Model reduction of large-scale dynamical systems

Lecture III: Krylov approximation and rational interpolation

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Outline

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Krylov approximation methods

Given $\Sigma = \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right)$, expand the transfer function around s_0:

$$\mathbf{H}(s) = \eta_0 + \eta_1(s - s_0) + \eta_2(s - s_0)^2 + \eta_3(s - s_0)^3 + \cdots$$

Moments at s_0: η_j, $j \geq 0$. Find $\hat{\Sigma} = \left(\begin{array}{c|c} \hat{\mathbf{A}} & \hat{\mathbf{B}} \\ \hline \hat{\mathbf{C}} & \hat{\mathbf{D}} \end{array}\right)$, with

$$\hat{\mathbf{H}}(s) = \hat{\eta}_0 + \hat{\eta}_1(s - s_0) + \hat{\eta}_2(s - s_0)^2 + \hat{\eta}_3(s - s_0)^3 + \cdots$$

such that for appropriate k:

$$\boxed{\eta_j = \hat{\eta}_j,\ j = 1, 2, \cdots, k}$$

Moment matching methods can be implemented in a **numerically stable** and **efficient way**.

Krylov approximation methods: Special cases

- $s_0 = \infty$ *Moments: Markov parameters*
  Problem: (partial) realization
  Solution computed through: Lanczos and Arnoldi procedures
- $s_0 = 0$
  Problem: Padé approximation
  Solution computed through: Lanczos and Arnoldi procedures
- *In general: arbirtary* $s_0 \in \mathbb{C}$
  Problem: Rational interpolation
  Solution computed through: Rational Lanczos

- **Computation of moments**: numerically problematic
- **Key fact for numerical reliability**: If $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ given
- **moment matching without moment computation**
  $\Rightarrow$ **iterative implementation.**

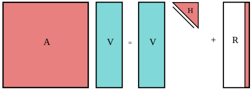
Outline

The Arnoldi procedure

Given is $\mathbf{A} \in \mathbb{R}^{n\times n}$, and $\mathbf{b} \in \mathbb{R}^n$. Let $\mathcal{R}_k(\mathbf{A}, \mathbf{b}) \in \mathbb{R}^{n\times k}$ be the reachability or Krylov matrix. It is *assumed* that $\mathcal{R}_k$ has full column rank equal to k.

Devise a process which is iterative and at the k^{th} step we have

$$\mathbf{A}\mathbf{V}_k = \mathbf{V}_k\mathbf{H}_k + \mathbf{R}_k,\ \mathbf{V}_k,\ \mathbf{R}_k \in \mathbb{R}^{n\times k},\ \mathbf{H}_k \in \mathbb{R}^{k\times k},\ k = 1, 2, \cdots, n$$



These quantities have to satisfy the following conditions at each step.

- The columns of $\mathbf{V}_k$ are orthonormal: $\mathbf{V}_k^*\mathbf{V}_k = \mathbf{I}_k,\ k = 1, 2, \cdots, n$.
- $\text{span col}\, \mathbf{V}_k = \text{span col}\, \mathcal{R}_k(\mathbf{A}, \mathbf{b}),\ k = 1, 2, \cdots, n$
- The residual R_k satisfies the Galerkin condition: $\mathbf{V}_k^*\mathbf{R}_k = 0$, $k = 1, 2, \cdots, n$.

This problem can be solved by the **Arnoldi procedure**.

Arnoldi: recursive implementation

Given: $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{b} \in \mathbb{R}^n$
Find: $\mathbf{V} \in \mathbb{R}^{n\times k}$, $\mathbf{f} \in \mathbb{R}^n$, and $\mathbf{H} \in \mathbb{R}^{k\times k}$, such that

$$\mathbf{AV} = \mathbf{VH} + \mathbf{f}\mathbf{e}_k^* \quad \text{where} \quad \mathbf{H} = \mathbf{V}^*\mathbf{AV},\ \mathbf{V}^*\mathbf{V} = \mathbf{I}_k,\ \mathbf{V}^*\mathbf{f} = 0,$$

with $\mathbf{H}$ in *upper Hessenberg* form.

1. $\mathbf{v}_1 = \frac{\mathbf{b}}{\|\mathbf{b}\|}$, $\mathbf{w} = \mathbf{A}\mathbf{v}_1$; $\alpha_1 = \mathbf{v}_1^*\mathbf{w}$
   $\mathbf{f}_1 = \mathbf{w} - \mathbf{v}_1\alpha_1$; $\mathbf{V}_1 = (\mathbf{v}_1)$; $\mathbf{H}_1 = (\alpha_1)$
2. For $j = 1, 2, \cdots, k-1$
   1. $\beta_j = \| \mathbf{f}_j \|$, $\mathbf{v}_{j+1} = \frac{\mathbf{f}_j}{\beta_j}$
   2. $\mathbf{V}_{j+1} = \left(\mathbf{V}_j \ \ \mathbf{v}_{j+1}\right)$, $\hat{\mathbf{H}}_j = \begin{pmatrix} \mathbf{H}_j \\ \beta_j \mathbf{e}_j^* \end{pmatrix}$
   3. $\mathbf{w} = \mathbf{A}\mathbf{v}_{j+1}$, $\mathbf{h} = \mathbf{V}_{j+1}^*\mathbf{w}$, $\mathbf{f}_{j+1} = \mathbf{w} - \mathbf{V}_{j+1}\mathbf{h}$
   4. $\mathbf{H}_{j+1} = \left(\hat{\mathbf{H}}_j \ \ \mathbf{h}\right)$

