

# Nonlinear Wigner-Ville spectrum estimation using wavelet soft-thresholding

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## ABSTRACT

The large variance of the Wigner-Ville distribution makes smoothing essential for producing readable estimates of the time-varying power spectrum of noise corrupted signals. Since linear smoothing trades reduced variance for increased bias of the signal components, we explore two nonlinear estimation techniques based on soft-thresholding in an orthonormal basis representation. Soft-thresholding provides considerable variance reduction without greatly impairing the time-frequency resolution of the estimated spectrum.

**Keywords:** Wigner-Ville distribution, wavelet transform, wavelet soft-thresholding.

## 1 INTRODUCTION

The concept of a time-varying power spectrum has proved indispensable in a wide range of applications involving nonstationary signals — from radar and sonar to speech, music, geophysics, and acoustics. A particularly successful approach to time-frequency analysis has been driven by the Wigner-Ville spectrum [1]. Given a nonstationary harmonizable random process  $z(t)$ , its Wigner-Ville spectrum  $\overline{W}_z(t, f)$  is defined as the Fourier transform of the symmetric time-varying autocorrelation function  $R_z(t, \tau) = E [z(t + \frac{\tau}{2}) z^*(t - \frac{\tau}{2})]$  or, equivalently, as the expectation of the empirical Wigner-Ville distribution  $W_z(t, f)$  of the process:

$$\overline{W}_z(t, f) = E [W_z(t, f)] \quad (1)$$

with

$$W_z(t, f) = \int z\left(t + \frac{\tau}{2}\right) z^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau \quad (2)$$

the empirical Wigner-Ville distribution.

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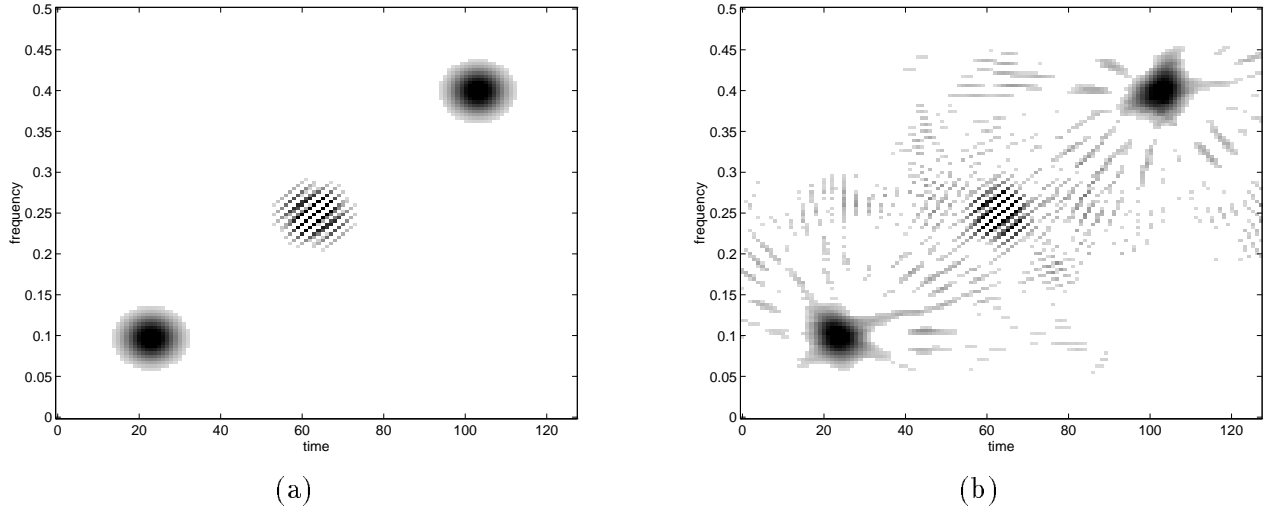


Figure 1: *Empirical Wigner-Ville distributions of (a) a deterministic two-tone test signal, and (b) the same test signal submerged in 6 dB SNR additive white Gaussian noise. While the time-varying spectrum estimate (b) is unbiased, it suffers from high variance. (Horizontal axis corresponds to time  $t$ ; vertical axis corresponds to frequency  $f$ .)*

When multiple realizations of the same random process are unavailable, the expectation in (1) must be ignored and the Wigner-Ville spectrum estimated from the empirical Wigner-Ville distribution of the available data (2). However, the empirical distribution alone does not function as an adequate estimator, due to its high variance (see Fig. 1). As in stationary spectrum estimation, we must smooth the empirical Wigner-Ville distribution to obtain a consistent, readable estimate of the time-varying spectrum.

