Nonstationary Signal Enhancement Using The Wavelet Transform

Vidya Venkatachalam Jorge L. Aravena
Department of Electrical and Computer Engineering
Louisiana State University, Baton Rouge, LA 70810.
Phone: (504) 388-5537, fax: (504) 388-5200,
email: aravena@gate.ee.lsu.edu

Abstract

Conventional signal processing typically involves frequency selective techniques which are highly inadequate for nonstationary signals. In this paper, we present an approach to perform time-frequency selective processing using the Wavelet Transform. The approach is motivated by the excellent localization, in both time and frequency, afforded by the wavelet basis functions. Suitably chosen wavelet basis functions are used to characterize the subspace of signals that have a given localized time-frequency support, thus enabling a time-frequency partitioning of signals. A practical implementation scheme using filter banks is also presented, and the effectiveness of the approach over conventional techniques is demonstrated.

1 Introduction

In several application areas, such as speech, audio, image and video subband coding systems, the signal properties and statistics vary temporally or spatially. Additionally, the signals may show high energy concentration in several localized time-frequency regions. For these signals, time-frequency decomposition, processing and reconstruction is widely used. This is done using distributions and transforms such as the Wigner Distribution, the Short Time Fourier Transform, the Gabor Transform, and more recently, the Wavelet Transform. The underlying idea in this type of analysis is to obtain a time-frequency energy distribution of the signal so that one can isolate, and process independently, components of the signal corresponding to regions of high energy concentration in the time-frequency plane.

Our research objective can be simply described as follows:

Given a nonstationary signal, which has its energy localized in **disjoint regions** *in the time-frequency plane. The main task is to develop procedures to manipulate the signal components in the different regions* **independently**, *and* **concurrently**, *with the intention of enhancing the signal.*

This paper focuses on the issue of partitioning the time-frequency plane into disjoint regions such that signal components with support in the different regions are effectively isolated. This is necessary for the independent manipulation of the different components in order to achieve the desired enhancement.

2 Background review

The authors in [1] addressed a similar problem and attempted to solve it through the use of the Wigner distribution (WD). While the WD has very desirable properties when applied to signals with only one region of concentration in the time-frequency plane, it has been acknowledged that it has some very serious limitations when dealing with multicomponent signals. In most cases, the WD is incapable of resolving two components in a signal due to the presence of excessive cross-terms. One needs very high resolution in order to distinguish two closely spaced components, but then, at high resolutions the WD may produce negative values which are difficult to interpret in energy terms. Thus, the WD is a poor candidate for the analysis of multicomponent nonstationary signals.

Another commonly used time-frequency representation is the Short Time Fourier Transform (STFT). If $F_x(t,f)$ is the STFT of a signal $x(t)$, the associated energy time-frequency distribution is given by the **spectrogram**(SP), which is defined as $SP_x(t,f) = | F_x(t,f) |^2$. The spectrogram is a very effective representation in most situations, even where the WD fails, namely, the resolution of multicomponent signals. It has little cross-term interference, and always assumes only positive values, providing for unambiguous interpretations. However, the shape of the window critically determines the time resolution afforded by the STFT. The main disadvantage is that it allows constant time-frequency resolution, and time events not separated by an interval larger than the window length cannot be effectively isolated by the STFT. Hence, though the STFT overcomes several problems associated with the WD, it is limited by the time resolution it can afford.

The Wavelet Transform (WT) overcomes the limitation in resolution by introducing a scale parameter which allows variations in the time-frequency resolution and generates a TDF which is always nonnegative. The continuous wavelet transform of a function $x(t) \in L_2(\mathcal{R})$ is defined as

$$T^w\left[x\right](a,b) = \langle x, \psi_{a,b} \rangle = \int_{-\infty}^{\infty} x(t)\psi^*_{a,b}dt \qquad (1)$$

where $\psi_{a,b}$ is defined as

$$\psi_{a,b} = \mid a \mid^{-\frac{1}{2}} \psi(\frac{t-b}{a})$$

The function, $\psi_{a,b}$, represents a dilated and shifted version of a basic function $\psi(t)$, called the *'mother wavelet'* or basic wavelet function. The inverse WT is computed as

$$x(t) = c^{-1} \int\int T^w\left[x\right](a,b)\psi_{a,b}\frac{dadb}{a^2} \qquad (2)$$

where $c = 2\pi \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega$. This implies that the WT is invertible whenever $c < \infty$. This rather weak condition for inversion is called the *admissibility condition.*

