voiced speech and (c) 512 samples of unvoiced speech.

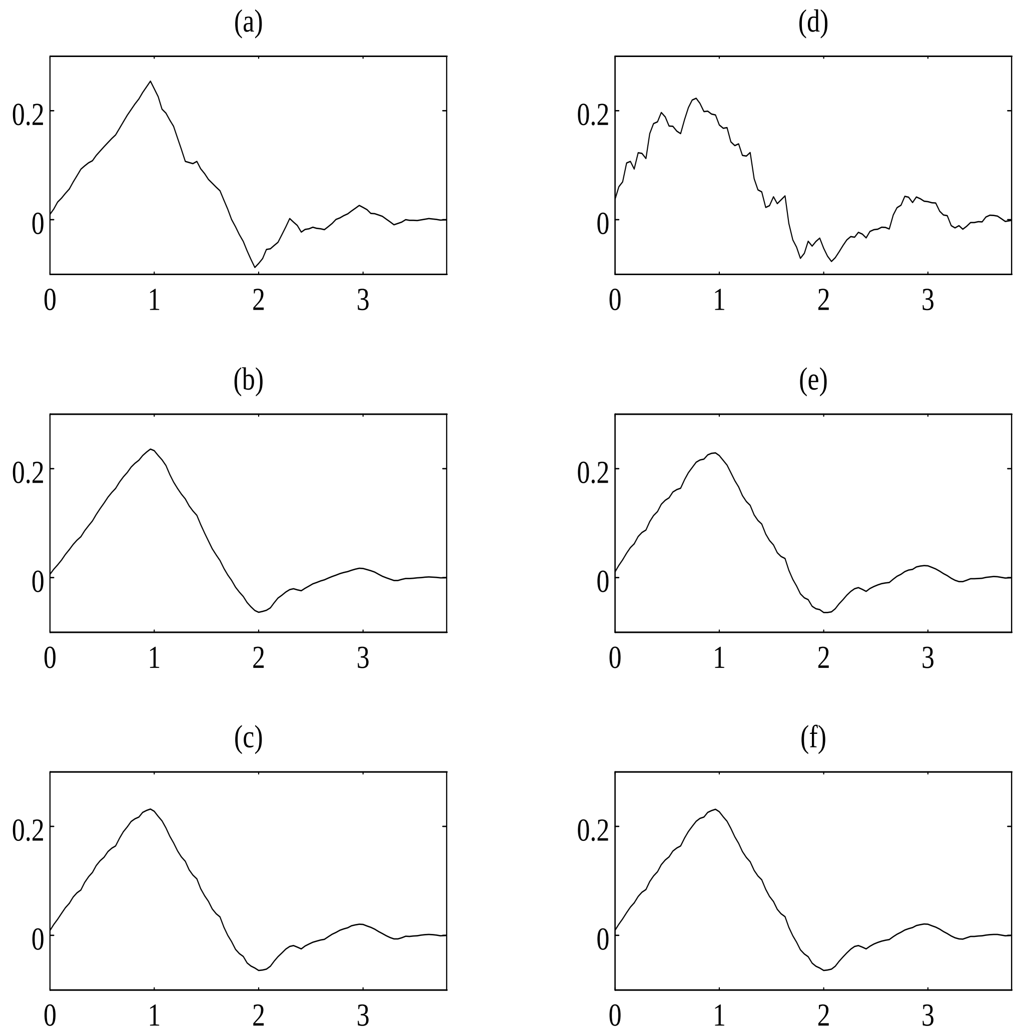

Figure 5.14: (a), (b) and (c) Optimal scaling functions for voiced speech and (d), (e) and (f) optimal scaling function for unvoiced speech. (a) and (d) corresponds to analysis at scale $J = 0$, (b) and (e) at scale $J = 1$ and, (c) and (f) at scale $J = 2$.

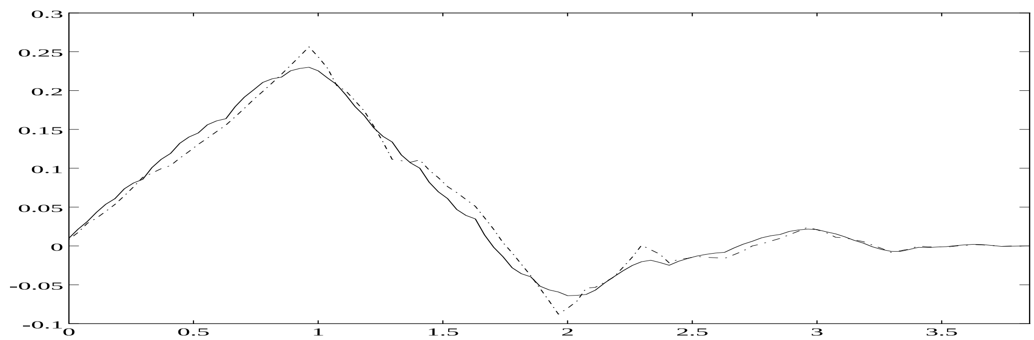

Figure 5.15: A comparison between the optimal design (solid line) at scale $J = 3$ and the 3-regular multiplicity 3 scaling function (dashed line).

