Model reduction of large-scale dynamical systems

Lecture II: Balanced truncation and the Lyapunov equation

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Outline

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System operators and norms

$$\Sigma = \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right) : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \Rightarrow \begin{cases} \mathbf{h}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\delta(t), t \geq 0 \\ \mathbf{H}(s) = \mathbf{D} + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \end{cases}$$

Convolution operator $\mathcal{S}$:

$\mathcal{S} : \mathbf{u} \mapsto \mathbf{y}$, $\mathbf{y}(t) = \int_{-\infty}^{\infty} \mathbf{h}(t-\tau)\mathbf{u}(\tau)d\tau$, $t \in \mathbb{R}$

Singular values: let $\underline{\sigma} := \inf_\omega \sigma_{min}(\mathbf{H}(j\omega))$ and $\bar{\sigma} := \sup_\omega \sigma_{max}(\mathbf{H}(j\omega))$.

$$\boxed{\sigma \text{ singular value of } \mathcal{S} \quad \Leftrightarrow \quad \sigma \in [\underline{\sigma}, \bar{\sigma}]}$$

Hankel operator $\mathcal{H}$:

$\mathcal{H} : \mathbf{u}_- \mapsto \mathbf{y}_+$, $\mathbf{y}_+(t) = \int_{-\infty}^{0} \mathbf{h}(t-\tau)\mathbf{u}_-(\tau)d\tau$, $t \in \mathbb{R}_-$

Singular values: solve for continuous-time **Lyapunov equations**

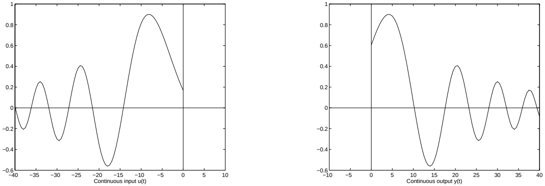
$$\boxed{\sigma_i^2(\mathcal{H}) = \lambda_i(\mathcal{H}^*\mathcal{H}) = \lambda_i(\mathcal{P}\mathcal{Q})} \quad \text{where} \quad \begin{cases} \mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^* + \mathbf{B}\mathbf{B}^* = \mathbf{0} \\ \mathbf{A}^*\mathcal{Q} + \mathcal{Q}\mathbf{A} + \mathbf{C}^*\mathbf{C} = \mathbf{0} \end{cases}$$

Hankel map

$\mathcal{H}$: "past" inputs $\Rightarrow$ "future"outputs

$$\mathbf{y}(t) = \int_{-\infty}^{0} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau = \mathbf{C}e^{\mathbf{A}t} \cdot \int_{0}^{\infty} e^{\mathbf{A}\tau}\mathbf{B}\mathbf{u}(-\tau)d\tau,$$

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0), \quad \mathbf{x}(0) = \int_{0}^{\infty} e^{\mathbf{A}\tau}\mathbf{B}\mathbf{u}(-\tau)d\tau.$$



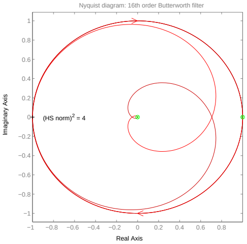
$$\mathbf{u}(t),\ t \in (-\infty, 0) \Longrightarrow \mathbf{x}(0) \Longrightarrow \mathbf{y}(t),\ t \in [0, \infty)$$

Hankel singular values and Nyquist plot

In the SISO case:

$$\left.\begin{array}{c}\text{Area of Nyquist diagram}\\ \text{including multiplicities}\end{array}\right\} = \pi \cdot \underbrace{(\sigma_1^2 + \cdots + \sigma_n^2)}_{(\text{Hilbert}-\text{Schmidt Norm of } \mathcal{H})^2} = \pi \|\mathcal{H}\|_{\text{HS}}^2$$

Example: 16th order low-pass Butterworth filter:



$$\sigma^2 = \begin{bmatrix} 9.9963e-001 \\ 9.9163e-001 \\ 9.2471e-001 \\ 6.8230e-001 \\ 3.1336e-001 \\ 7.7116e-002 \\ 1.0359e-002 \\ 8.4789e-004 \\ 4.5242e-005 \\ 1.5903e-006 \\ 3.6103e-008 \\ 5.0738e-010 \\ 4.1111e-012 \\ 1.6892e-014 \\ 7.5197e-017 \\ 3.2466e-017 \end{bmatrix}$$

Outline

The system gramians

Follow POD method: assume stable system

Impulse response: $\mathbf{h}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B},\ t \geq 0$

input-to-state map $\mathbf{x}$: $e^{\mathbf{A}t}\mathbf{B}$; state-to-output map η: $\mathbf{C}e^{\mathbf{A}t}$
input $\delta(t) \to \mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{B}$; initial condition $\mathbf{x}(0) \to$ output $\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$:

Corresponding grammians:

$$\mathcal{P} = \sum_t \mathbf{x}(t)\mathbf{x}(t)^* = \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^* e^{\mathbf{A}^* t}\, dt$$
$$\mathcal{Q} = \sum_t \eta(t)^*\eta(t) = \int_0^\infty e^{\mathbf{A}^* t}\mathbf{C}^*\mathbf{C}e^{\mathbf{A}t}\, dt$$

State transformation $\hat{\mathbf{x}} = \mathbf{T}\mathbf{x} \Rightarrow \hat{\mathcal{P}} = \mathbf{T}\mathcal{P}\mathbf{T}^*$, $\hat{\mathcal{Q}} = \mathbf{T}^{-*}\mathcal{Q}\mathbf{T}^{-1} \Rightarrow$

$\lambda_i(\mathcal{P}\mathcal{Q})$ are **input-output invariants of** $\Sigma = \sigma_i^2(\mathcal{H})$

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Outline

Balanced truncation

Basis change: $\bar{\mathbf{x}} = \mathbf{Tx}$, $\det T \neq 0$; let $\mathcal{P} = \mathbf{UU}^*$ and $\mathcal{Q} = \mathbf{LL}^*$ where $\mathbf{U}$ is upper and $\mathbf{L}$ lower triangular. Compute the SVD $\mathbf{U}^*\mathbf{L} = \mathbf{ZSY}^*$.
A balancing transformation is: $\mathbf{T} = \mathbf{S}^{\frac{1}{2}}\mathbf{Z}^*\mathbf{U}^{-1} = \mathbf{S}^{-\frac{1}{2}}\mathbf{Y}^*\mathbf{L}^*$.

$$\boxed{\mathcal{P} = \mathcal{Q} = S = \mathrm{diag}\,(\sigma_1, \cdots, \sigma_n)}$$

Partition (in the balanced basis):

$$\mathbf{A} = \left(\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right), \mathbf{B} = \left(\begin{array}{c} \mathbf{B}_1 \\ \hline \mathbf{B}_2 \end{array}\right), \mathbf{C} = (\mathbf{C}_1 \mid \mathbf{C}_2), \mathbf{S} = \left(\begin{array}{c|c} \Sigma_1 & 0 \\ \hline 0 & \Sigma_2 \end{array}\right)$$

ROM obtained by balanced truncation $\hat{\Sigma} = \left(\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{B}_1 \\ \hline \mathbf{C}_1 & \end{array}\right)$

Σ_2 contains the small Hankel singular values.
• Let the distinct singular values of Σ be σ_i, with multiplicities m_i, $i = 1, \cdots, q$. Let Σ_1 have singular values σ_i, $i = 1, \cdots, k$, with the corresponding multiplicities m_i, $i = 1, \cdots, k$, $k < q$: The following error bound holds:

$$\|\Sigma - \Sigma_1\|_\infty \leq 2\,(\sigma_{k+1} + \cdots + \sigma_q)$$

If $\Sigma_2 = \sigma_n$, equality holds.

Interpretation of balanced basis

Given state $\mathbf{x}$:

$\mathcal{E}_r$: min. input energy steering $\mathbf{0} \to \mathbf{x} \quad \Rightarrow \quad \mathcal{E}_r = \mathbf{x}^* \mathcal{P}^{-1} \mathbf{x}$
$\mathcal{E}_o$: output observation energy $\mathbf{x} \to \mathbf{0} \quad \Rightarrow \quad \mathcal{E}_o = \mathbf{x}^* \mathcal{Q} \mathbf{x}$

States difficult to reach $=$ states requiring large amount of energy to reach.
States difficult to observe $=$ states yielding small amounts of observation energy.

Resulting principle for model reduction: eliminate states which are difficult to reach and/or difficult to observe.

However, states which are difficult to reach may not be difficult to observe and vice-versa (properties are basis dependent).

Balancing implies:

$$\Rightarrow \quad \mathcal{E}_r \cdot \mathcal{E}_o = 1$$

Outline

Numerical aspects

We need thus to solve for the gramians from

$$\mathbf{A}^*\mathcal{Q} + \mathcal{Q}\mathbf{A} + \mathbf{C}^*\mathbf{C} = \mathbf{0} \quad \text{and} \quad \mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^* + \mathbf{B}\mathbf{B}^* = \mathbf{0},$$

and then find “dominant spaces” X and Y such that

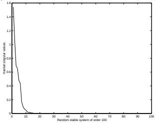
$$\mathbf{Y}^*\mathbf{X} = \mathbf{I}_n, \quad \mathcal{P}\mathcal{Q}\mathbf{X} = \mathbf{X}\Lambda^2, \quad \mathbf{Y}^*\mathcal{P}\mathcal{Q} = \Lambda^2\mathbf{Y}^*$$

Problem : behaviour of Hankel singular values

It is known that $\mathcal{H} > 0 \;\Rightarrow\; \frac{\lambda_1(\mathcal{H})}{\lambda_n(\mathcal{H})} \approx 4^n$.