Up to the present, linear, convolution-based estimates of the Wigner-Ville spectrum have predominated. These estimates generalize the time-invariant periodogram and lie in Cohen's class of time-frequency representations [1, 2]. Representations from Cohen's class are parameterized by a kernel function  $\phi$  and can be written as

$$C_z(t, f) = \iint W_z(u, v) \phi(t - u, f - v) du dv. \quad (3)$$

Taking Fourier transforms of (3) yields an alternate interpretation of these linear estimates as weighting the narrowband ambiguity function  $A_z$  of the signal by a smoothing function  $\Phi$

$$W_z \xrightarrow{2\text{-d FT}} A_z \xrightarrow{\text{weight}} \Phi A_z \xrightarrow{2\text{-d FT}^{-1}} C_z. \quad (4)$$

The functions  $A_z$  and  $\Phi$  correspond to the Fourier transforms (FTs) of the Wigner-Ville distribution and smoothing kernel  $\phi$ , respectively.

While lowpass kernels smooth and thus reduce variance in Wigner-Ville spectrum estimates, they also smear the signal components that we wish to view at full resolution. Figure 2 illustrates two popular Cohen's class estimates. The spectrogram

$$\left| \int z(\tau) w^*(\tau - t) e^{-j2\pi f \tau} d\tau \right|^2$$

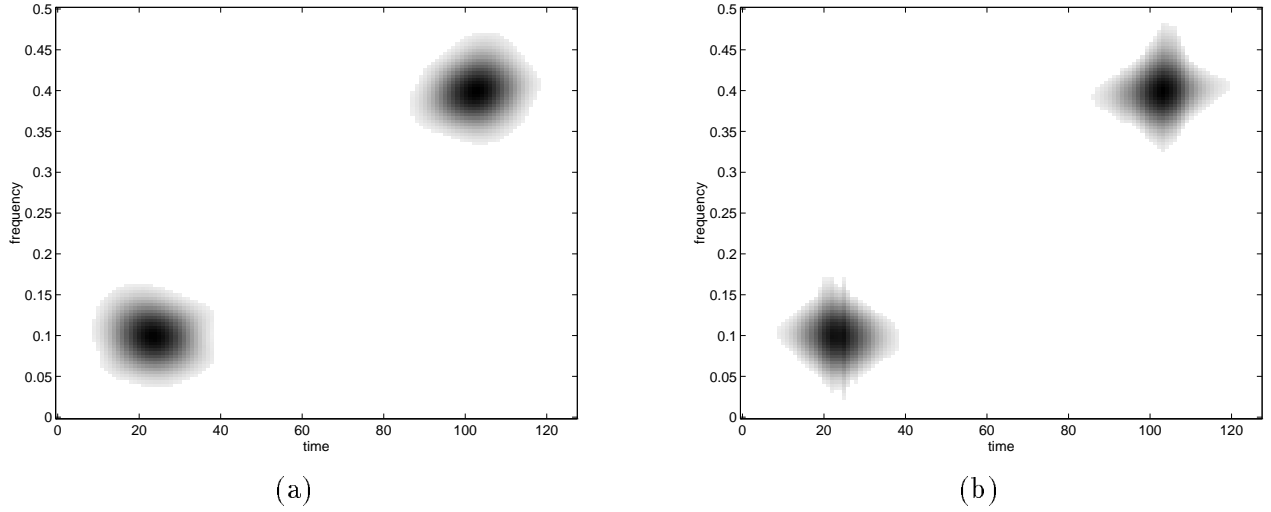


Figure 2: The (a) spectrogram and (b) Choi-Williams distribution of the noisy test signal from Fig. 1 illustrate the bias-variance tradeoff inherent in all linearly smoothed Cohen’s class spectrum estimates.

shown in Fig. 2(a) corresponds to  $\phi =$  the Wigner-Ville distribution of the analysis window  $w$ , while the Choi-Williams distribution shown in Fig. 2(b) corresponds to  $\phi =$  the Fourier transform in  $\theta, \tau$  of the function  $\exp(-\theta^2 \tau^2 / \rho)$  [2]. Both estimates achieve reduced variance at the expense of increased bias.