The auto-correlation and cross-correlation functions are defined by

$$\Psi_{i,j}(t) = \int_{\mathrm{R}} d\tau \, \psi_i(\tau)\psi_j(t+\tau). \tag{5.46}$$

For compactly supported WTFs (in general) there is no explicit (analytical) formula for the values of the scaling function. One usually obtains (samples of) the scaling function from the scaling vector by one of the methods in Section 5.2. Despite this fact it turns out that the correlation functions can be computed exactly. Assuming that samples of $\psi_i(t)$ at the M-adic rationals are given, corresponding samples of $\Psi_{i,j}(t)$ can be obtained by approximating the integral in Eqn. 5.46. This approximation process working on the *approximate* samples of $\psi_i(t)$ obtained by the *infinite product method* gives the *exact* samples of the correlation functions! The two approximation processes - that of obtaining the samples of ψ_i and that of computing the step approximation cancel each other to give the samples of the correlation functions exactly.

In the infinite product method (from Eqn. 5.11)

$$\mathcal{Z}(\psi_i(M^{-J}n))(z) \approx M^{J/2} H_i\left(z^{M^{J-1}}\right) \prod_{i=0}^{J-2} H_0\left(z^{M^i}\right).$$

If we define $\mathcal{H}_{i,j}(z) = H_i(z)H_j(z^{-1})$, then the $\mathcal{Z}$-transforms of the approximate samples of the correlation function are given by

$$\mathcal{Z}(\tilde{\Psi}_{i,j}(M^{-J}n))(z) \approx \mathcal{H}_{i,j}\left(z^{M^{J-1}}\right)\prod_{i=0}^{J-2}\mathcal{H}_0\left(z^{M^i}\right).$$

For ON wavelet bases we show that the correlations computed above are in fact *exact.* It is interesting to note that in the infinite product method the the approximation of the samples is good only for large enough J, while for all J the samples of the correlation functions are exact. Recall that for an ON wavelet basis (from Eqn. 4.35)

$$\sum_k \left|\widehat{\psi}_i(\omega + 2\pi k)\right|^2 = 1. \tag{5.47}$$

Now consider the Fourier transform of $\Psi_{i,j}$:

$$\widehat{\Psi}_{i,j}(\omega) = \left|\widehat{\psi}_{i,j}(\omega)^2\right| = \left[\frac{1}{M}\mathcal{H}_{i,j}\left(\frac{\omega}{M}\right)\right]\left\{\prod_{i=2}^{J}\left[\frac{1}{M}\mathcal{H}_{0,0}\left(\frac{\omega}{M^i}\right)\right]\right\}\left|\widehat{\psi}_0\left(\frac{\omega}{M^J}\right)\right|^2.$$

By sampling $\Psi_{i,j}(t)$ at the points $M^{-J}n$ this Fourier transform gets periodized ([95])

$$\frac{1}{M^J}\sum_n \Psi_{i,j}(M^{-J}n)e^{-\frac{\imath\omega n}{M^J}} = \frac{1}{M}\mathcal{H}_{i,j}\left(\frac{\omega}{M}\right)\left\{\prod_{i=2}^{J}\frac{1}{M}\mathcal{H}_{0,0}\left(\frac{\omega}{M^i}\right)\right\}\sum_k\left|\widehat{\psi}_0\left(\frac{\omega}{M^J}+2\pi k\right)\right|^2$$

Eqn. 5.47 now implies that

$$\sum_n \Psi_{i,j}(M^{-J}n)e^{-\frac{\imath\omega n}{M^J}} = \mathcal{H}_{i,j}\left(\frac{\omega}{M}\right)\prod_{i=2}^{J}\mathcal{H}_{0,0}\left(\frac{\omega}{M^i}\right),$$

and therefore

$$\mathcal{Z}(\Psi_{i,j}(M^{-J}n)) = \mathcal{H}_{i,j}\left(z^{M^{J-1}}\right)\prod_{i=0}^{J-2}\mathcal{H}_{0,0}\left(z^{M^i}\right).$$

5.6 Wavelet-Galerkin Approximation of Analog Filters

Many applications in signal processing and in numerical analysis involve estimating Af or $A^{-1}f$ where $f : \mathbb{R} \mapsto \mathbb{R}$ and $A : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is a linear-translation-invariant (LTI) operator. Computational considerations require accurate approximation of f and A by *discretized* versions f_d and A_d. Analytical properties of K-regular, multiplicity M wavelets provide accurate discrete approximations of functions by *truncated* wavelet expansions of the form

$$f(t) \approx \sum_k f_{\Delta t,k} \sqrt{\frac{1}{\Delta t}} \psi_0 \left(\frac{t}{\Delta t} - k \right).$$