The variable **a** is referred to as the *scale parameter*. It determines the dilation performed on the basic wavelet and it is inversely proportional to the frequency ([2]). Later we will establish a formal correspondence between the two. The variable **b** determines the time location of the wavelet; hence it is called the *time parameter*. The wavelets thus provide a natural localization of a given function $x(t) \in L_2(\mathcal{R})$ (finite energy signals) in time and frequency. One can visualize the WT, $T^w\left[x\right](\cdot,b)$, as the extracting of the signal information in varying neighborhoods of **b**; or examining it with a continuous family of bandpass filters. As the width of the time window decreases (the scale reduces), the bandpass increases. The effect of this is that the WT offers good spectral and poor temporal resolutions at low frequencies, which is useful for analyzing low frequency components of long duration; and good temporal and poor spectral resolutions at high frequencies which is useful for analyzing signal components of high frequency and short duration. Since most nonstationary signals encountered in practice are of these forms, the WT provides a very good representation for these signals. This inherent property of the WT is the foundation of the proposed use of the wavelets for nonstationary signal enhancement.

3 Time-frequency Partitioning using the WT

In this section, we address the first part of the enhancement problem, namely the partitioning of the time-frequency plane into different disjoint regions such that the signal components are localized in these regions.

The main issue here is one of characterizing the subspace of functions having support in a given time-frequency region R using suitable basis functions. For this, we need an index to measure the energy concentration of a signal component in a particular region. For the WT, this index is defined in terms of the **scalogram** associated with the WT of a function $x(t)$ and defined as

$$SC_x(a,b) = \mid T^w\left[x\right](a,b) \mid^2$$

The signal energy over a given support R is defined as

$$E_x^{(R)} = \int\int_{(a,b)\in R} SC_x(a,b)\frac{dadb}{a^2}$$

The basis behind our proposed approach is the natural localization property afforded by the wavelets. In the time domain they are designed to have good windowing characteristics; compact support is a desirable property. On the other hand, the admissibility condition effectively imposes a frequency band-pass characteristic. Thus, if $\psi(t)$ is an admissible wavelet with Fourier transform, $\Psi(\omega)$, one can associate to it a time support interval BT_ψ centered at a time t_ψ and a frequency support BW_ψ centered at a frequency ω_ψ. Hence, $\psi(t)$ has time-frequency support $BT_\psi \times BW_\psi$ centered as (t_ψ, ω_ψ).

The wavelet $\psi_{a,b}$ has been time shifted by **b** and dilated by **a**. Hence it has a time-frequency support

$$S_{a,b} = aBT_\psi \times \frac{BW_\psi}{a}$$

and centered at the point

$$(t,\omega)_{a,b} = \left(t_\psi + b, \frac{\omega_\psi}{a}\right)$$

The same argument can be used to establish a correspondence between scale and frequency. Let ω_ψ denote the center frequency corresponding to the wavelet function $\psi(t)$. Let ω denote the center frequency for $\psi_{a,b}(t)$. Then, the scale a is related to the frequency ω given by

$$\omega = \frac{\omega_\psi}{a}$$

With this assignment, the scalogram, as a function of ω, does correspond to a time frequency energy density function.

$$\begin{aligned} E_x &= c^{-1}\int_\omega \int_t SC_x(\frac{\omega_\psi}{\omega}, t)\left(\frac{\omega}{\omega_\psi}\right)^2 \frac{\omega_\psi}{\omega^2} d\omega dt \\ &= (c\omega_\psi)^{-1}\int_\omega \int_t SC_x(\frac{\omega_\psi}{\omega}, t) d\omega dt \\ \textit{and}\ E_x^{(R)} &= (c\omega_\psi)^{-1}\int \int_{(\omega,t)\in R} SC_x(\frac{\omega_\psi}{\omega}, t) d\omega dt \end{aligned}$$

Thus, we can obtain the energy of the signal $x(t)$ in a region R in the time-frequency plane using the WT. The crucial feature in this approach then is to determine the a, b which correspond to a region R (using the relationship developed above), and then determine the corresponding WT coefficients, in order to compute the energy concentration in R. This problem is discussed below.