Properties of Arnoldi

- $\mathbf{H}_k$ is obtained by projecting A onto the span of the columns of $\mathbf{V}_k$: $\mathbf{H}_k = \mathbf{V}_k^* \mathbf{A} \mathbf{V}_k$.
- The remainder $\mathbf{R}_k$ has rank one and can be written as $\mathbf{R}_k = \mathbf{r}_k \mathbf{e}_k^*$, where $\mathbf{e}_k$ is the k^{th} unit vector; thus $\mathbf{r}_k \perp \mathcal{R}_k$.
- This further implies that $\mathbf{v}_{k+1} = \frac{\mathbf{r}_k}{\|\mathbf{r}_k\|}$, where $\mathbf{v}_{k+1}$ is the $(k+1)^{\text{st}}$ column of $\mathbf{V}$. Consequently, $\mathbf{H}_k$ is an **upper Hessenberg** matrix.

$$\mathbf{H}_k = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} & \cdots & h_{1,k-1} & h_{1,k} \\ h_{2,1} & h_{2,2} & h_{2,3} & \cdots & h_{2,k-1} & h_{2,k} \\ & h_{3,2} & h_{3,3} & & h_{3,k-1} & h_{3,k} \\ & & & \ddots & \vdots & \vdots \\ & & & & h_{k-1,k-1} & h_{k-1,k} \\ & & & & h_{k,k-1} & h_{k,k} \end{bmatrix}$$

- Let $\mathbf{p}_k(\lambda) = \det(\lambda \mathbf{I}_k - \mathbf{H}_k)$, be the characteristic polynomial of $\mathbf{H}_k$. This monic polynomial is the solution of the following minimization problem

$$\mathbf{p}_k = \arg\min \|\mathbf{p}(\mathbf{A})\mathbf{b}\|_2$$

where the minimum is taken over all *monic* polynomials $\mathbf{p}$ of degree k. Since $\mathbf{p}_k(\mathbf{A})\mathbf{b} = \mathbf{A}^k\mathbf{b} + \mathcal{R}_k \cdot \underline{\mathbf{p}}$, where $\underline{\mathbf{p}}_{i+1}$ is the coefficient of λ^i of the polynomial $\mathbf{p}_k$, it also follows that the coefficients of $\mathbf{p}_k$ provide the least squares fit between $\mathbf{A}^k\mathbf{b}$ and the columns of $\mathcal{R}_k$.

- There holds

$$\mathbf{r}_k = \frac{1}{\|\mathbf{p}_{k-1}(\mathbf{A})\mathbf{b}\|} \mathbf{p}_k(\mathbf{A})\mathbf{b}, \quad \mathbf{H}_{k,k-1} = \frac{\|\mathbf{p}_k(\mathbf{A})\mathbf{b}\|}{\|\mathbf{p}_{k-1}(\mathbf{A})\mathbf{b}\|}$$

An alternative way of looking at Arnoldi

Consider a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, a starting vector $\mathbf{b} \in \mathbb{R}^{n}$, and the corresponding reachability matrix $\mathcal{R}_n = [\mathbf{b}\ \ \mathbf{A}\mathbf{b}\ \ \cdots\ \ \mathbf{A}^{n-1}\mathbf{b}]$. The following relationship holds true:

$$\mathbf{A}\mathcal{R}_n = \mathcal{R}_n\mathbf{F}\ \ \text{where}\ \ \mathbf{F} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & \cdots & 0 & -\alpha_2 \\ & & \vdots & & \\ 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{pmatrix}$$

and $\chi_{\mathbf{A}}(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$, is the characteristic polynomial of $\mathbf{A}$. Compute the QR factorization of $\mathcal{R}_n$:

$$\mathcal{R}_n = \mathbf{V}\mathbf{U},\ \ \mathbf{V}^*\mathbf{V} = I_n,\ \ \mathbf{U}\ \text{upper triangular}$$

It follows that

$$\mathbf{AVU} = \mathbf{VUF} \;\Rightarrow\; \mathbf{AV} = \mathbf{V}\underbrace{\mathbf{UFU}^{-1}}_{\bar{\mathbf{A}}} \;\Rightarrow\; \mathbf{AV} = \mathbf{V}\bar{\mathbf{A}}$$

Since $\mathbf{U}$ is upper triangular, so is $\mathbf{U}^{-1}$; furthermore $\mathbf{F}$ is upper Hessenberg. Therefore $\bar{\mathbf{A}}$ being the product of an upper triangular times an upper Hessenberg times an upper triangular matrix is *upper Hessenberg*.

The k-step Arnoldi factorization can now be obtained by considering the first k columns of the above relationship, to wit:

$$[\mathbf{AV}]_k = \left[\mathbf{V}\bar{\mathbf{A}}\right]_k \;\Rightarrow\; \mathbf{A}[\mathbf{V}]_k = [\mathbf{V}]_k\bar{\mathbf{A}}_{kk} + \mathbf{f}\mathbf{e}_k^*$$

where $\mathbf{f}$ is a multiple of the $(k+1)$-st column of $\mathbf{V}$. Notice that $\bar{\mathbf{A}}_{kk}$ is still upper Hessenberg, while the columns of $[\mathbf{V}]_k$ provide an orthonormal basis for the space spanned by the first k columns of the reachability matrix $\mathcal{R}_n$.