The corresponding Galerkin projection of the operator $A_{\Delta t} = P_{\Delta t} A P_{\Delta t}$, where $P_{\Delta t}$ denotes orthogonal projection onto the space $V_{\Delta t} = Span \left\{ \sqrt{\frac{1}{\Delta t}} \psi_0 (\frac{t}{\Delta t} - k) \;\mid\; k \in \mathbf{Z} \right\}$, provides a discrete approximation to A and admits a representation by convolution of the sequence of expansion coefficients $f_{\Delta x,k}$ with a suitable kernel sequence $t_{\Delta x,k}$. This section derives three results for the approximation of LTI operators in wavelet bases. The Wavelet-Galerkin discretization of LTI operators have good numerical properties, arising from the vanishing moments property of wavelets. If the WTF is K-regular, then the first K moments of the wavelets vanish.

When an operator is approximated by Galerkin projections, the action of the operator on an appropriate subspace of $L^2(\mathbb{R})$ is studied. The closer the subspace is to $L^2(\mathbb{R})$, the better the approximation. Hence, if there is a chain of successive approximation spaces to $L^2(\mathbb{R})$, then the corresponding Galerkin projections give successively better discretizations of the operator. Wavelet multiresolution analysis gives one such chain, $\{W_{0,s} \mid t \in \mathbb{R}\}$, where $W_{0,s} = Span\,\{\psi_0(M^s t - k)\}$, where $\psi_0(t)$ is the scaling function associated with a multiplicity M WTF (or orthonormal basis). In this section this multiresolution chain of spaces will be labeled as follows:

$$V_{\Delta t} = W_{0,\log_M(\Delta t)}.$$

As $\Delta t \rightarrow 0$, $V_{\Delta t}$ approaches $L^2(\mathbb{R})$ and as $\Delta t \rightarrow \infty$, $V_{\Delta t}$ approaches $\{0\}$.

5.6.1 Approximation Characterization

If $g = Af$, standard discretization methods for computing g include (when A is a differential operator) the finite-difference (FD) and the method of Galerkin projections. In both methods the function f is represented by an approximation defined by a finite set of parameters. If f_d denotes this vector of parameters and A_d and g_d denote discretizations of A and g respectively, then f_d satisfies the algebraic equation $g_d = A_d f_d$. For FD methods, f_d and g_d are samples of f and g and A_d is a finite-difference operator matrix. Here we discuss the discretization using Galerkin projections onto $V_{\Delta t}$ corresponding to a K-regular, multiplicity M orthonormal wavelet basis (or WTF).

We will assume that the LTI operator A is represented by a convolution kernel $a(t)$. It is well known that in the FD discretization A_d is acts as a discrete convolution on the *samples* of f and can be represented by a sequence $a_{\Delta t,k}$. With the WG discretization, we show that, under mild assumptions, A_d acts as a discrete convolution on the *samples* of f. The approximation properties of the kernel $a_{\Delta t,k}$, corresponding to a grid-spacing of Δt (either in FD or in WG) are related to the smoothness of f and the moments of the kernels $a(t)$ and $a_{\Delta t,k}$ as follows: Let

$$(Af)(t) = \int_{\mathbf{R}} d\lambda\, a(\lambda) f(t-\lambda) = a * f \tag{5.48}$$

$$(A_d f_d)(t) = \sum_n a_{\Delta t,n} f(t - n\Delta x) = a_{\Delta t} * f \tag{5.49}$$

$$m(a(t),k) = \int_{\mathbf{R}} dt\, a(t) t^k \tag{5.50}$$

$$\mu(a_{\Delta t},k) = \sum_n a_{\Delta t,n} n^k \tag{5.51}$$

Expanding f as a Taylor series around t,

$$(Af)(t) = \sum_k (-1)^k \frac{m(a(t),k)}{k!} f^{(k)}(t) \tag{5.52}$$

$$(Af)(t) = \left[\sum_k (-1)^k \frac{m(a(t),k)}{k!} \delta^k(t)\right] * f(t)$$

and similarly

$$(A_d f_d)(t) = \sum_k (-1)^k \frac{\mu(a_{\Delta t},k)(\Delta t)^k}{k!} f^{(k)}(t) \tag{5.53}$$

$$(A_d f_d)(x) = \left[\sum_k (-1)^k \frac{\mu(a_{\Delta x},k)(\Delta x)^k}{k!} \delta^k(x)\right] * f.$$

If $A = \frac{d^p}{dx^p}$, then $a(t) = \delta^{(p)}(t)$, where $\delta^{(p)}$ is the p^{th} derivative of the Dirac delta measure.

$$\int_{\mathbf{R}} d\lambda\, \delta^{(p)}(\lambda) f(t-\lambda) = f^{(p)}(t). \tag{5.54}$$