Since all linearly smoothed time-varying spectrum estimates share a common smearing vs. noise reduction (bias vs. variance) tradeoff, in this paper we will explore two nonlinear Wigner-Ville spectrum estimation techniques [3]. The *soft-thresholding* approaches we will consider can provide considerable variance reduction without greatly impairing the time-frequency resolution of the estimates. Wavelet soft-thresholding algorithms — multiresolution denoising techniques applicable to a wide range of signals and images — construct nonlinear estimates of signals or images embedded in additive white Gaussian noise using a simple three-step procedure [4]: (1) compute the wavelet transform of the data; (2) translate (“soft-threshold”) the wavelet coefficients towards zero by a set threshold value; (3) invert the modified wavelet coefficients to obtain the final estimate.

When applied to the empirical Wigner-Ville distribution, wavelet soft-thresholding corresponds to nonlinear, scalar processing of the coefficients of the distribution in a wavelet basis representation

$$W_z \xrightarrow{2\text{-d WT}} B_z \xrightarrow{\text{threshold}} \Gamma_\gamma(B_z) \xrightarrow{2\text{-d WT}^{-1}} D_z.$$

Here  $B_z$  represents the wavelet transform (WT) coefficients of  $W_z$ ,  $\Gamma_\gamma$  represents soft-thresholding with threshold  $\gamma$ , and  $D_z$  represents the smoothed spectrum estimate. In contrast, Cohen’s class estimates result from linear scalar processing of the coefficients of the empirical Wigner-Ville distribution in the sinusoidal Fourier basis representation (see (4)). Figure 3 illustrates a wavelet soft-thresholded Wigner-Ville distribution for the same noisy test signal utilized in Figs. 1 and 2. Unlike the spectrogram and Choi-Williams distribution, this time-varying spectrum estimate offers reduced variance without degraded resolution.

After a brief review of wavelet soft-thresholding in Section 2, we discuss its application to the time-varying spectral analysis of unknown deterministic signals embedded in noise in Section 3. Since wavelet

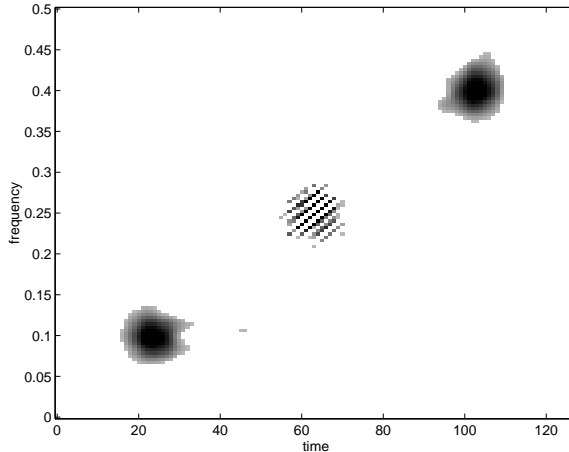


Figure 3: *Wavelet soft-thresholded Wigner-Ville distribution of the noisy test signal from Fig. 1. This nonlinearly smoothed time-varying spectrum estimate boasts the low bias of the Wigner-Ville distribution with the reduced variance of the linearly smoothed Cohen's class estimates.*

processing of the Wigner-Ville distribution sacrifices some of its desirable properties, in Section 4 we introduce soft-thresholding of the ambiguity function representation from (4). After further examples in Section 5, we close in Section 6 with some preliminary conclusions. While tantalizing, we will find that since the Wigner-Ville distribution of a noisy signal does not conform to the standard additive white Gaussian noise model, the application (or misapplication!) of soft-thresholding techniques to time-frequency analysis remains as ad hoc as previous nonlinear schemes such as Wigner-Ville distribution thresholding, median filtering, and so on [5].

