A SIGNAL TRANSFORM COVARIANT TO SCALE CHANGES

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Abstract

A unitary signal transformation that is covariant by translation to scale changes (dilations and compressions) in the signal is formulated and justified. Unlike the Mellin transform, which is invariant to scale changes, this new transform is a true indicator of the scale content of a signal.

1 INTRODUCTION

The concept of scale has been a subject of considerable interest recently, due to the introduction of the wavelet transform and classes of bilinear time-scale distributions [1], [2], [3]. Since these tools analyze signals only in terms of their joint time-scale content, it is relevant to consider also transforms indicating solely the scale content of signals. Traditional concepts such as the time content and frequency content of a signal $x(t)$ can be expressed in terms of $x(t)$ and its Fourier transform $(\mathbf{F}x)(f)$.[1] However, the situation with scale is more confused, with a variety of transforms proposed to indicate scale content.

Most approaches to date have been based on or have resulted in the Mellin transform [2]

$$(\mathcal{M}x)(c) \;=\; \int_0^\infty x(u)\, e^{-j2\pi c \log u}\, \frac{du}{\sqrt{u}}. \qquad (1)$$

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[1]The bracket notation $(\mathcal{V}g)(v)$ represents the result of operating on the function (signal) g with the operator $\mathcal{V}$ and then evaluating at the point v.

It is natural to insist that a transform indicating the scale content of a signal must change as the scale of the signal is changed. This requirement immediately eliminates the Mellin transform from contention as a scale indicating transform, since it is *invariant* to scale changes. That is, defining the scale change or dilation operator as $(\mathcal{D}_d x)(t) = x(e^{-d}t)\, e^{-d/2}$, we have

$$|(\mathcal{M}\mathcal{D}_d\, x)(c)| \;=\; |(\mathcal{M}x)(c)|. \tag{2}$$

The aim of this paper is to derive and justify the signal transform

$$(\mathcal{S}x)(\sigma) \;=\; e^{\sigma/2}\, x(e^{\sigma}t_0) \tag{3}$$

that is covariant by translation to scale changes

$$(\mathcal{S}\mathcal{D}_d\, x)(\sigma) \;=\; (\mathcal{S}x)(\sigma - d). \tag{4}$$

The derivation of this scale transform will utilize some simple concepts from unitary operator theory and eigenanalysis.

2 MATHEMATICAL BACKGROUND

Define the time-shift and frequency-shift operators as $(\mathcal{T}_t\, g)(u) = g(u-t)$ and $(\mathcal{F}_f\, g)(u) = e^{j2\pi f u}\, g(u)$, respectively. The time, frequency, and dilation operators are unitary. Since a unitary operator $\mathcal{V}$ maps the signal space of square-integrable functions $L^2(\mathbb{R})$ back onto itself in a way that preserves exactly its structure, it can be interpreted as simply a "relabeling operator" that takes every function $x \in L^2(\mathbb{R})$ and gives it a new name $\mathcal{V}x$. This relabeling is equivalent to changing the frame of reference or to changing bases. Two operators, $\mathcal{A}$ and $\mathcal{B}$, are defined as *unitarily equivalent* if they are equivalent modulo a change of basis, that is, if $\mathcal{B} = \mathcal{U}^{-1}\mathcal{A}\mathcal{U}$ for some unitary transformation $\mathcal{U}$ [4], [5].

Given a linear operator $\mathcal{Q}$ on $L^2(\mathbb{R})$, solution of the eigenequation

$$\left(\mathcal{Q}\, \mathbf{e}_q^{\mathcal{Q}}\right)(\tau) \;=\; \lambda_q^{\mathcal{Q}}\, \mathbf{e}_q^{\mathcal{Q}}(\tau) \tag{5}$$

yields the eigenfunctions $\left\{\mathbf{e}_q^{\mathcal{Q}}(\tau)\right\}$ and the eigenvalues $\left\{\lambda_q^{\mathcal{Q}}\right\}$ of $\mathcal{Q}$, both of which are indexed by the parameter q. If $\mathcal{Q}$ is unitary, then the eigenfunctions form an orthonormal basis for $L^2(\mathbb{R})$. In this case, the basis expansion onto these eigenfunctions yields another unitary operator, which we will refer to as the $\mathcal{Q}$–Fourier transform $\mathbf{F}_{\mathcal{Q}}$. The forward transform of a signal $x \in L^2(\mathbb{R})$ is given by

$$(\mathbf{F}_{\mathcal{Q}}\, x)(q) \;=\; \int x(u)\, \mathbf{e}_q^{\mathcal{Q}*}(u)\, du. \tag{6}$$

Note that $\mathbf{F}_{\mathcal{Q}}$ is invariant (up to a phase shift) to the operator $\mathcal{Q}$; that is, $|(\mathbf{F}_{\mathcal{Q}}\mathcal{Q}\, x)(q)| = |(\mathbf{F}_{\mathcal{Q}}\, x)(q)|$.

The eigenfunctions of the time operator, $\mathbf{e}_f^{\mathcal{T}}(u) = e^{j2\pi f u}$, yield the usual Fourier transform $\mathbf{F}_{\mathcal{T}} = \mathbf{F}$, while the eigenfunctions of the frequency operator, $\mathbf{e}_t^{\mathcal{F}}(u) = \delta(u-t)$, yield the (trivial) signal transform $(\mathbf{F}_{\mathcal{F}}\, x)(t) = x(t)$. The eigenfunctions of the dilation operator on $L^2(\mathbb{R}_+)$ are the "hyperbolic chirp" functions

$$\mathbf{e}_c^{\mathcal{D}}(u) \;=\; u^{-1/2}\, e^{j2\pi c \log u}, \qquad u > 0. \tag{7}$$

Since these functions are by definition invariant to dilations, the Mellin transform $\mathbf{F}_{\mathcal{D}}$ in (1) is likewise invariant.