The symmetric Lanczos procedure

If $\mathbf{A} = \mathbf{A}^*$ then the Arnoldi procedure is the same as the symmetric Lanczos procedure. In this case $\mathbf{H}_k$ is tridiagonal:

$$\mathbf{H}_k = \begin{bmatrix} \alpha_1 & \beta_2 & & & & \\ \beta_2 & \alpha_2 & \beta_3 & & & \\ & \beta_3 & \alpha_3 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & & \alpha_{k-1} & \beta_k \\ & & & & \beta_k & \alpha_k \end{bmatrix}$$

This matrix shows that the vectors in the Lanczos procedure satisfy a *three term recurrence* relationship

$$\mathbf{A}\mathbf{v}_i = \beta_{i+1}\mathbf{v}_{i+1} + \alpha_i\mathbf{v}_i + \beta_i\mathbf{v}_{i-1},\ i = 1, 2, \cdots, k-1$$

Remark. If the remainder $\mathbf{r}_k = 0$, the procedure has terminated, in which case if $(\lambda, \mathbf{x})$ is an eigenpair of $\mathbf{H}_k$, $(\lambda, \mathbf{V}_k x)$ is an eigenpair of $\mathbf{A}$ (since $\mathbf{H}_k\mathbf{x} = \lambda\mathbf{x}$ implies $\mathbf{A}\mathbf{V}_k\mathbf{x} = \mathbf{V}_k\mathbf{H}_k\mathbf{x} = \lambda\mathbf{V}_k\mathbf{x}$).

Two-sided Lanczos

The two-sided Lanczos procedure. Given $\mathbf{A} \in \mathbb{R}^{n \times n}$ which is not symmetric, and two vectors $\mathbf{b}, \mathbf{c}^* \in \mathbb{R}^n$, devise a process which is iterative and the k^{th} step there holds:

$$\mathbf{A}\mathbf{V}_k = \mathbf{V}_k\mathbf{H}_k + \mathbf{R}_k, \;\; \mathbf{A}^*\mathbf{W}_k = \mathbf{W}_k\mathbf{H}_k + \mathbf{S}_k, \quad k = 1, 2, \cdots, n.$$

- Biorthogonality: $\mathbf{W}_k^*\mathbf{V}_k = \mathbf{I}_k$,
- $\text{span col}\,\mathbf{V}_k = \text{span col}\,\mathcal{R}_k(\mathbf{A}, \mathbf{b})$, $\text{span col}\,\mathbf{W}_k = \text{span col}\,\mathcal{R}_k(\mathbf{A}^*, \mathbf{c}^*)$,
- Galerkin conditions: $\mathbf{V}_k^*\mathbf{S}_k = 0$, $\mathbf{W}_k^*\mathbf{R}_k = 0$, $k = 1, 2, \cdots, n$.

Remarks.

- The second condition of the second item above can also be expressed as $\text{span rows}\,\mathbf{W}_k^* = \text{span rows}\,\mathcal{O}_k(c, A)$, where $\mathcal{O}_k$ is the observability matrix of the pair $(\mathbf{c}, \mathbf{A})$.
- The *assumption* for the solvability of this problem is $\det \mathcal{O}_k(\mathbf{c}, \mathbf{A})\mathcal{R}_k(\mathbf{A}, \mathbf{b}) \neq 0$, $k = 1, 2, \cdots, n$.
- The associated *Lanczos polynomials* are defined as $\mathbf{p}_k(\lambda) = \det(\lambda \mathbf{I}_k - \mathbf{H}_k)$, and the induced inner product is defined as $\langle \mathbf{p}(\lambda), \mathbf{q}(\lambda) \rangle = \langle \mathbf{p}(\mathbf{A}^*)\mathbf{c}^*, \mathbf{q}(A)\mathbf{b} \rangle = \mathbf{c}\,\mathbf{p}(A) \cdot \mathbf{q}(A)\,\mathbf{b}$.

Two-sided Lanczos: recursive implementation

Given: the triple $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{b}, \mathbf{c}^* \in \mathbb{R}^n$
Find: $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n\times k}$, , $\mathbf{g} \in \mathbb{R}^{\mathbf{n}}$, and $\mathbf{H} \in \mathbb{R}^{k\times k}$, such that

$$\mathbf{AV} = \mathbf{VH} + \mathbf{f}\mathbf{e}_k^*, \;\; \mathbf{A}^*\mathbf{W} = \mathbf{W}\mathbf{H}^* + \mathbf{g}\mathbf{e}_k^* \quad \text{where}$$

$\mathbf{H} = \mathbf{V}^*\mathbf{AW}$, $\mathbf{V}^*\mathbf{W} = \mathbf{I}_k$, $\mathbf{W}^*\mathbf{f} = 0$, $\mathbf{V}^*\mathbf{g} = 0$. The projections π_L and π_U above, are given by $\mathbf{V}^*$, $\mathbf{W}$, respectively.