## 2 WAVELET SOFT-THRESHOLDING

The wavelet transform of a one-dimensional continuous-time signal  $s(t)$  is defined as

$$B_s(m, k) = 2^{-k/2} \int s(t) \psi^*(2^{-k}t - m) dt. \quad (5)$$

When the dilates and translates of the wavelet function  $\psi(t)$  form an orthonormal basis, we have the signal representation, or inverse wavelet transform

$$s(t) = \sum_{m,k} B_s(m, k) 2^{-k/2} \psi(2^{-k}t - m). \quad (6)$$

Roughly speaking, the wavelet transform of a smooth signal is concentrated in a relatively small number of wavelet coefficients. On the other hand, the transform of a white noise signal spreads out over all coefficients.

The wavelet thresholding concept arose from combining these two observations with the conventional wisdom that simple thresholding performs well as a data recovery technique whenever the data lies above the noise floor. Wavelet thresholding addresses the following data recovery problem (stated in one dimension for simplicity): Recover the smooth, discrete-time signal  $s(i)$ ,  $i = 1, \dots, N$ , given the corrupted observations  $z(i) = s(i) + n(i)$ , where  $n(i)$  is a white Gaussian sequence of zero mean and variance  $\sigma^2$ . The algorithm of Donoho and Johnstone [4] runs as follows:

1. Compute the wavelet transform of  $s+n$  using a discrete-time, finite-data analog to (5) (an interval adapted filterbank).
2. Translate all wavelet coefficients  $B_{s+n}(m, k)$  towards zero by the amount  $\gamma = \sqrt{2 \log(N)/N} \sigma$ .
3. Invert the thresholded coefficients using the discrete-time, finite-data analog to (6).

A bidimensional wavelet transform [6] extends this procedure to image data in two dimensions.

### 3 WAVELET SOFT-THRESHOLDING THE WIGNER-VILLE DISTRIBUTION

In addition to being straightforward and intuitively reasonable, wavelet soft-thresholding possesses two remarkable properties [4], both potentially useful for Wigner-Ville spectrum estimation of unknown deterministic signals embedded in noise. First, with high probability, the data estimate is at least as smooth as the desired noise-free data. Thus, given a smooth set of Wigner-Ville distribution signal components embedded in noise, wavelet denoising should not introduce artifacts that could be interpreted as new components. Second, the estimate achieves almost the minimax mean-square error over every one of a wide variety of smoothness measures, including many where linear estimators do not and cannot achieve the minimax value. Thus, for time-varying spectral analysis of noisy signals, nonlinear smoothing of the empirical Wigner-Ville distribution should offer higher performance than linear smoothing. Simulations support this intuition; Figure 3 illustrates a wavelet soft-thresholded Wigner-Ville distribution for the same noisy test signal utilized in Figs. 1 and 2.

Unfortunately, it appears difficult to go beyond simulations for justifying wavelet soft-thresholding in this context, because our data recovery model does not match that for which the algorithm was developed. In particular, the Wigner-Ville distribution of the noise corrupted signal  $z = s + n$ , given by

$$W_{s+n} = W_s + W_n + 2 \operatorname{Re} W_{s,n}$$

where the last term involves the cross-Wigner-Ville distribution

$$W_{s,n}(t, f) = \int s\left(t + \frac{\tau}{2}\right) n^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau,$$

corresponds to data  $W_s$  plus interference  $W_n + 2 \operatorname{Re} W_{s,n}$ . This interference is anything but Gaussian and white:  $W_{s,n}$  is Gaussian, yet highly correlated and signal-dependent, while  $W_n$  is neither Gaussian nor uncorrelated. Further complicating matters, note that since  $W_{s,n}$  has variance proportional to  $\|s\|^2 \sigma^2$  and  $W_n$  has variance proportional to  $\sigma^4$ , one term will dominate depending on the particular value of SNR.