3.1 Determining the subspace corresponding to a region R

Consider the wavelet family $\{\psi_{a,b}\}$. Each wavelet has a well defined support $S_{a,b}$ in the time frequency plane, where

$$S_{a,b} = aBT_\psi \times \frac{BW_\psi}{a}$$

Hence, essentially all its energy is concentrated in that region. By a suitable choice of a, b (corresponding to wavelets which are well localized within the region R), we can 'cover' the region R using a finite number of these wavelet functions. Then, the projection of the signal onto the subspace spanned by these wavelet functions would determine the signal concentration $E_x^{(R)}$ in that particular region. The reconstruction of the signal using only these coefficients in the inverse WT should give the component concentrated in that region. Thus, this can be used to isolate components into different time-frequency regions.

There are problems here regarding overlapping of the different $\psi_{a,b}$s, and a spillover effect if the $\psi_{a,b}$s have substantial support outside of the region R.

The previous analysis assumes that the region $S_{a,b}$ can be used to estimate the time frequency support of the wavelet $\psi_{a,b}$. One can make a more detailed analysis by estimating the *support of the wavelet transform of* $\psi_{a,b}$. For this purpose, we note that

$$T^w[\psi_{a,b}](\alpha, \beta) = \langle \psi_{a,b}, \psi_{\alpha,\beta} \rangle$$

This inner product will be zero if the two wavelets have *disjoint time support*. Hence one can estimate the support by determining conditions for the two intervals to overlap. It is not difficult to show that the two wavelets will have common time support if one of the following two conditions is satisified

$$b - aT_\psi \le \beta - \alpha T_\psi \le b + aT_\psi$$

$$b - aT_\psi \le \beta + \alpha T_\psi \le b + aT_\psi$$

For simplicity in the notation we have used $T_\psi = BT_\psi/2$, with BT_ψ the time support of the basic wavelet. The first equation constrains the left end of the support for $\psi_{\alpha,\beta}$ to be inside the support for $\psi_{a,b}$ while the second establishes a similar constraint for the right endpoint. This defines a romboidal region centered at (a, b), and which can be then converted to an equivalent region in the time-frequency plane.

It is well known that the WT offers a significant amount of redundant information, which may be useful to improve performance in noisy environments. However, from a computational point of view one may gain efficiency by eliminating redundancy. A natural implementation of this idea is to generate an orthonormal basis (ONB) for a given subspace using suitable wavelet functions. One approach to obtain such a basis is through a *multiresolution*. This is essentially a ladder of subspaces

$$\ldots, \subset V_1 \subset V_0 \subset V_{-1} \subset \ldots$$

defined by a unit norm *analysis function*, $\phi(t)$, solution of a two scale equation

$$\phi(t) = \sum_n h_n \sqrt{2}\phi(2t - n)$$

Moreover, defining $\phi_{m,n} = 2^{-\frac{m}{2}}\phi(\frac{t}{2^m} - n)$, the collection $\{\phi_{m,n}\}_n$ is an ONB for V_m. One of the tenets of multiresolution analysis ([3]) is that to every multiresolution it is possible to associate a (unit norm) admissible wavelet, $\psi(t)$, of the form

$$\psi(t) = \sum_n g_n \sqrt{2}\phi(2t - n)$$

such that the collection

$$\psi_{m,n} = 2^{-\frac{m}{2}}\psi(\frac{t}{2^m} - n);\ m, n \in \mathcal{Z}$$

defines another orthonormal basis for L_2 and the subspaces

$$W_m = \overline{span\{\psi_{m,n}, n \in \mathcal{Z}\}};\ m \in \mathcal{Z}$$

constitute an orthogonal decomposition such that $V_m = V_{m+1} \oplus W_{m+1}$. For this reason, the space, W_m, is referred to as the *subspace with details of level m*.
A given finite energy signal $x(t)$ can be expressed as

$$x(t) = \sum_{m,n \in \mathcal{Z}} \langle x, \psi_{m,n} \rangle \, \psi_{m,n}$$