But for all pass transfer function $\Lambda = \mathbf{I}_N$.

Typical behaviour for stable system :



One has to be careful with numerical calculation of gramians (ill conditioning).

The Lyapunov equation in the Schur basis

Consider the *Lyapunov equation*

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^* + \mathbf{B}\mathbf{B}^* = \mathbf{0},$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathcal{P} = \mathcal{P}^* \in \mathbb{R}^{n\times n}$, $(\mathbf{A}, \mathbf{B})$ is reachable, and $\mathbf{A}$ has no eigenvalues on the imaginary axis.

We transform $\mathbf{A}$ in Schur form, i.e. in upper triangular for by orthogonal similarity: $\mathbf{A} \leftarrow \mathbf{TAT}^*$, $\mathbf{B} \leftarrow \mathbf{TB}$, $\mathcal{P} \leftarrow \mathbf{T}^*\mathcal{P}\mathbf{T}$. Then partition $\mathbf{A}$, $\mathbf{B}$, and $\mathcal{P}$ compatibly where $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are upper triangular:

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix}, \ \mathcal{P} = \begin{pmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{12}^* & \mathcal{P}_{22} \end{pmatrix}$$

Lemma

Under the above assumptions the following hold:

(a) *The pair* $\mathbf{A}_{22}$, $\mathbf{B}_2$, *is reachable.*

(b) $\mathcal{P}_{22}$ *is non-singular.*

(c) *The pair* $(\mathbf{A}_{11}, \hat{\mathbf{B}}_1)$ *is reachable, where* $\hat{\mathbf{B}}_1 = \mathbf{B}_1 - \mathcal{P}_{12}\mathcal{P}_{22}^{-1}\mathbf{B}_2$.

Dense Lyapunov solvers (exact)

Stewart's Lyapunov solver ($O(N^3)$ flops) :

1. transform to Schur basis
2. solve for columns of $\mathcal{P}$ recursively
3. return to original coordinates $\mathcal{P} \leftarrow \mathbf{T}\mathcal{P}\mathbf{T}^*$ if needed

Hammarling's Lyapunov solver for $\mathcal{R}e\lambda_i(\mathbf{A}) < 0$, ($O(N^3)$ flops) :

1. transform to Schur basis and write the Cholesky factorization $\mathcal{P} = \mathbf{U}\mathbf{U}^*$:

$$\mathbf{A}\mathbf{U}\mathbf{U}^* + \mathbf{U}\mathbf{U}^*\mathbf{A} + \mathbf{B}\mathbf{B}^* = \mathbf{0}$$

   with $\mathbf{A}$, $\mathbf{U}$ in upper triangular form
2. solve for columns of $\mathbf{U}$ recursively
3. return to original coordinates $\mathcal{P} = \mathbf{T}\mathbf{U}^*\mathbf{U}\mathbf{T}^*$ if needed

Advantages of Hammarlings method: better conditioned $\kappa(\mathbf{U}) = \sqrt{\kappa(\mathcal{P})}$ and back transformation is not always needed.

Hammarling's square-root alorithm

If $\mathbf{A}$ is stable then $\mathcal{P} > 0$. In this case in the Schur basis, we can write $\mathcal{P} = \mathbf{U}\mathbf{U}^*$, where $\mathbf{U}$ is upper triangular. Hammarling observed that $\mathbf{U}$ can be computed *without explicitly computing* $\mathcal{P}$ first. We have

$$\mathcal{P} = \begin{pmatrix} \hat{\mathcal{P}}_{11}^{1/2} & \mathcal{P}_{12}\mathcal{P}_{22}^{-1/2} \\ \mathbf{0} & \mathcal{P}_{22}^{1/2} \end{pmatrix} \begin{pmatrix} \hat{\mathcal{P}}_{11}^{1/2} & \mathbf{0} \\ \mathcal{P}_{22}^{-1/2}\mathcal{P}_{12}^* & \mathcal{P}_{22}^{1/2} \end{pmatrix} = \mathbf{U}\mathbf{U}^*.$$