Nevertheless, as very little is known about the probability density of  $W_n$ , ad hoc methods such as thresholding must suffice until a more complete theory for Wigner-Ville spectrum estimation can be developed.<sup>1</sup> Some progress has been made with the stationary power spectrum [8]; a similar approach

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<sup>1</sup>Obviously, estimation of the Wigner-Ville spectrum by first denoising the signal and then computing the Wigner-Ville distribution of the result avoids any problems with the additive white Gaussian noise requirement. However, signal components modulated to lower frequencies are “smoother” than their counterparts at higher frequencies, and hence they are preferentially treated by the wavelet transform before thresholding. This favoritism is demonstrated in Fig. 4, where we show the empirical Wigner-Ville distribution of the denoised test signal. For modulated signals, the Wilson bases [7] represent a better alternative to wavelets for soft-thresholding.

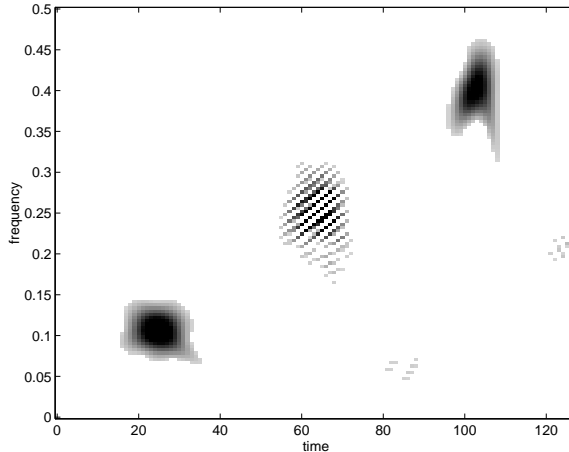


Figure 4: *Wigner-Ville spectrum estimate of the noisy test signal from Fig. 1 obtained by first wavelet soft-thresholding the data and then computing the empirical Wigner-Ville distribution of the result. Note the distortion in the high frequency signal component due to mismatch with the wavelet denoising basis.*

might prove useful here and result in an explicit formula for the threshold  $\gamma$ . At present, we take  $\gamma$  either as a free parameter or adjust it automatically to optimize some measure of estimation performance as in [9, 10].

The denoising provided by wavelet soft-thresholding comes at some expense in terms of the desirable mathematical properties of the Wigner-Ville spectrum [1, 2]. First, due to the nonlinearity of the processing, the energy preservation and marginal properties fail to hold true. Second, the time-frequency shift covariance property is lost: Because the discrete wavelet transform is not covariant to shifts, a time-frequency shift in the empirical Wigner-Ville distribution will result in a different thresholding pattern and thus a slightly different estimated spectrum. (Fortunately, the new shift-invariant soft-thresholding methods of Coifman and Donoho [11] and Lang et al [12] restore this important property — see Fig. 5.) Third, with separable wavelet processing, the rotation covariance property of the Wigner-Ville distribution abandons us as well (although it should be noted that nonadaptive linear smoothing cannot retain rotation covariance either).

#### 4 SOFT-THRESHOLDING THE AMBIGUITY FUNCTION

The wavelet transform proves so useful as a soft-thresholding basis transformation, because wavelets form unconditional bases for an incredible variety of signal spaces, including most of those related to smoothness [4]. Sinusoids are more limited in their utility for soft-thresholding, because they do not form unconditional bases for most of these spaces. Nevertheless, in light of the loss of desirable TFR properties mentioned in the previous section, it appears reasonable to consider also the Fourier basis for Wigner-Ville spectrum estimation.