The coefficients $DW_x[m,n] = T^w[x](2^m, 2^m n) = \langle x, \psi_{m,n} \rangle, m, n \in \mathcal{Z}$ are referred to as the *discrete wavelet transform of* $x(t)$. It follows immediately that the total energy of the signal can be expressed as

$$E_x = \sum_{m,n \in \mathcal{Z}} \mid DW_x[m,n] \mid^2$$

Suppose $\{\psi_{m_i,n_i}\}_{i=1}^N$ for suitably chosen m_i, n_i is the collection of orthonormal wavelets which have their support in a given region, R. These wavelet functions with localized support in the region R are effectively an ONB for the collection of functions with energy concentrated in that particular region. Thus, determining the component of a signal with energy in R is equivalent to determining the wavelet coefficients for $\{\psi_{m_i,n_i}\}_{i=1}^N$.
Let $x_R(t)$ denote the component of a signal $x(t)$ with support in R

$$\begin{aligned} x_R(t) &= \sum_{i=1}^N \langle x, \psi_{m_i,n_i} \rangle \, \psi_{m_i,n_i} \\ &= \sum_{i=1}^N T^w\,[x]\,(2^{m_i}, n_i 2^{m_i}) \psi_{m_i,n_i} \end{aligned}$$

The energy of the signal $x_R(t)$ in the time-frequency plane is then given by

$$\begin{aligned} E_{x_R} &= \sum_{i=1}^N \mid \frac{T^w\,[x]\,(2^{m_i}, n_i 2^{m_i})}{2^{m_i}} \mid^2 \\ &= E_{x_R}^{(R)} \end{aligned}$$

This is consistent with our initial assumption that $x_R(t)$ is the component of $x(t)$ with support in R. Thus, this approach isolates those signal components that have localized high energy concentrations.
Since the basis is orthonormal, we avoid the problem of overlapping and obtain a more accurate representation of the signal concentration in the region R. In this analysis, the choice of the mother wavelet function becomes critically important in determining the quality of the covering.

4 Filter Bank Implementation

Filter banks are efficient in implementing the parsimonious time-frequency partitioning created by a multiresolution. Filter banks are widely used for the computation of the WT on a dyadic grid, i.e. $a = 2^m$ and $b = n2^m$. As a result, we obtain the mapping of the continuous $1-D$ function $x(t)$ to a $2-D$ discrete grid defined by m, n in the time-scale plane.
Suppose we have a function $x(t) \in V_0$. We can then define a unique representation x_d for $x(t)$ in l_2 using the frame operator $\mathcal{F}$ defined by

$$x_d[n] = \mathcal{F}x = \langle x, \phi_{0,n} \rangle$$

This sequence is used as the input in the filter bank implementation scheme. The output sequence $\tilde{x_d}$ is then mapped to an $L_2(\Re)$ function $\tilde{x}(t)$ using the adjoint frame operator $\mathcal{F}^*$ defined as

$$x(t) = \mathcal{F}^* x_d = \sum_m \sum_n x_d[n] \psi_{m,n}$$

Figure 1 shows the typical ladder structure used in the computation of the wavelet coefficients and the corresponding filter bank representation. The analysis filters and the synthesis filters (not shown) are designed based on the $\phi(t)$ and the $\psi(t)$ respectively, and the decimation factor in the kth channel is given by $I_k = 2^k$. Consequently, we obtain at the kth channel output of the analysis bank, vectors d_k such that $d_k(n) = \langle x, \psi_{k,n} \rangle$, which is the WT coefficient corresponding to a unique location in the time-frequency plane defined by the support of $\psi_{k,n}$. By isolating only those channel coefficients which correspond to the $\psi_{k,n}$ which form an ONB for a region R, we can directly obtain the signal energy concentration in R, and isolate the signal components which have a high energy concentration in R. This achieves the desired time-frequency partitioning.
Once the partitioning is accomplished, we can manipulate the isolated signal components independently. This effectively implies modifying the WT coefficients in the region R (selected analysis channel outputs) in a desired manner before reconstruction in the synthesis bank.