Therefore

$$m(a(t), k) = \begin{cases} p! & k = p \\ 0 & \text{otherwise} \end{cases}$$

and with $\Delta t = 1$

$$\mu(a_{\Delta t}, k) = \begin{cases} p! & k = p \\ 0 & k > p \end{cases}$$

with

$$a_{\Delta t, n} = a_{1,n}/(\Delta t)^p. \tag{5.55}$$

We now have the following theorem:

Theorem 39 Let $T = \frac{d^p}{dx^p}$. Let $f \in C^p(\mathbb{R})$ and let $\left|f^{(c)}(x)\right|$ be bounded for some $c \geq p$ and let $q > p$ be the smallest positive integer such that

$$m(t_{\Delta x}, q) \neq 0. \tag{5.56}$$

Then, if either c is finite or infinite, and $c \geq q$

$$|A_d f_d(x) - (Af)(x)| \leq |\mu(a_1, q)| \frac{(\Delta x)^{q-p}}{q!} \left|f^{(q)}\right|. \tag{5.57}$$

Proof: Follows from Eqn. 5.55, Eqn. 5.52 and Eqn. 5.49. □

Note that theorem is valid for *any* discretization method, FD, WG or otherwise.

5.6.2 The Virtual Expansion Theorem

We have seen that when Af is discretized suitably, the approximation error is characterized by the moments of $a(t)$ and $a_{1,k}$. Now we consider the WG discretization with a grid-spacing of Δt, or projections onto $V_{\Delta t}$. Then $A_d = P_{\Delta t} A P_{\Delta t}$.

$$P_{\Delta t} f = \sum_k f_{\Delta t, k} \psi_{0, \Delta t, k} \tag{5.58}$$

and hence

$$g_d = A_d f_d = \sum_{l,k} \psi_{0,\Delta t,l} \int_{\mathbf{R}} dt\, \psi_{0,\Delta t,l}(t) f_{\Delta t,k} A \psi_{0,\Delta t,k}(t) \tag{5.59}$$

or in terms of expansion coefficients

$$g_{\Delta t,k} = \langle g_d, \psi_{0,\Delta t,k} \rangle = \sum_l a_{\Delta t,k-l} f_{\Delta t,l} = f_{\Delta t,k} * a_{\Delta t}, \tag{5.60}$$

where

$$a_{\Delta t,k} = \langle \psi_{0,\Delta t,0}, A\psi_{0,\Delta t,k} \rangle \tag{5.61}$$

Thus A_d acts as a discrete convolution just as in the FD case, only that it acts on the expansion coefficients of f, rather than the samples of f. In order to apply the approximation result of the previous section we show that under realistic assumptions the discrete sequence $a_{\Delta x,k}$ can be thought to act on the samples of f. In the WG method, we have to compute the projection coefficients $f_{\Delta t,k}$ for some fixed grid-spacing Δt from f. Typically f is specified by its samples at the grid-spacing of Δt. Hence f has to be interpolated or approximated from the samples before the inner products $f_{\Delta t,k}$ are computed. If, for fixed k, a Taylor series approximation for f is used around the point $k\Delta t$, then it is seen $f_{\Delta t,k}$ are related to $f(k\Delta t)$ by a convolution [58].

Theorem 40 Let f be approximated locally (around $k\Delta t$) by $p_{n,k}(t)$, the polynomial obtained by truncating the Taylor series expansion of f around $k\Delta t$ to n terms. Let

$$f_{\Delta t,k} = \langle p_{n,k}(t), \psi_{0,\Delta t,k}(t) \rangle . \tag{5.62}$$

Then there exists a sequence $e(k), k = 0, 1, \ldots, n-1$ such that

$$f_{\Delta t,k} = \sum_l f(l\Delta t) e(k-l) = f(l\Delta t) * e(l). \tag{5.63}$$

Proof: The polynomial $p_{n,k}(t)$ is the Lagrange interpolant through the N points $f(l\Delta t)$ for $l \in \{k, k+1, \ldots, k+N-1\}$. □

If the Fourier Transform of e does not vanish, then by deconvolution of $f_{\Delta t,k}$, $f(k\Delta t)$ can be obtained. From Eqn. 5.62 and Eqn. 5.60 we have

$$g_{\Delta t,k} = f(l\Delta t) * e(l) * a_{\Delta t,l} = g(l\Delta t) * e(l)$$

and hence

$$g(l\Delta t) = f(l\Delta t) * a_{\Delta t,l}$$

Therefore, we have proved the following Virtual Expansion Theorem.

Theorem 41 Let $a_{\Delta t,k}$ be the convolution operator corresponding to WG discretization. Then, under the assumption of local polynomial approximation, in terms of the samples of f, A acts as a convolution with the same convolution kernel $a_{\Delta t,k}$ that acts on the expansion coefficients.

Notice that this result makes sense even if e is not invertible.