The resulting scheme fits in the framework of (4), but with a now signal-dependent kernel  $\Phi_z$  that soft-thresholds the ambiguity function of the signal

$$W_z \xrightarrow{2\text{-d FT}} A_z \xrightarrow{\text{threshold}} \Gamma_\gamma(A_z) \xrightarrow{2\text{-d FT}^{-1}} E_z. \quad (7)$$

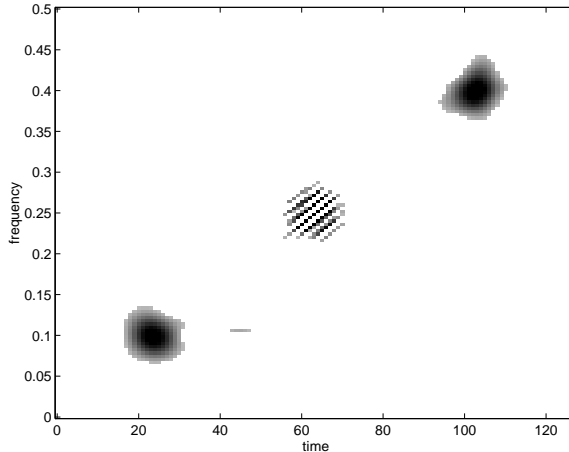


Figure 5: *Wavelet soft-thresholded Wigner-Ville spectrum estimate of the noisy test signal from Fig. 1 obtained using the redundant wavelet soft-thresholding algorithm of [12]. In addition to shift covariance, better denoising performance results.*

Note that  $E_z$  belongs to Cohen’s class and is time-frequency shift covariant; additional constraints can be imposed on the thresholding to ensure that it satisfies other properties such as energy preservation and marginals [2], if desired. Furthermore, the Rényi information measures [10] can be utilized to optimize the threshold value. Figure 6 illustrates a Wigner-Ville spectrum estimate obtained by soft-thresholding the ambiguity function of the noisy test signal; it closely resembles the estimates of Figs. 3 and 5. Less ad hoc approaches to signal-dependent kernel design are detailed in [9, 13].<sup>2</sup>

## 5 FURTHER EXAMPLES

In Fig. 7, we estimate the time-varying spectrum of 2.5 msec of a noise corrupted echolocation chirp emitted by the large brown bat, *Eptesicus fuscus*. While the empirical Wigner-Ville distribution exhibits low bias and the spectrogram exhibits low variance, only the two wavelet soft-thresholded estimates boast both simultaneously.

## 6 CONCLUSIONS

While our results are preliminary and admittedly somewhat ad hoc, nonlinear smoothing techniques have potential for providing time-frequency analyses with Wigner-Ville-like resolution down to low SNRs. The hallmarks of the wavelet soft-thresholding technique — simplicity, use of information across scales, smoothness preservation, and near optimality for additive white Gaussian noise — remain tantalizing, but more work is required in order to justify its application to Wigner-Ville spectrum estimation. It is likely that a detailed analysis of the correlated, nonGaussian interference will inspire modifications to the algorithm, with a corresponding performance increase.

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<sup>2</sup>In fact, the algorithm (7) is closely related to a relaxed version of the “1/0” optimal kernel design from [9, 14] — “Linear Program 3” from [14] in particular.

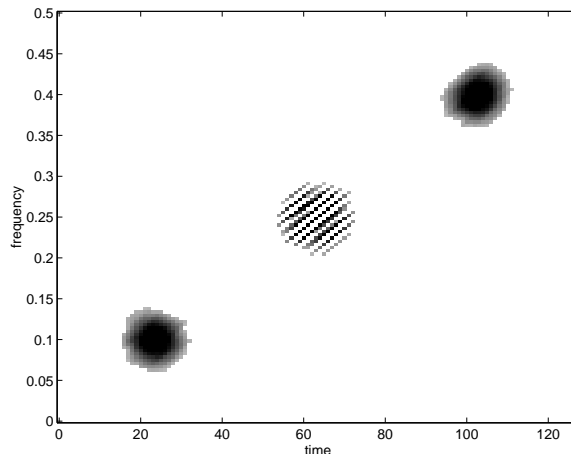


Figure 6: *Wigner-Ville spectrum estimate of the noisy test signal from Fig. 1 obtained by soft-thresholding the ambiguity function of the signal as in (7). The resulting distribution is a shift-covariant member of Cohen’s class.*

Finally, we note that soft-thresholded representations of time  $t$  and scale  $a$  (related to the continuous wavelet transform and the scalogram [6]) can be obtained simply by processing the reparameterized empirical Wigner-Ville distribution  $W_z(t, f_0/a)$ .