1. $\beta_1 := \sqrt{|\mathbf{b}^*\mathbf{c}^*|}$, $\gamma_1 := \operatorname{sgn}(\mathbf{b}^*\mathbf{c}^*)\beta_1$
   $\mathbf{v}_1 = \mathbf{b}/\beta_1$, $\mathbf{w}_1 := \mathbf{c}^*/\gamma_1$
2. For $j = 1, \cdots, k$, set
   1. $\alpha_j = \mathbf{w}_j^*\mathbf{A}\mathbf{v}_j$
   2. $\mathbf{r}_j = \mathbf{A}\mathbf{v}_j - \alpha_j\mathbf{v}_j - \gamma_j\mathbf{v}_{j-1}$, $\mathbf{q}_j = \mathbf{A}^*\mathbf{w}_j - \alpha_j\mathbf{w}_j - \beta_j\mathbf{w}_{j-1}$
   3. $\beta_{j+1} = \sqrt{|\mathbf{r}_j^*\mathbf{q}_j|}$, $\gamma_{j+1} = \operatorname{sgn}(\mathbf{r}_j^*\mathbf{q}_j)\beta_{j+1}$
   4. $\mathbf{v}_{j+1} = \mathbf{r}_j/\beta_{j+1}$, $\mathbf{w}_{j+1} = \mathbf{q}_j/\gamma_{j+1}$

Properties of two-sided Lanczos

- $\mathbf{H}_k$ is obtained by projecting A as follows: $\mathbf{H}_k = \mathbf{W}_k^* \mathbf{A} \mathbf{V}_k$.
- The remainders $\mathbf{R}_k$, $\mathbf{S}_k$ have rank one and can be written as $\mathbf{R}_k = \mathbf{r}_k \mathbf{e}_k^*$, $\mathbf{S}_k = \mathbf{q}_k \mathbf{e}_k^*$.
- This further implies that $\mathbf{v}_{k+1}$, $\mathbf{w}_{k+1}$ are scaled versions of $\mathbf{r}_k$, $\mathbf{q}_k$ respectively Consequently, $\mathbf{H}_k$ is a **tridiagonal** matrix.
- The generalized Lanczos polynomials $\mathbf{p}_k(\lambda) = \det(\lambda \mathbf{I}_k - \mathbf{H}_k)$, $k = 0, 1, \cdots, n-1$, $\mathbf{p}_0 = 1$, are orthogonal: $\langle \mathbf{p}_i, \mathbf{p}_j \rangle = 0$, for $i \neq j$.
- The columns of $\mathbf{V}_k$, $\mathbf{W}_k$ and the Lanczos polynomials satisfy the following three-term recurrences

$$
\begin{array}{lcllcl}
\gamma_k \mathbf{v}_{k+1} & = & (\mathbf{A} - \alpha_k)\mathbf{v}_k & - & \beta_{k-1}\mathbf{v}_{k-1} \\
\beta_k \mathbf{w}_{k+1} & = & (\mathbf{A}^* - \alpha_k)\mathbf{w}_k & - & \gamma_{k-1}\mathbf{w}_{k-1} \\
\gamma_k \mathbf{p}_{k+1}(\lambda) & = & (\lambda - \alpha_k)\mathbf{p}_k(\lambda) & - & \beta_{k-1}\mathbf{p}_{k-1}(\lambda) \\
\beta_k \mathbf{q}_{k+1}(\lambda) & = & (\lambda - \alpha_k)\mathbf{q}_k(\lambda) & - & \gamma_{k-1}\mathbf{q}_{k-1}(\lambda)
\end{array}
$$

Example: symmetric Lanczos

Consider the following symmetric matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

With the starting vector $\mathbf{b} = [1 \;\; 0 \;\; 0 \;\; 0]^*$, we obtain

$$\mathbf{V}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{H}_1 = [2], \mathbf{R}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{V}_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{6}} \end{bmatrix}, \mathbf{H}_2 = \begin{bmatrix} 2 & \sqrt{6} \\ \sqrt{6} & \frac{8}{3} \end{bmatrix}, \mathbf{R}_2 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{54}} \\ 0 & \frac{-1}{\sqrt{54}} \\ 0 & \frac{1}{\sqrt{54}} \end{bmatrix}$$

$$\mathbf{V}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \mathbf{H}_3 = \begin{bmatrix} 2 & \sqrt{6} & 0 \\ \sqrt{6} & \frac{8}{3} & \frac{1}{\sqrt{18}} \\ 0 & \frac{1}{\sqrt{18}} & \frac{4}{3} \end{bmatrix}, \mathbf{R}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-\sqrt{3}}{2} \end{bmatrix}$$

$$\mathbf{V}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \end{bmatrix}, \mathbf{H}_4 = \begin{bmatrix} 2 & \sqrt{6} & 0 & 0 \\ \sqrt{6} & \frac{8}{3} & \frac{1}{\sqrt{18}} & 0 \\ 0 & \frac{1}{\sqrt{18}} & \frac{4}{3} & \frac{\sqrt{3}}{\sqrt{2}} \\ 0 & 0 & \frac{\sqrt{3}}{\sqrt{2}} & 0 \end{bmatrix}, \mathbf{R}_4 = 0_{4\times 4}$$

where

$$\mathbf{A}\mathbf{V}_k = \mathbf{V}_k\mathbf{H}_k + \mathbf{R}_k,\ \mathbf{V}_k^*\mathbf{R}_k = 0 \ \Rightarrow\ \mathbf{H}_k = \mathbf{V}_k^*\mathbf{A}\mathbf{V}_k,\ k = 1, 2, 3, 4.$$