5.6.3 Wavelet Approximation

If the WTF is K-regular, then the fact that the first K moments of the wavelets vanish can be used to show that differential operators are well-approximated in orthonormal wavelet bases [39]. Let the first K moments of wavelets vanish. Consider the WG discretization of the operator $\frac{d^p}{dx^p}$. Then,

$$Af = f^{(p)} = f * \delta^{(p)} \tag{5.64}$$

$$a_{1,k} = \left\langle \psi_0(t), \psi_0^{(p)}(t-k) \right\rangle = \frac{1}{2\pi} \int_{\mathbf{R}} \left| \widehat{\psi}_0(\omega) \right|^2 (i\omega)^p \tag{5.65}$$

or equivalently,

$$a_{1,k} = \frac{1}{2\pi} \int_0^{2\pi} d\omega \, i^p \sum_k (\omega + 2\pi k)^p \left| \widehat{\psi}_0(\omega + 2\pi k) \right|^2 e^{i\omega k} \tag{5.66}$$

Hence the Fourier Transform of $t_{1,k}$, namely $T(\omega)$ is given by

$$A(\omega) = i^p \sum_k (\omega + 2\pi k)^p \left| \widehat{\psi}_0(\omega + 2\pi k) \right|^2 \tag{5.67}$$

When ψ_0 is K-regular, for small ω

$$A(\omega) = i^p \omega^p + O(\omega^{2K-p}) \tag{5.68}$$

and hence

$$\mu(t_{1,k}, n) = (i\frac{d}{d\omega})^n A(\omega)|_{\omega=0} = p!\delta(p-n), n < 2K - p. \tag{5.69}$$

Therefore we have proved the following Theorem.

> **Theorem 42** Let $\psi_0(t)$ generate a multiplicity M, K-regular WTF. If $A = \frac{d^p}{dt^p}$ where $p < 2K - p$, then, for the WG discretization, for even p,
>
> $$\mu(t_1, q) = p!\delta(p-q) \text{ for } q < 2K - p, \tag{5.70}$$
>
> and for odd p,
>
> $$\mu(t_1, q) = p!\delta(p-q) \text{ for } q < 2K + 1 - p. \tag{5.71}$$

When p is odd, the sequence $t_{1,k}$ is odd symmetric and when p is even $t_{1,k}$ is even symmetric. Hence in the odd case $\mu(t_{1,k}, 2K-p)$ is also zero.

Table 5.5 shows the error bound coefficients in the WG approximation of differential operators in the multiplicity $M = 2$ case for different values of K and different orders of derivatives p. For comparison the error bound coefficient for finite-difference approximation is also given.

Table 5.5: Wavelet-Galerkin Method: Error Bound Coefficients for $(\frac{d}{dt})^p$

p	FD	$K = 3$	$K = 4$	$K = 5$
1	$.1667\ \Delta^2$	$.0078\ \Delta^6$	$.0017\ \Delta^8$	$.0005\Delta^{10}$
2	$.0833\ \Delta^2$	$.1460\ \Delta^4$	$.2042\ \Delta^6$	$.0050\ \Delta^8$
3	$.2900\ \Delta^2$		$.0210\ \Delta^6$	$.0059\ \Delta^8$
4	$.1667\ \Delta^2$			$.7735\ \Delta^6$

5.7 Wavelet-Based Lowpass/Bandpass Interpolation

Lowpass and bandpass interpolation schemes based on orthonormal wavelet bases is presented with a precise description of the interpolation classes (i.e., signals for which the interpolation is exact). If the wavelet basis is K-regular then the interpolation class includes polynomials of degree $2K-1$. The interpolation classes are best described in the wavelet transform domain. In the time domain the interpolation schemes may be interpreted as the Wavelet-Galerkin approximation of the shift operator on $L^2(\mathbb{R})$ restricted to W_i. This interpretation gives an efficient recursive M-adic interpolation scheme. The Fourier Transform of the lowpass interpolating function is also (a positive) interpolating function. The nature of the corresponding interpolating class is not well understood.

5.7.1 Wavelet-Based Interpolation

Given a sequence $x(n)$, we want to construct a continuous time signal $x(t)$ such that its integral samples agree with $x(n)$. In general - since sampling is a irreversible process - there are infinitely many interpolants $x(t)$. However, by restricting the interpolating class, by invoking additional assumptions like bandlimitedness etc., one can even ensure the uniqueness of $x(t)$. Let $\mathcal{F}$ be an interpolating class with such a uniqueness property. Then every $x(t) \in \mathcal{F}$ is of the form

$$x(t) = \sum_n x(n) g_n(t), \tag{5.72}$$

where $\{g_n\}$, the *interpolating set*, satisfies $g_n(k) = \delta(k-n)$. The choice of the family $\{g_n(t)\}$ completely determines $\mathcal{F}$ and hence the interpolation technique and its analytic properties. A desirable property (which we call translation invariance) of the interpolation technique would be that the sequence $x(n-k)$, for fixed k, interpolates to $x(t-k)$. This constrains the interpolating set to arise from a single function $g(t)$. That is

$$g_n(t) = g(t-n) \quad \text{and} \quad g(n) = \delta(n). \tag{5.73}$$

Given $g(t)$ satisfying Eqn. 5.73, one has translation-invariant interpolation technique.