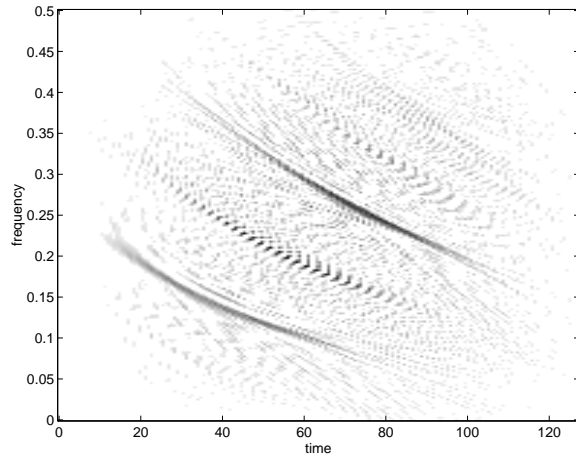
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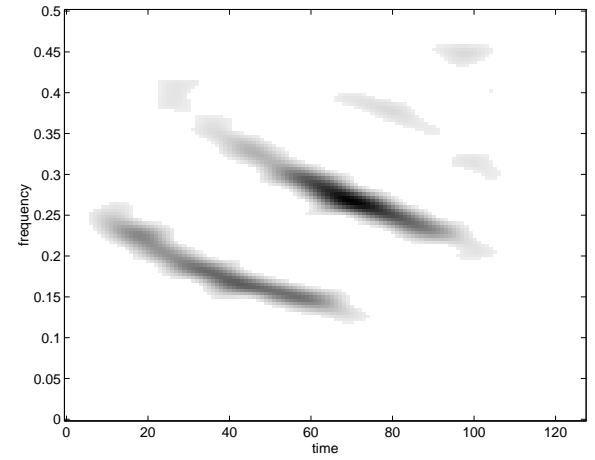
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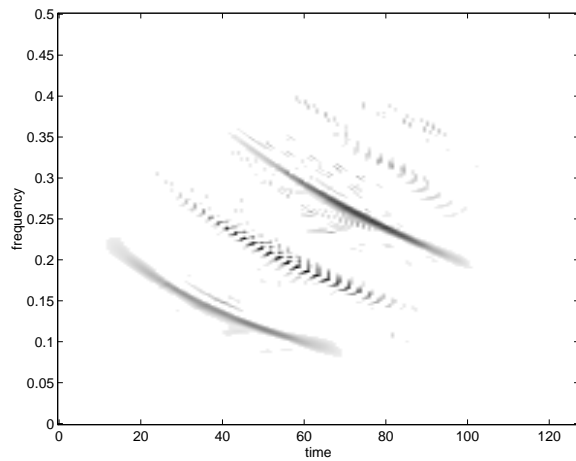




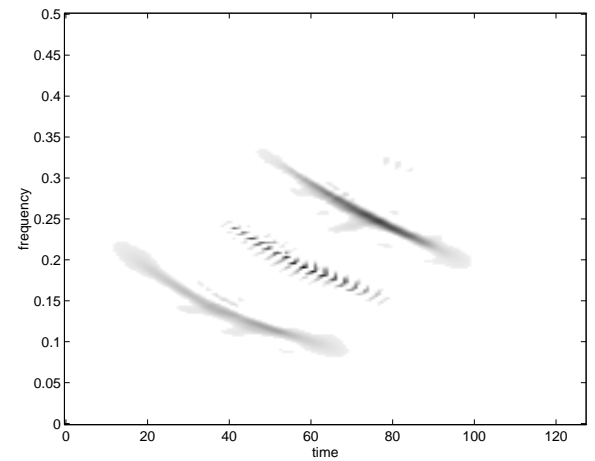
(a)



(b)



(c)



(d)

Figure 7: Wigner-Ville spectrum estimates of a noise corrupted bat echolocation chirp. (a) Empirical Wigner distribution. (b) Linearly smoothed spectrogram. (c) Wavelet soft-thresholded Wigner-Ville distribution, threshold  $\gamma_1$ . (d) Wavelet soft-thresholded Wigner-Ville distribution, threshold  $\gamma_2 > \gamma_1$ .

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