Outline

Arnoldi and moment matching

The Arnoldi factorization can be used for model reduction as follows. Recall the QR factorization of the reachability matrix $\mathcal{R}_k \in \mathbb{R}^{n\times k}$; a projection $\mathbf{V}\mathbf{V}^*$ can then be attached to this factorization:

$$\mathcal{R}_k = \mathbf{V}\mathbf{U} \;\Rightarrow\; \mathbf{V} = \mathcal{R}_k \mathbf{U}^{-1}$$

where $\mathbf{V} \in \mathbb{R}^{n\times k}$, $\mathbf{V}^*\mathbf{V} = \mathbf{I}_k$, and $\mathbf{U}$ is upper triangular. The reduced order system is:

$$\bar{\Sigma} = \left(\begin{array}{c|c} \bar{\mathbf{A}} & \bar{\mathbf{B}} \\ \hline \bar{\mathbf{C}} & \end{array}\right) \quad \text{where} \quad \boxed{\bar{\mathbf{A}} = \mathbf{V}^*\mathbf{A}\mathbf{V}},\; \boxed{\bar{\mathbf{B}} = \mathbf{V}^*\mathbf{B}},\; \boxed{\bar{\mathbf{C}} = \mathbf{C}\mathbf{V}}$$

Theorem. $\bar{\Sigma}$ as defined above satisfies the equality of the Markov parameters $\hat{\eta}_i = \eta_i$, $i = 1, \cdots, k$. Furthermore, $\bar{\mathbf{A}}$ is in Hessenberg form, and $\bar{\mathbf{B}}$ is a multiple of the unit vector $\mathbf{e}_1$.

Proof. First notice that since $\mathbf{U}$ is upper triangular, $\mathbf{v}_1 = \frac{\mathbf{B}}{\|\mathbf{B}\|}$, and since $\mathbf{V}^*\mathcal{R}_k = \mathbf{U}$ it follows that $\bar{\mathbf{B}} = \mathbf{u}_1 = \| \mathbf{B} \| \, \mathbf{e}_1$; therefore $\bar{\mathbf{B}} = \mathbf{V}^*\mathbf{B}$. $\mathbf{V}\mathbf{V}^*\mathbf{B} = \mathbf{V}\bar{\mathbf{B}} = \mathbf{B}$, hence $\bar{\mathbf{A}}\bar{\mathbf{B}} = \mathbf{V}^*\mathbf{A}\mathbf{V}\mathbf{V}^*\mathbf{B} = \mathbf{V}^*\mathbf{A}\mathbf{B}$; in general, since $\mathbf{V}\mathbf{V}^*$ is a projection along the columns of $\mathcal{R}_k$, we have $\mathbf{V}\mathbf{V}^*\mathcal{R}_k = \mathcal{R}_k$; moreover: $\hat{\mathcal{R}}_k = \mathbf{V}^*\mathcal{R}_k$; hence

$$(\hat{\eta}_1 \ \cdots \ \hat{\eta}_k) = \hat{\mathbf{C}}\hat{\mathcal{R}}_k = \mathbf{C}\mathbf{V}\mathbf{V}^*\mathcal{R}_k = \mathbf{C}\mathcal{R}_k = (\eta_1 \ \cdots \ \eta_k)$$

Finally, the upper triangularity of $\mathbf{U}$ implies that $\mathbf{A}$ is in Hessenberg form. ∎

Remark.

Similarly, one can show that reduction by means the two-sided Lanczos procedure preserves $2k$ Markov parameters.

Remarks

- The number of operations is $\mathcal{O}(k^2 n)$ vs. $\mathcal{O}(n^3)$, which implies **efficiency**. The requirement for memory is large if k is relatively large.
- Only *matrix-vector* multiplications are required. No matrix factorizations and/or inversions. There is no need to compute the transformed n-th order model and then truncate. This eliminates **ill-conditioning**.
- **Drawbacks:** • Numerical issue: Arnoldi/Lanczos methods loose orthogonality. This comes from the instability of the classical Gram-Schmidt procedure. Remedy: re-orthogonalization.
  • no global error bound.
  • $\hat{\Sigma}$ tends to approximate the high frequency poles of Σ. Remedy: match expansions around other frequencies $\Rightarrow$ **rational Lanczos**. ▪

Outline

Partial realization by projection

Given a system $\Sigma = (\mathbf{A}, \mathbf{B}, \mathbf{C})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B}, \mathbf{C}^* \in \mathbb{R}^n$, We seek a lower dimensional model $\hat{\Sigma} = (\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$, where $\hat{\mathbf{A}} \in \mathbb{R}^{k\times k}$, $\hat{\mathbf{B}}, \hat{\mathbf{C}}^* \in \mathbb{R}^k$, $k < n$, such that $\hat{\Sigma}$ preserves some properties of the original system, through appropriate projection methods. In other words, we seek $\mathbf{V} \in \mathbb{R}^{n\times k}$ and $\mathbf{W} \in \mathbb{R}^{n\times k}$ such that $\mathbf{W}^*\mathbf{V} = \mathbf{I}_k$, and the reduced system is given by:
$\hat{\mathbf{A}} = \mathbf{W}^*\mathbf{A}\mathbf{V}$, $\hat{\mathbf{B}} = \mathbf{W}^*\mathbf{B}$, $\hat{\mathbf{C}} = \mathbf{C}\mathbf{V}$.