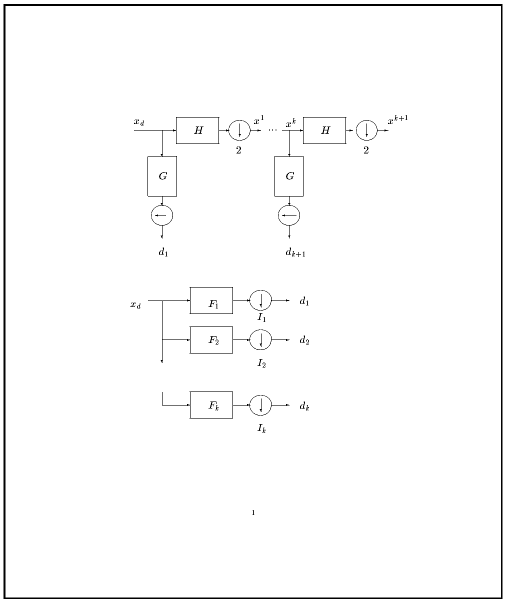

Figure 1: Computation of the Discrete Wavelet Transform and Filter Bank Representation

5 Experimental Results

In this section, we demonstrate the effectiveness of the proposed time-frequency partitioning approach in the processing of nonstationary signals corrupted by noise. The test signal has four well defined components shown in Figure 2. Components a and b (resp. c and d) are chirp signals with disjoint time support and the same energy frequency distribution. Components a and c (resp. b and d) have overlapping time support but approximately disjoint frequency supports. For demonstration purposes, we have used the uncorrupted signal to determine the appropriate time frequency regions. For this purpose, we have used Daubechies' D_{10}, compact support wavelet. For each component, we determine the region R such that the signal has 85% of its energy in that region. This is used to define the wavelets spanning the subspace of signals with support in R. In practice, this analysis determines which coefficients of the discrete wavelet transform must be used in the reconstruction of the signal.

For the multicomponent and noise corrupted signal, we computed the wavelet coefficients and used only those corresponding to wavelets in the selected region to generate the *enhanced signal*. Effectively, this approach completely eliminates the noise outside the time-frequency support of the signal. Figure 3 shows the multicomponent signal, the noise corrupted signal and representative reconstructed components.

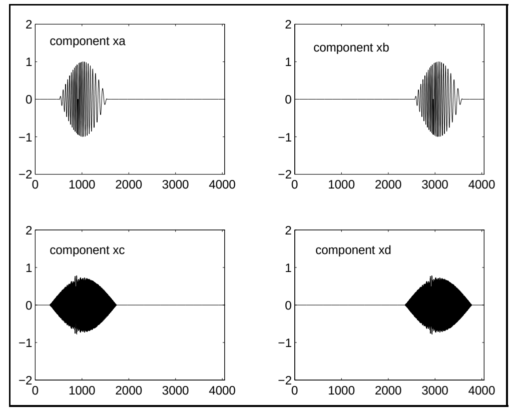

Figure 2: Components of Test Signal

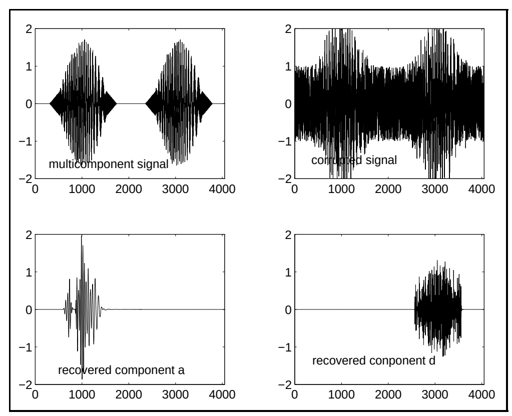

Figure 3: Enhancement with Wavelet Transform

References

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