The problem is now to successively compute the smaller pieces of $\mathcal{P}$, namely: $\mathcal{P}_{22} \in \mathbb{R}^{(n-k)\times(n-k)}$, $\mathcal{P}_{12} \in \mathbb{R}^{k\times(n-k)}$, and finally $\hat{\mathcal{P}}_{11} \in \mathbb{R}^{k\times k}$, where $k < n$. To that effect, we have the three equations

$$\left.\begin{array}{l} \mathbf{A}_{22}\mathcal{P}_{22} + \mathcal{P}_{22}\mathbf{A}_{22}^* + \mathbf{B}_2\mathbf{B}_2^* = \mathbf{0}, \\ \mathbf{A}_{11}\mathcal{P}_{12} - \mathcal{P}_{12}\left[\mathcal{P}_{22}^{-1}\mathbf{A}_{22}\mathcal{P}_{22} + \mathcal{P}_{22}^{-1}\mathbf{B}_2\mathbf{B}_2^*\right] + \mathbf{A}_{12}\mathcal{P}_{22} + \mathbf{B}_1\mathbf{B}_2^* = \mathbf{0}, \\ \mathbf{A}_{11}\hat{\mathcal{P}}_{11} + \hat{\mathcal{P}}_{11}\mathbf{A}_{11}^* + \hat{\mathbf{B}}_1\hat{\mathbf{B}}_1^* = \mathbf{0}. \end{array}\right\}$$

We can thus solve the first (Lyapunov) equation for $\mathcal{P}_{22}$. Then, we solve the second equation which is a Sylvester equation for $\mathcal{P}_{12}$. Then, since $(\mathbf{A}_{11}, \hat{\mathbf{B}}_1)$ is reachable, the problem of solving a Lyapunov equation of size n is reduced to that of solving a Lyapunov equation of smaller size $k < n$.

Hammarling's square-root algorithm

In practice, we choose $k = n - 1$; thus we compute successively the last row/column of the square root factors. From the first equation we obtain

$$\mathcal{P}_{22} = -\frac{\mathbf{B}_2\mathbf{B}_2^*}{\mathbf{A}_{22} + \mathbf{A}_{22}^*} \in \mathbb{R}$$

The middle equation implies $\left[\mathbf{A}_{11} + \mathbf{A}_{22}^*\mathbf{I}_{n-1}\right] \mathcal{P}_{12} = \mathbf{A}_{12} \frac{\mathbf{B}_2\mathbf{B}_2^*}{\mathbf{A}_{22}+\mathbf{A}_{22}^*} - \mathbf{B}_1\mathbf{B}_2^*$; thus

$$\mathcal{P}_{12} = \left[\mathbf{A}_{11} + \mathbf{A}_{22}^*\mathbf{I}_{n-1}\right]^{-1} \left[\frac{\mathbf{B}_2}{\mathbf{A}_{22} + \mathbf{A}_{22}^*}\mathbf{A}_{12} - \mathbf{B}_1\right] \mathbf{B}_2^* \in \mathbb{R}^{n-1}$$

This yields the last column of the upper triangular square root factor **U**. There remains to determine the upper triangular square root factor $\hat{\mathcal{P}}_{11}^{1/2}$ which satisfies the Lyapunov equation given by the third equation. Since the pair $(\mathbf{A}_{11}, \hat{\mathbf{B}}_1)$ is reachable, and the problem is reduced to dimension *one less*.

Dense Lyapunov solvers (approximate)

Given $z \in \mathbb{C}$ with $\mathcal{R}e(z) \neq 0$, the *sign iteration* is:

$$z_{n+1} = \frac{1}{2}\left(z_n + \frac{1}{z_n}\right), \; z_0 = z.$$

The fixed points of this iterations are ± 1. Therefore if $\mathcal{R}e(z) > 0$, $\lim_{n\to\infty} z_n = 1$, and if $\mathcal{R}e(z) < 0$, $\lim_{n\to\infty} z_n = -1$.

Consider $\mathbf{Z} \in \mathbb{R}^{n\times n}$ with eigenvalue decomposition $\mathbf{Z} = \mathbf{V}\Lambda\mathbf{V}^{-1}$, $\Lambda = \mathrm{diag}(\Lambda_+, \Lambda_-)$, where Λ_+, Λ_- contain Jordan blocks corresponding to the r, $n-r$ eigenvalues with positive, negative real parts, respectively. The **sign function** of this matrix is $\mathbf{Z}_\sigma = \mathbf{V}\mathrm{diag}(\mathbf{I}_r, -\mathbf{I}_{n-r})\mathbf{V}^{-1}$. This can be used to solve the Sylvester and the Lyapunov equations.

For matrices the sign function can be obtained by the following iteration: $\mathbf{Z} \in \mathbb{C}^{r\times r}$: $\mathbf{Z}_{n+1} = (\mathbf{Z}_n + \mathbf{Z}_n^{-1})/2$, $\mathbf{Z}_0 = \mathbf{Z}$. Fixed points in this case, are matrices which satisfy $\mathbf{Z}^2 = \mathbf{I}_r$, i.e., matrices which are diagonalizable and their eigenvalues are ± 1.