Lemma

With $\mathbf{V} = [\mathbf{B},\ \mathbf{A}\mathbf{B},\ \cdots,\ \mathbf{A}^{k-1}\mathbf{B}] = \mathcal{R}_k(\mathbf{A}, \mathbf{B})$ and $\mathbf{W}$ any left inverse of V, $\hat{\Sigma}$ is a partial realization of Σ and matches k Markov parameters.

From a numerical point of view, one would not use **V** as defined above since usually the columns of **V** are almost linearly dependent. As it turns out any matrix whose column span is the same as that of **V** can be used.

Proof.

We have $\hat{\mathbf{C}}\hat{\mathbf{B}} = \mathbf{C}\mathbf{V}\mathbf{W}^*\mathbf{B} = \mathbf{C}\mathcal{R}_k(\mathbf{A}, \mathbf{B})\mathbf{e}_1 = \mathbf{C}\mathbf{B}$; furthermore

$$\hat{\mathbf{C}}\hat{\mathbf{A}}^j\hat{\mathbf{B}} = \mathbf{C}\mathcal{R}_k(\mathbf{A}, \mathbf{B})\mathbf{W}^*\mathbf{A}^j\mathcal{R}_k(\mathbf{A}, \mathbf{B})\mathbf{e}_1 = \mathbf{C}\mathcal{R}_k(\mathbf{A}, \mathbf{B})\mathbf{W}^*\mathbf{A}^j\mathbf{B} = \mathbf{C}\mathcal{R}_k(\mathbf{A}, \mathbf{B})\mathbf{e}_{j+1} = \mathbf{C}\mathbf{A}^j\mathbf{B},\ j = 1, \cdots, k-1.$$

□

Rational interpolation by projection

Suppose now that we are given k distinct points $s_j \in \mathbb{C}$. $\mathbf{V}$ is defined as the generalized reachability matrix

$$\mathbf{V} = \left[(s_1\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}, \;\cdots\;, \;(s_k\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\right],$$

and as before, let $\mathbf{W}$ be any left inverse of $\mathbf{V}$. Then

Lemma

$\hat{\Sigma}$ defined above, interpolates the transfer function of Σ at the s_j, that is

$$\mathbf{H}(s_j) = \mathbf{C}(s_j\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B} = \hat{\mathbf{C}}(s_j\mathbf{I}_k - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}} = \hat{\mathbf{H}}(s_j), \;\; j = 1, \cdots, k.$$

Proof.

The following string of equalities leads to the desired result:

$$\begin{aligned}
\hat{\mathbf{C}}(s_j\mathbf{I}_k - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}} &= \mathbf{C}\mathbf{V}(s_j\mathbf{I}_k - \mathbf{W}^*\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^*\mathbf{B} \\
&= \mathbf{C}\left[(s_1\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}, \;\cdots\;, \;(s_k\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\right]\left(\mathbf{W}^*(s_j\mathbf{I}_n - \mathbf{A})\mathbf{V}\right)^{-1}\mathbf{W}^*\mathbf{B} \\
&= \left[\mathbf{C}(s_1\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}, \;\cdots\;, \;\mathbf{C}(s_k\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\right]\left([\cdots \;\mathbf{W}^*\mathbf{B}\; \cdots]\right)^{-1}\mathbf{W}^*\mathbf{B} \\
&= \left[\mathbf{C}(s_1\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}, \;\cdots\;, \;\mathbf{C}(s_k\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\right]\mathbf{e}_j \\
&= \mathbf{C}(s_j\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}.
\end{aligned}$$

□

Matching points with multiplicity

We now wish to match the value of the transfer function at a given point $s_0 \in \mathbb{C}$, together with $k-1$ derivatives. For this we define the *generalized reachability matrix*

$$\mathbf{V} = \left[(s_0\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B},\ (s_0\mathbf{I}_n - \mathbf{A})^{-2}\mathbf{B},\ \cdots,\ (s_0\mathbf{I}_n - \mathbf{A})^{-k}\mathbf{B} \right],$$

together with any left inverse **W** thereof.

Lemma

$\hat{\Sigma}$ interpolates the transfer function of Σ at s_0, together with $k-1$ derivatives at the same point, $j = 0, 1, \cdots, k-1$:

$$\frac{(-1)^j}{j!}\ \frac{d^j}{ds^j}\mathbf{H}(s)\bigg|_{s=s_0} = \mathbf{C}(s_0\mathbf{I}_n - \mathbf{A})^{-(j+1)}\mathbf{B} = \hat{\mathbf{C}}(s_0\mathbf{I}_k - \hat{\mathbf{A}})^{-(j+1)}\hat{\mathbf{B}} = \frac{(-1)^j}{j!}\ \frac{d^j}{ds^j}\hat{\mathbf{H}}(s)\bigg|_{s=s_0}$$

Proof.