A general way to construct such a function $g(t)$ is to start with an orthonormal family $\{p(t-n)\}$, determined by integral translates of a function $p(t)$. Let $r(t)$ be the autocorrelation of $p(t)$. Then,

$$\int_{\mathbf{R}} dt\, p(t)p(t+n) = \delta(n) = r(n) \tag{5.74}$$

making $r(t)$ a candidate function for interpolation. Such a function $p(t)$ can be constructed relatively easily in the Fourier domain where the orthonormality of $\{p(t-n)\}$ is equivalent to a *positive partition of unity.* That is,

$$\sum_k |\widehat{p}(\omega + 2\pi k)|^2 = \sum_k \widehat{r}(\omega + 2\pi k) = 1, \tag{5.75}$$

where $\widehat{p}(\omega)$ is the Fourier Transform of $p(t)$. An interesting fact to be noted is that if for some ω_0, $\widehat{r}(\omega_0) = 1$, then $\widehat{r}$ in itself forms a *positive* interpolating function with $g(t) = \widehat{r}(2\pi(t - \omega_0))$.

Clearly the autocorrelations of the scaling function and wavelets of a K-regular multiplicity M orthonormal wavelet basis satisfy Eqn. 5.74, and therefore give rise to interpolation schemes. In fact for $i \in \mathcal{R}(M)$, the M interpolation schemes are given by

$$x_{\Psi_i}(t) = \sum_n x(n)\Psi_i(t-n). \tag{5.76}$$

Let $\mathcal{F}_{\Psi_i}$ denote the i^{th} interpolating class.

5.7.2 The Wavelet-Galerkin Interpretation

Since the scaling function $\psi_0(t)$ is a lowpass function, $\Psi_0(t)$ gives rise to a lowpass interpolation scheme. Since the wavelets $\{\psi_i(t)\}$ are bandpass functions, each of $\Psi_i(t)$ gives rise to a bandpass interpolation scheme. To get a better understanding of the interpolation scheme, we study it in the context of the Wavelet-Galerkin approximation restricted to the space $W_{i,0}$ of the shift operator T_τ acting on $L^2(\mathbb{R})$. By means of the Virtual Expansion Theorem (Theorem 5.63) it follows that

$$(T_\tau x)(n) = x(n+\tau) = \sum_k T_\tau(k) x(n-k), \tag{5.77}$$

where

$$T_\tau(k) = \int_{\mathbf{R}} \psi_i(t)\psi_i(t+k+\tau)\, dt = \Psi_i(k+\tau). \tag{5.78}$$

Notice that Eqn. 5.77 and Eqn. 5.76 are the same with the identification $t = n + \tau$. Thus the interpolation scheme with $\Psi_i(t)$ is precisely the Wavelet-Galerkin approximation of the shift operator ([39]) on $L^2(\mathbb{R})$ restricted to $Span\,\{\psi_i(t-n)\}$.

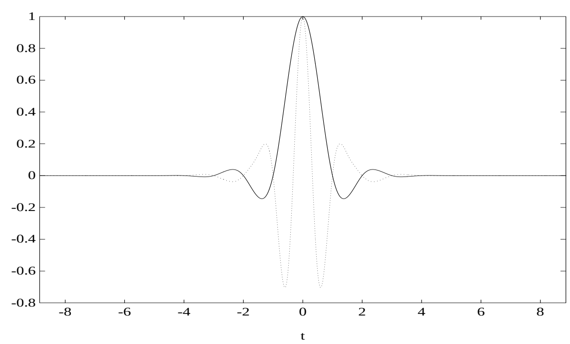

Figure 5.16: $\Psi_0(t)$ and $\Psi_1(t)$: $M = 2$, $N = 10$ Daubechies wavelet

5.7.3 Efficient M-adic Interpolation

The Wavelet-Galerkin interpretation, besides giving insight into the interpolation scheme, also gives an efficient numerical algorithm. First, consider the direct method to obtain $x(t)$ from $x(n)$ where one has to know $\Psi_i(t)$ for all $t \in \mathbb{R}$. One can recursively compute $\Psi_i(t)$ recursively at the M-adic rationals. In fact

$$\sum_n \Psi_i(M^{-J}n) z^{-n} = \left[\prod_{i=0}^{J-2} \mathcal{H}_0(z^{M^i})\right] \mathcal{H}_i(z^{M^{J-1}}) \tag{5.79}$$

and therefore

$$\sum_n \Psi_i(M^{-J}n) z^{-n} = \mathcal{H}_0(z) \left[\sum_n \Psi_i(M^{-(J-1)}n) z^{-Mn}\right],$$