3 THE SCALE TRANSFORM

The derivation of the scale transform follows directly from the unitary equivalence of the time, frequency, and dilation operators. The key observation is as follows. The transform $\mathbf{F}_{\mathcal{T}}$ based on the *time operator* eigenfunctions indicates the *frequency content* of a signal x, because it is covariant to frequency shifts $\mathcal{F}x$ of the signal. Similarly, the transform $\mathbf{F}_{\mathcal{F}}$ based on the *frequency operator* eigenfunctions indicates the *time content* of a signal, because it is covariant to time shifts $\mathcal{T}x$ of the signal. Therefore, the transform $\mathbf{F}_{\mathcal{H}}$ covariant to the scale operator will be based on the eigenfunctions of an operator $\mathcal{H}$ that is dual to the scale operator in the same sense that the time and frequency operators are dual.

Fortunately, the time and dilation operators are unitary equivalent, allowing the direct computation of $\mathcal{H}$. The time operator (on $L^2(\mathbb{R})$) and the dilation operator (on $L^2(\mathbb{R}_+)$) are related by

$$\mathcal{D}_d \;=\; \mathcal{E}^{-1}\, \mathcal{T}_d\, \mathcal{E}, \tag{8}$$

where $\mathcal{E} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ is the unitary exponential axis warping

$$(\mathcal{E}g)(u) \;=\; e^{u/2}\, g(e^u\, t_0) \tag{9}$$

and t_0 is an arbitrary positive reference time. Since the dual operator to time shift is frequency shift, the dual operator to dilation is therefore $\mathcal{H}_h = \mathcal{E}^{-1}\mathcal{F}_h\, \mathcal{E}$. The eigenfunctions of $\mathcal{H}$ are given by

$$\mathbf{e}_\sigma^{\mathcal{H}}(u) \;=\; \left(\mathcal{E}^{-1}\mathbf{e}_\sigma^{\mathcal{F}}\right)(u) \;=\; e^{\sigma/2}\, \delta(u - e^\sigma t_0), \qquad u > 0. \tag{10}$$

The expansion (6) onto these eigenfunctions yields the scale transform $\mathbf{F}_{\mathcal{H}} = \mathcal{S}$ given in (3),

$$(\mathcal{S}x)(\sigma) \;=\; e^{\sigma/2}\, x(e^\sigma t_0), \tag{11}$$

for signals in $L^2(\mathbb{R}_+)$. This transform clearly possesses the dilation covariance property (4).

4 INTERPRETATION

Although almost disappointingly simple, the transform (3) has the attributes of a scale indicating transform. Just as the value $x(t)$ of the time function at the point t indicates the signal content at the reference point $t = 0$ when the signal is translated by $-t$ seconds, the value $(\mathcal{S}x)(\sigma)$ of the scale transform at the point σ indicates the signal content at the reference point $t = t_0$ when the signal is dilated by the factor $-\sigma$. Furthermore, by analogy to the "pure frequency" functions $\mathbf{e}_f^{\mathcal{T}}(u) = e^{j2\pi f u}$ of the $\mathbf{F}_{\mathcal{T}}$ Fourier transform and the "pure time" functions $\mathbf{e}_t^{\mathcal{F}}(u) = \delta(u-t)$ of the $\mathbf{F}_{\mathcal{F}}$ transform, the functions $\mathbf{e}_\sigma^{\mathcal{H}}(u)$ in (10) play the role of "pure scale" functions having no spread about their inherent scale σ.

The scale transform can be applied to measure scale content in any domain, so long as the variable t of the function $x(t)$ is interpreted appropriately. One particularly enlightening interpretation considers t as a spatial variable in an imaging system, with $x(t)$ the distribution of an object in the direction perpendicular to the image plane. Then, σ represents a "zoom" parameter and the spread of $\mathcal{S}x$ indicates the amount of focus change required to successively bring all of the object into focus at the image plane.

While $\mathcal{S}$ is defined only for one-sided signals in $L^2(\mathbb{R}_+)$, signals in $L^2(\mathbb{R})$ can be handled by computing separate scale transforms along the positive and negative time axes. Alternatively, one can consider two values $\pm t_0$ of the reference time parameter.

5 CONCLUSIONS

Scale is a relative quantity (this thing is twice as large as that thing), and so we should anticipate difficulties with a transform indicating purely the scale content of a signal. The scale transform derived here is simply a warped version of the signal being analyzed, and thus it is instantaneous in the sense that its value $(\mathcal{S}x)(\sigma)$ at the point σ depends only on the signal value $x(e^\sigma t_0)$ at the point $e^\sigma t_0$. This is contrary to most other popular signal transforms, including the Fourier and Mellin transforms, which utilize integral averages over the totality of the signal time axis to compute a single value of the transform. However, the instantaneity of the scale transform is not completely heretical, since the concept of scale is bound closely to the physical domain where it is applied.

Finally, we note that the concept of duality introduced in Section 3 can be related to the concept of orthogonality of operators developed in detail in [5]. This approach is utilized to generalize the results of this paper to other operators in [6].

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