Let **V** be as defined above, and **W** be such that $\mathbf{W}^*\mathbf{V} = \mathbf{I}_k$. It readily follows that the projected matrix $s_0\mathbf{I}_r - \hat{\mathbf{A}}$ is in companion form (expression on the left) and therefore its powers are obtained by shifting its columns to the right:

$$s_0\mathbf{I}_k - \hat{\mathbf{A}} = \mathbf{W}^*(s_0\mathbf{I}_n - \mathbf{A})\mathbf{V} = [\mathbf{W}^*\mathbf{B},\ \mathbf{e}_1,\ \cdots,\ \mathbf{e}_{k-1}] \Rightarrow (s_0\mathbf{I}_k - \hat{\mathbf{A}})^{\ell} = [\underbrace{*\cdots *}_{\ell-1},\ \mathbf{W}^*\mathbf{B},\ \mathbf{e}_1,\ \cdots,\ \mathbf{e}_{k-\ell}].$$

Consequently $[\mathbf{W}^*(s_0\mathbf{I}_n - \mathbf{A})\mathbf{V}]^{-\ell}\mathbf{W}^*\mathbf{B} = \mathbf{e}_\ell$, which finally implies

$$\hat{\mathbf{C}}(s_0\mathbf{I}_k - \hat{\mathbf{A}})^{-\ell}\hat{\mathbf{B}} = \mathbf{CV}\,[\mathbf{W}^*(s_0\mathbf{I} - \mathbf{A})\mathbf{V}]^{-\ell}\ \mathbf{W}^*\mathbf{B} = \mathbf{CV}\mathbf{e}_\ell = \mathbf{C}(s_0\mathbf{I}_n - \mathbf{A})^{-\ell}\mathbf{B},\ \ell = 1, 2, \cdots, k.$$

□

General result: rational Krylov

A projector which is composed of any combination of the above three cases achieves matching of an appropriate number of Markov parameters and moments. Let the partial reachability matrix be

$$\mathcal{R}_k(\mathbf{A}, \mathbf{B}) = \left[\mathbf{B} \;\; \mathbf{AB} \;\; \cdots \;\; \mathbf{A}^{k-1}\mathbf{B}\right],$$

and partial generalized reachability matrix be:

$$\mathcal{R}_k(\mathbf{A}, \mathbf{B}; \sigma) = \left[(\sigma\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B} \;\; (\sigma\mathbf{I}_n - \mathbf{A})^{-2}\mathbf{B} \;\; \cdots \;\; (\sigma\mathbf{I}_n - \mathbf{A})^{-k}\mathbf{B}\right].$$

Rational Krylov

(a) If $\mathbf{V}$ as defined in the above three cases is replaced by $\bar{\mathbf{V}} = \mathbf{VR}$, $\mathbf{R} \in \mathbb{R}^{k\times k}$, $\det \mathbf{R} \neq 0$, and $\mathbf{W}$ by $\bar{\mathbf{W}} = \mathbf{R}^{-1}\mathbf{W}$, the same matching results hold true.
(b) Let $\mathbf{V}$ be such that

$$\operatorname{span\,col} \mathbf{V} = \operatorname{span\,col} \left[\mathcal{R}_k(\mathbf{A}, \mathbf{B}) \;\; \mathcal{R}_{m_1}(\mathbf{A}, \mathbf{B}; \sigma_1) \;\; \cdots \;\; \mathcal{R}_{m_\ell}(\mathbf{A}, \mathbf{B}; \sigma_\ell)\right],$$

and $\mathbf{W}$ any left inverse of $\mathbf{V}$. The reduced system matches k Markov parameters and m_i moments at $\sigma_i \in \mathbb{C}$, $i = 1, \cdots, \ell$.

Two-sided projections: the choice of **W**

Let $\mathcal{O}_k(\mathbf{C},\mathbf{A}) \in \mathbb{R}^{k\times n}$, be the partial observability matrix consisting of the first k rows of $\mathcal{O}_n(\mathbf{C},\mathbf{A}) \in \mathbb{R}^{n\times n}$. The first case is

$$\mathbf{V} = \mathcal{R}_k(\mathbf{A},\mathbf{B}),\ \ \mathbf{W} = (\underbrace{\mathcal{O}_k(\mathbf{C},\mathbf{A})\mathcal{R}_k(\mathbf{A},\mathbf{B})}_{\mathcal{H}_k})^{-1}\,\mathcal{O}_k(\mathbf{C},\mathbf{A}).$$

Lemma

Assuming that $\det \mathcal{H}_k \neq 0$, $\hat{\Sigma}$ *is a partial realization of* Σ *and matches* $2k$ *Markov parameters.*

Given $2k$ distinct points $s_1,\cdots,s_{2k}$, we will make use of the following generalized reachability and observability matrices:

$$\tilde{\mathbf{V}} = \left[(s_1\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\ \ \cdots\ \ (s_k\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\right],\ \ \tilde{\mathbf{W}} = \left[(s_{k+1}\mathbf{I}_n - \mathbf{A}^*)^{-1}\mathbf{C}^*\ \ \cdots\ \ (s_{2k}\mathbf{I}_n - \mathbf{A}^*)^{-1}\mathbf{C}^*\right].$$

Lemma

Assuming that $\det \tilde{\mathbf{W}}^*\tilde{\mathbf{V}} \neq 0$, *the projected system* $\hat{\Sigma}$ *where* $\mathbf{V} = \tilde{\mathbf{V}}$ *and* $\mathbf{W} = \tilde{\mathbf{W}}(\tilde{\mathbf{V}}^*\tilde{\mathbf{W}})^{-1}$ *interpolates the transfer function of* Σ *at the* $2k$ *points* s_i.