where $\mathcal{H}_0(z) = H_0(z)H_0(z^{-1})$. The above recursion computes $\Psi_i(t)$ at the M-adic rationals immediately gives an M-adic recursive scheme to compute $x_{\Psi_i}(t)$ at the M-adic rationals. From Eqn. 5.76,

$$x_{\Psi_i}\left(\frac{k}{M^J}\right) = \sum_n x(n)\Psi_i\left(\frac{k - M^J n}{M^J}\right)$$

and therefore if $X(z)$ is the $\mathcal{Z}$-transform of $x(n)$,

$$\begin{aligned}\sum_n x_{\Psi_i}(M^{-J}n)z^{-n} &= X(z^{M^J})\left[\sum_n \Psi_i(M^{-J}n)z^{-n}\right] \\ &= X(z^{M^J})\left[\prod_{i=0}^{J-2}\mathcal{H}_i(z^{2^i})\right]\mathcal{H}_0(z^{M^{J-1}}). \qquad (5.80)\end{aligned}$$

If $X_{\Psi_i,J}(z)$ is the $\mathcal{Z}$-transform of $x_{\Psi_i}(\frac{n}{M^J})$ as a sequence in n, then,

$$X_{\Psi_i,J}(z) = X_{\Psi_i,J-1}(z), \qquad (5.81)$$

with the initialization

$$X_{\Psi_i,1}(z) = X(z^M)\mathcal{H}_i(z). \qquad (5.82)$$

Therefore $\mathcal{H}_i(z)$ is an M-adic interpolation filter. Fig. 5.7.2 shows $\Psi_0(t)$ and $\Psi_1(t)$ for the case $M = 2$ and $K = 5$, and $N = 10$. Notice that $\Psi_0(t)$ and $\Psi_1(t)$ resemble the *sinc(t)* function and $sinc(2t) - sinc(t)$ respectively, which are the ideal lowpass and bandpass interpolants.

5.7.4 Interpolating Classes $\mathcal{F}_{\Psi_i}$

The WG framework gives the theoretical basis that makes the interpolation schemes *lowpass* and *bandpass* respectively as the case may be. Moreover, if a K-regular orthonormal wavelet basis is used, then the lowpass interpolation is exact for polynomials of degree $2K - 1$. We now characterize signal classes $\mathcal{F}_i$ for which the interpolation is exact. Clearly from the above observation, the class $\mathcal{F}_0$ includes polynomials of arbitrary large degree by suitable choice of the scaling vector. The interpolating

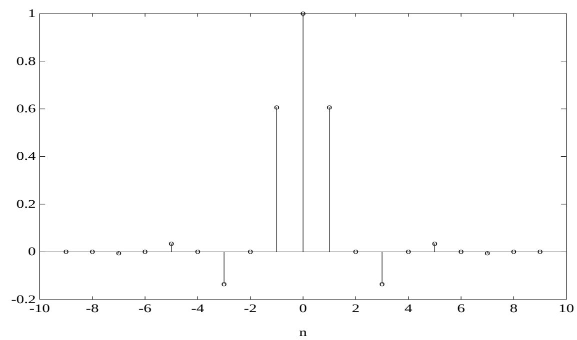

Figure 5.17: Lowpass Interpolation Filter : $\mathcal{H}_0(z)$, $M = 2$, $N = 10$

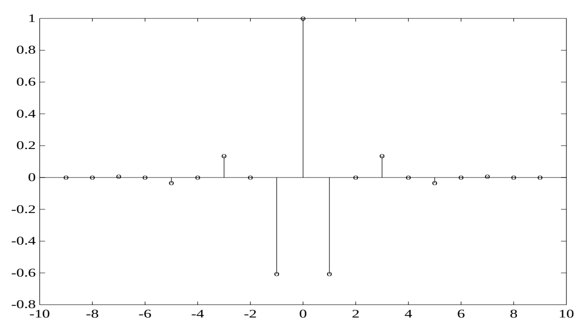

Figure 5.18: Bandpass Interpolation Filter : $\mathcal{H}_1(z)$, $M = 2$, $N = 10$

classes are precisely described in the wavelet transform domain. Recall that a function $f(t) \in W_{i,0}$ iff, there is a sequence $x(n)$ such that

$$f(t) = \sum_n x(n)\psi_i(t-n). \tag{5.83}$$

Now we define the Discrete-Scale Wavelet Transform (DSWT) similar to the DWT, only that the time parameter is allowed to be continuous. That is,

$$Wf(i,j,\tau) = \int_{\mathbf{R}} dt\, f(t) M^{j/2}\psi_i(M^j t - \tau) \tag{5.84}$$

Then, the DSWT of a function $f(t) \in W_i$ at scale $j=0$ is given by

$$\begin{aligned} Wf(i,0,\tau) &= \sum_n x(n) \int_{\mathbf{R}} dt\, \psi_i(t-n)(\psi_i(t-\tau) \\ &= \sum_n x(n)\Psi_i(\tau - n). \end{aligned} \tag{5.85}$$