If the matrix has eigenvalues ± 1 but is not diagonalizable, convergence to a diagonalizable matrix with eigenvalues ± 1 is achieved in finitely many steps.

The sign iteration is invariant under similarity, i.e. if the j^{th} iterate of $\mathbf{Z}$ is $\mathbf{Z}_j$, the j^{th} iterate of $\mathbf{VZV}^{-1}$ is $\mathbf{VZ}_j\mathbf{V}^{-1}$. The following holds.

Lemma

Let $\mathbf{J}$ *be a Jordan block, i.e.* $\mathbf{J} = \lambda\mathbf{I}_r + \mathbf{N}$*, where* λ *was positive real part, and* $\mathbf{N}$ *is the nilpotent matrix with* 1*s above the diagonal and zeros elsewhere. The sign iteration of* $\mathbf{J}$ *converges to* $\mathbf{I}_r$*.*

Proof.

Since the matrix is upper triangular, the elements on the diagonal will converge to 1. Furthermore, this limit, denoted by $\mathbf{J}'$, has Toeplitz structure. Thus each iteration applied to $\mathbf{J}'$, will bring zeros to the successive super-diagonals. Thus $\mathbf{J}'$ (which is upper triangular and Toeplitz with ones on the diagonal) will converge in r steps to the identity matrix. □

Corollary

If $\mathbf{Z} \in \mathbb{C}^{r \times r}$, *and* $\mathcal{R}e\lambda_i(\mathbf{Z}) < 0$ $(\mathcal{R}e\lambda_i(\mathbf{Z}) > 0)$, *the sign iteration converges to* $\mathbf{Z}_n \to -\mathbf{I}_r$ $(\mathbf{Z}_n \to +\mathbf{I}_r)$, *respectively.*

Proof.

Let $\mathbf{Z} = \mathbf{V}\Lambda\mathbf{V}^{-1}$ be the EVD of $\mathbf{Z}$. Since the iteration is not affected by similarity we need to consider the iterates of the Jordan blocks $\mathbf{J}_i$ of Λ. The proposition implies that each Jordan block converges to $\pm\mathbf{I}$, depending on whether the real part of the corresponding eigenvalue is positive or negative. □

We will now consider a matrix of the following type

$$\mathbf{Z} = \begin{pmatrix} \mathbf{A} & -\mathbf{C} \\ \mathbf{0} & -\mathbf{B} \end{pmatrix}, \mathbf{A} \in \mathbb{R}^{n\times n}, \mathcal{R}e\lambda_i(\mathbf{A}) < 0, \mathbf{B} \in \mathbb{R}^{k\times k}, \mathcal{R}e\lambda_i(\mathbf{B}) < 0, \mathbf{C} \in \mathbb{R}^{n\times k}.$$

Lemma

The iteration $\mathbf{Z}_{n+1} = (\mathbf{Z}_n + \mathbf{Z}_n^{-1})/2,\ \mathbf{Z}_0 = \mathbf{Z}$*, converges to*

$$\lim_{j\to\infty} \mathbf{Z}_j = \begin{pmatrix} -\mathbf{I}_n & 2\mathbf{X} \\ \mathbf{0} & \mathbf{I}_k \end{pmatrix} \quad \text{where} \quad \mathbf{AX} + \mathbf{XB} = \mathbf{C}.$$

Proof.

Let $\mathbf{ZV} = \mathbf{V}\Lambda$, with $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2)$, be the eigenvalue decomposition of $\mathbf{Z}$. $\mathbf{V}$ is upper block triangular

$$\begin{pmatrix} \mathbf{A} & -\mathbf{C} \\ \mathbf{0} & -\mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{0} & \mathbf{V}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{0} & \mathbf{V}_{22} \end{pmatrix} \begin{pmatrix} \Lambda_1 & \mathbf{0} \\ \mathbf{0} & \Lambda_2 \end{pmatrix}$$

This readily implies that the solution of the Sylvester equation is $\mathbf{V}_{12}\mathbf{V}_{22}^{-1}$. The block triangular structure of $\mathbf{Z}$ is preserved during the iterations, i.e. $\mathbf{Z}_j$ is also block upper triangular. Furthermore, by the corollary above, the limits of the $(1,1)$ and $(2,2)$ blocks are $-\mathbf{I}_n$, $\mathbf{I}_k$, respectively. Thus $\mathbf{Z}_j$ as $j \to \infty$ has the form claimed. There remains to show that $\mathbf{X}$ satisfies the Sylvester equation as stated. This follows since $\lim_{j\to\infty} \mathbf{Z}_j\mathbf{V} = \mathbf{V}\lim_{j\to\infty} \Lambda_j$, and therefore

$$\begin{pmatrix} -\mathbf{I}_n & 2\mathbf{X} \\ \mathbf{0} & \mathbf{I}_k \end{pmatrix} \begin{pmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{0} & \mathbf{V}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{0} & \mathbf{V}_{22} \end{pmatrix} \begin{pmatrix} -\mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_k \end{pmatrix}$$

This implies $\mathbf{X} = \mathbf{V}_{12}\mathbf{V}_{22}^{-1}$ is indeed the solution to the Sylvester equation. □

Remarks

(a) The above result shows that the solution of the Sylvester equation where **A**, and **B** have eigenvalues either in the left or the right half planes, can be solved by means of the sign iteration.

(b) The above result also shows that given a matrix $\mathbf{Z} = \mathbf{V}\Lambda\mathbf{V}^{-1} \in \mathbb{C}^{n\times n}$, with k eigenvalues in the left-half plane, the sign iteration yields the matrix $\mathbf{Z}_\infty = \mathbf{V}\mathrm{diag}(-\mathbf{I}_k, \mathbf{I}_{n-k})\mathbf{V}^{-1}$. Therefore $\Pi_\pm = \frac{1}{2}(\mathbf{I}_n \pm \mathbf{Z}_\infty)$ yields the spectral projectors **Z** onto its unstable/stable eigenspaces, respectively.

(c) For Lyapunov equations $\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^* = \mathbf{Q}$, the starting matrix is

$$\mathbf{Z} = \begin{pmatrix} \mathbf{A} & -\mathbf{Q} \\ \mathbf{0} & -\mathbf{A}^* \end{pmatrix}, \quad \mathbf{A} \in \mathbb{R}^{n\times n}, \quad \mathcal{R}e\lambda_i(\mathbf{A}) < 0 \;\Rightarrow\; \mathbf{Z}_j = \begin{pmatrix} \mathbf{A}_j & -\mathbf{Q}_j \\ \mathbf{0} & -\mathbf{A}_j^* \end{pmatrix}$$

The iterations can also be written as follows

$$\mathbf{A}_{j+1} = \frac{1}{2}\left(\mathbf{A}_j + \mathbf{A}_j^{-1}\right), \; \mathbf{A}_0 = \mathbf{A}; \;\; \mathbf{Q}_{j+1} = \frac{1}{2}\left(\mathbf{Q}_j + \mathbf{A}_j^{-1}\mathbf{Q}_j\mathbf{A}_j^{-*}\right), \; \mathbf{Q}_0 = \mathbf{Q}.$$

The limits of these iterations are $\mathbf{A}_\infty = -\mathbf{I}_n$ and $\mathbf{Q}_\infty = 2\mathcal{P}$, where $\mathcal{P}$ is the solution of the Lyapunov equation.

(d) The convergence of the sign iteration is ultimately *quadratic*.

(e) Often, in the solution of the Lyapunov equation the constant term is provided in factored form $\mathbf{Q} = \mathbf{R}\mathbf{R}^*$. As a consequence we can obtain the $(j+1)^{\mathrm{st}}$ iterate in factored form

$$\mathbf{Q}_{j+1} = \mathbf{R}_{j+1}\mathbf{R}_{j+1}^* \;\text{ where }\; \mathbf{R}_{j+1} = \frac{1}{\sqrt{2}}\left[\mathbf{R}_j, \;\; \mathbf{A}_j^{-1}\mathbf{R}_j\right] \;\Rightarrow\; \mathbf{Q}_\infty = \mathbf{R}_\infty\mathbf{R}_\infty^* = 2\mathcal{P}$$

Thus the solution is obtained in factored form. However, $\mathbf{R}_\infty$ has infinitely many columns, although its rank cannot exceed n. In order to avoid this, a rank revealing RQ factorization is needed: $\mathbf{R}_j = \mathbf{T}_j\mathbf{U}_j$, where $\mathbf{T}_j = [\Delta_j^*, \;\; \mathbf{0}]^*$, Δ_j is upper triangular and $\mathbf{U}_j\mathbf{U}_j^* = \mathbf{I}_j$. Thus at the j^{th} step $\mathbf{R}_j$ can be replaced by $\mathbf{T}_j$ which has exactly as many columns as the rank of $\mathbf{R}_j$.

(f) The sign function was first used for solving the Lyapunov equation by Roberts. For recent results see Benner.