In other words, the the class $\mathcal{F}_i$ is precisely described by

$$\mathcal{F}_{\Psi_i} = \{x(t) | x(t) = Wf(i,0,\tau) \text{ for some } f \in W_{i,0}\}. \tag{5.86}$$

Notice also that the Fourier Transform of the *scaling function* $\widehat{\Psi}_0(\omega)$ satisfies $\widehat{\Psi}_0(0) = 1$, and therefore from Eqn. 5.75, $g(t) = \widehat{\Psi_0}(2\pi t)$ is also an interpolating function. Fig. 5.19 shows the M-adic, K-regular, lowpass interpolation filter for $M=4$, and $K=4$. The length of the scaling vector in this case is $N=16$. h_0. The corresponding function $\Psi_0(t)$ is shown in Fig. 5.20

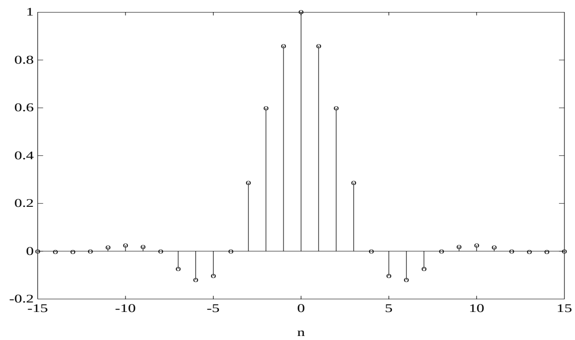

Figure 5.19: M-adic Lowpass Interpolation Filter : $N = 4$, $M = 4$

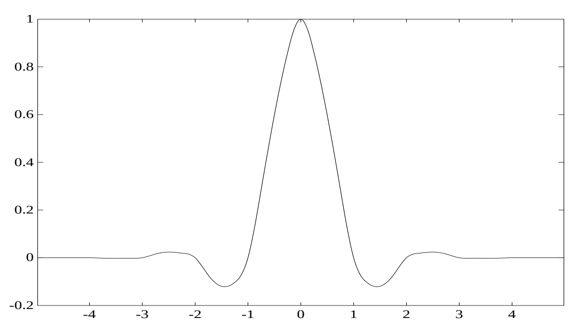

Figure 5.20: $\Psi_{0,0}(t)$: $N = 16$, $M = 4$

Chapter 6

Conclusion

Multidimensional multirate systems can be analyzed using the Aryabhatta/Bezout identity over integer matrices as a fundamental tool. This identity gives rise to the Representatives' Mapping Theorem - a new result in the theory of integer matrices - which allows one to swap upsamplers and downsamplers in a multirate system. In conjunction with the generalized polyphase representation, this result is used to reduce the multidimensional rational sampling rate filter bank problem into a multidimensional uniform sampling rate filter bank problem (which does not yet have a clean solution [54, 93, 11]).

Perfect reconstruction (PR) filter banks and transmultiplexers are two important multirate systems. PR filter bank theory (in one dimension) has a rich algebraic structure that allows one to design design various classes of filter banks with desired properties - unitary filter banks, modulated filter banks, unitary FIR filter banks with symmetry, etc. A theory of modulated filter banks (MFBs) with arbitrary length filters is described. Every M-channel PR MFB decomposes into a set of approximately $M/2$ two channel PR filter banks. If the MFB is unitary, so are the constituent two channel filter banks. This result is used to parameterize all FIR unitary MFBs. Efficient algorithms for design and implementation of MFBs are described. PR filter banks also provide a natural change of bases in separable Hilbert spaces. Under mild conditions PR filter banks give rise to wavelet frames for $L^2(\mathbb{R})$. If the PR filter bank is unitary then the the wavelet frame is also a tight frame. By appropriate choice of the unitary filter bank one can also ensure that the wavelet basis is orthonormal. Regular multiplicity M orthonormal wavelet bases can also be constructed by using state-space techniques. Because of this relationship between filter banks and wavelets, the rich algebraic structure of filter banks can be used to construct corresponding classes of wavelet bases - like modulated wavelet bases.

Efficient algorithms exist for computations in filter banks and WTFs. In wavelet analysis, the samples of a signal themselves give a third order approximation to the scaling function expansion coefficients if the wavelet basis is K regular with $K \geq 2$. Wavelets can be used for lowpass/bandpass interpolation and for the approximation of linear-translation invariant operators using Galerkin projections. The optimal wavelet for representing a given signal or classes of signals can be designed easily if the signal classes are assumed to be bandlimited. When most orthonormal wavelet bases are used to analyze bandlimited signals, the scaling function expansion coefficients at all scales above a certain scale can be considered to be generalized samples of the signal (wavelet sampling theorem). The signal in this case can be recovered from the scaling expansion coefficients.

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