(g) For the solution of large sparse Lyapunov equations the ADI ieration can be used (will not be discussed). ■

Examples

Consider the number z=2. We perform the iteration $z_{n+1} = \frac{1}{2}(z_n + \frac{1}{z_n})$, $z_0 = 2$. The iterates converge to 1. The error $e_n = z_n - 1$ is as follows:

$e_1 = 2.5000\,10^{-1}$
$e_2 = 2.5000\,10^{-2}$
$e_3 = 3.0488\,10^{-4}$
$e_4 = 3.6461\,10^{-8}$
$e_5 = 1.0793\,10^{-15}$
$e_6 = 5.8246\,10^{-31}$
$e_7 = 1.6963\,10^{-61}$
$e_8 = 1.4388\,10^{-122}$

This shows that convergence is fast (the exponent of the error doubles at each iteration). ■

Examples

Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 0 & -1 \\ -1 & 2 \\ -2 & -7 \end{bmatrix}.$$

We form the matrix

$$\mathbf{Z} = \left[\begin{array}{c|c} \mathbf{A} & -\mathbf{C} \\ \hline \mathbf{0} & -\mathbf{B} \end{array}\right].$$

Since both **A** and **B** are composed of one Jordan block with eigenvalues -1, the iteration will converge in finitely many steps. Indeed:

$$\mathbf{Z}_1 = \left[\begin{array}{ccc|cc} -3/2 & -1 & -1/2 & 3 & 3 \\ 1/2 & 0 & 1/2 & 1 & -1 \\ -1/2 & -1 & -3/2 & -1 & 3 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right], \ \mathbf{Z}_2 = \left[\begin{array}{ccc|cc} -1 & 0 & 0 & 2 & 2 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & -2 & 2 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Therefore the solution of the Sylvester equation $\mathbf{AX} + \mathbf{XB} + \mathbf{C} = \mathbf{0}$ is

$$\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

■

SLICOT toolbox

SLICOT provides (high performance) routines linked to MATLAB with Control Toolbox functions :

Linear Matrix Equation Solver	
sldisy	Solves discrete-time Sylvester equations
sllyap	Solves Lyapunov equations
slstei	Solves Stein equations
slstly	Solves stable factorized Lyapunov equations
slstst	Solves stable factorized Stein equations
slsylv	Solves Sylvester equations

Generalized Linear Matrix Equation Solver	
slgely	Solves generalized Lyapunov equations
slgesg	Solves generalized pairs of matrix equations
slgest	Solves generalized Stein equations
slgsly	Solves generalized stable Lyapunov equations
slgstst	Solves generalized stable Stein equations
slgesy	Solves generalized Sylvester equations

Algebraic Riccati Equation Solver	
slcaregs	Solves CARE with generalized Schur method
slcares	Solves CARE with Schur method
sldaregs	Solves DARE with generalized Schur method
sldares	Solves DARE with Schur method
sldaregsv	Solves DARE with double generalized Schur

SLICOT toolbox

Controllability, Observability	
slconf	Computes controllability staircase form
slminr	Computes minimal realization sub-system
slobsf	Computes observability staircase form

Transformation Routines	
slsbal	Balances the system
slsdec	Transforms to ordered block diagonal form
slsorsf	Transforms to ordered Schur form
slsrsf	Transforms state matrix to Schur form

Coprime Factorization Routines	
lcf	State-space left coprime factorization
rcf	State-space right coprime factorization
lcfid	State-space left inner coprime factorization
rcfid	State-space right inner coprime factorization

Approximation Routines	
bta	balancing-free square-root truncation
btabal	square-root truncation
bta_cf	balancing-free square-root truncation cop. fact.
btabal_cf	square-root truncation with cop. fact.
spa	balancing-free square-root Sing. Pert. Appr.
spabal	square-root Sing. Pert. Appr.
spa_cf	balancing-free square-root SPA with copr. fact.
spabal_cf	square-root SPA with copr. fact.
hna	for HNA with additive spectral decomposition

Free via `http://www.slicot.org/`

Outline

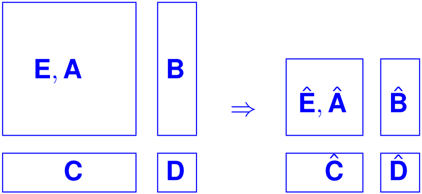
Linear Time Invariant Systems (Implicit)

$$\left\{\begin{array}{l} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t). \end{array}\right. \Rightarrow \mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$$

$\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^p$, $\mathbf{x}(t) \in \mathbb{R}^n$, $n \ll m, p$. Find a low order system driven by the same input but with small output error $\|\mathbf{y}(t) - \hat{\mathbf{y}}(t)\|$:

$$\left\{\begin{array}{l} \hat{\mathbf{E}}\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t) \\ \hat{\mathbf{y}}(t) = \hat{\mathbf{C}}\hat{\mathbf{x}}(t) + \hat{\mathbf{D}}\mathbf{u}(t), \end{array}\right. \Rightarrow \hat{\mathbf{H}}(s) = \hat{\mathbf{C}}(s\hat{\mathbf{E}} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}},$$

where $\hat{\mathbf{y}}(t) \in \mathbb{R}^p$, $\hat{\mathbf{x}}(t) \in \mathbb{R}^n$.



Without loss of generality we may assume $\mathbf{D} = \hat{\mathbf{D}} = \mathbf{0}$.

New definition of gramians

Reachability and observability gramian can be defined as

$$\mathcal{P} = \int_0^{+\infty} (e^{\mathbf{E}^{-1}\mathbf{A}t}\mathbf{E}^{-1}\mathbf{B})(e^{\mathbf{E}^{-1}\mathbf{A}t}\mathbf{E}^{-1}\mathbf{B})^* dt, \;\; \mathcal{Q} = \int_0^{+\infty} (\mathbf{C}\mathbf{E}^{-1}e^{\mathbf{A}\mathbf{E}^{-1}t})^*(\mathbf{C}\mathbf{E}^{-1}e^{\mathbf{A}\mathbf{E}^{-1}t}) dt.$$

By Parseval's theorem these are also equal to

$$\mathcal{P} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\jmath\omega\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}\mathbf{B}^*(\jmath\omega\mathbf{E}-\mathbf{A})^{-*} d\omega, \;\; \mathcal{Q} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\jmath\omega\mathbf{E}-\mathbf{A})^{-*}\mathbf{C}^*\mathbf{C}(\jmath\omega\mathbf{E}-\mathbf{A})^{-1} d\omega.$$

Modified gramians can be computed from the “symmetric” formulae:

$$\mathbf{A}\hat{\mathcal{P}}\mathbf{E}^* + \mathbf{E}\hat{\mathcal{P}}\mathbf{A}^* + \mathbf{B}\mathbf{B}^* = \mathbf{0}, \;\; \mathbf{A}^*\hat{\mathcal{Q}}\mathbf{E} + \mathbf{E}^*\hat{\mathcal{Q}}\mathbf{A} + \mathbf{C}^*\mathbf{C} = \mathbf{0}.$$

There holds $\mathcal{P} = \hat{\mathcal{P}}$ and $\mathcal{Q} = \mathbf{E}^*\hat{\mathcal{Q}}\mathbf{E}$ and the Hankel singular values are:

$$\sigma_i(\mathcal{H}) = \sqrt{\lambda_i(\hat{\mathcal{P}}\mathbf{E}^*\hat{\mathcal{Q}}\mathbf{E})}$$

Some system norms

Σ system	$\mathcal{S}$ Convolution op	$\mathcal{H}$ Hankel op	$\mathbf{H}(s)$ Transfer func	$\mathbf{h}(t)$ Impulse resp	Expression/ Interpretation
$\\|\Sigma\\|_{\mathcal{H}_2}$ $\mathcal{H}_2$-norm			$\\|\mathbf{H(s)}\\|_{\mathcal{H}_2}$ Hardy space norm	$\\|\mathbf{h(t)}\\|_{\mathcal{L}_2}$	$\sqrt{\text{trace } C\mathcal{P}C^*} = \sqrt{\text{trace } B^*\mathcal{Q}B}$ RMS value of **y** for white noise **u**
$\\|\Sigma\\|_{\mathcal{H}_\infty}$ $\mathcal{H}_\infty$-norm	$\\|\mathcal{S}\\|_{2-\text{ind}}$		$\\|\mathbf{H}(s)\\|_{\mathcal{H}_\infty}$ Hardy space norm		$\sup_\omega \sigma_{\max}\mathbf{H}(j\omega)$ Peak of amplitude Bode plot
$\\|\Sigma\\|_1$ 1-norm	$\\|\mathcal{S}\\|_{1-\text{ind}}$				$\left[\int \lvert\mathbf{h}_{ij}\rvert\right]_1 = \max_i \sum_j \int \lvert\mathbf{h}_{ij}\rvert$ Peak column gain
$\\|\Sigma\\|_\infty$ ∞-norm	$\\|\mathcal{S}\\|_{\infty-\text{ind}}$				$\left[\int \lvert\mathbf{h}_{ij}\rvert\right]_\infty = \max_j \sum_i \int \lvert\mathbf{h}_{ij}\rvert$ Peak row gain
$\\|\Sigma\\|_H$ Hankel norm		$\\|\mathcal{H}\\|_{2-\text{ind}}$			$\sqrt{\lambda_{\max}(\mathcal{P}\mathcal{Q})}$
$\\|\Sigma\\|_{HS}$ Hilbert-Schmidt		$\\|\mathcal{H}\\|_F$ Frobenius norm			$\sqrt{\sum_k \lambda_k(\mathcal{P}\mathcal{Q})}$ Area of Nyquist plot

Remarks: For SISO systems the following hold.

(a) The $\mathcal{H}_2$ norm of Σ is equal to the $2, \infty$ induced norm of the convolution operator $\mathcal{S}$.

(b) The ∞ norm of Σ is equal to the total variation of the step response.