Remarks

(a) The same procedure as above can be used to approximate *implicit systems*, i.e., systems that are given in a generalized form $\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$, where $\mathbf{E}$ may be singular. The reduced system is given by

$$\hat{\mathbf{E}} = \mathbf{W}^*\mathbf{E}\mathbf{V},\ \hat{\mathbf{A}} = \mathbf{W}^*\mathbf{A}\mathbf{V},\ \hat{\mathbf{B}} = \mathbf{W}^*\mathbf{B},\ \hat{\mathbf{C}} = \mathbf{C}\mathbf{V},$$

where

$$\mathbf{W}^* = \begin{bmatrix} \mathbf{C}(s_{k+1}\mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(s_{2k}\ \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix},\ \mathbf{V} = \left[(s_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\ \ \cdots\ \ (s_k\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\right]$$

(b) Sylvester equations and projectors. The solution of an appropriate Sylvester equation $\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{H} + \mathbf{B}\mathbf{G} = \mathbf{0}$, provides a projector that interpolates the original system $\mathbf{C}, \mathbf{A}, \mathbf{B}$ at minus the eigenvalues of $\mathbf{H}$. Therefore the projectors above can be obtained by solving Sylvester equations. ■

Outline

Choice of Krylov projection points:
Optimal $\mathcal{H}_2$ model reduction

Recall: the $\mathcal{H}_2$ norm of a stable system is:

$$\|\Sigma\|_{\mathcal{H}_2} = \left(\int_{-\infty}^{+\infty} \mathbf{h}^2(t)dt\right)^{1/2}$$

where $\mathbf{h}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}$, $t \geq 0$, is the impulse response of Σ.

Goal: construct a *Krylov projector* such that

$$\Sigma_k = \arg \min_{\substack{\deg(\hat{\Sigma}) = r \\ \hat{\Sigma}\ :\ \text{stable}}} \left\|\Sigma - \hat{\Sigma}\right\|_{\mathcal{H}_2} = \left(\int_{-\infty}^{+\infty} (\mathbf{h} - \hat{\mathbf{h}})^2(t)dt\right)^{1/2}$$

First-order necessary optimality conditions

Let $(\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$ solve the optimal $\mathcal{H}_2$ problem and let $\hat{\lambda}_i$ denote the eigenvalues of $\hat{\mathbf{A}}$. The necessary conditions are

$$\mathbf{H}(-\hat{\lambda}_i^*) = \hat{\mathbf{H}}(-\hat{\lambda}_i^*) \;\text{ and }\; \frac{d}{ds}\mathbf{H}(s)\Big|_{s=-\hat{\lambda}_i^*} = \frac{d}{ds}\hat{\mathbf{H}}(s)\Big|_{s=-\hat{\lambda}_i^*}$$

Thus the reduced system has to match the first two moments of the original system at the *mirror images* of the eigenvalues of $\hat{\mathbf{A}}$.

The $\mathcal{H}_2$ norm: if $\mathbf{H}(s) = \sum_{k=1}^{n} \frac{\phi_k}{s-\lambda_k} \;\Rightarrow\;$ $\boxed{\|\mathbf{H}\|_{\mathcal{H}_2}^2 = \sum_{k=1}^{n} c_k\, \mathbf{H}(-\lambda_i^*)}$

Corollary. With $\hat{\mathbf{H}}(s) = \sum_{k=1}^{r} \frac{\hat{\phi}_k}{s-\hat{\lambda}_k}$, the $\mathcal{H}_2$ norm of the error system, is

$$\mathcal{J} = \left\|\mathbf{H} - \hat{\mathbf{H}}\right\|_{\mathcal{H}_2}^2 = \sum_{i=1}^{n} \phi_i \left[\mathbf{H}(-\lambda_i) - \hat{\mathbf{H}}(-\lambda_i)\right] + \sum_{j=1}^{r} \hat{\phi}_j \left[\hat{\mathbf{H}}(-\hat{\lambda}_j) - \mathbf{H}(-\hat{\lambda}_j)\right]$$

Conclusion. The $\mathcal{H}_2$ error is due to the *mismatch* of the transfer functions $\mathbf{H} - \hat{\mathbf{H}}$ at the *mirror images* of the full-order and reduced system poles λ_i, $\hat{\lambda}_i$.

An iterative algorithm

Let the system obtained after the $(j-1)^{\rm st}$ step be $(\mathbf{C}_{j-1}, \mathbf{A}_{j-1}, \mathbf{B}_{j-1})$, where $\mathbf{A}_{j-1} \in \mathbb{R}^{k\times k}$, $\mathbf{B}_{j-1}, \mathbf{C}_{j-1}^* \in \mathbb{R}^k$. At the $j^{\rm th}$ step the system is obtained as follows

$$\mathbf{A}_j = (\mathbf{W}_j^*\mathbf{V}_j)^{-1}\mathbf{W}_j^*\mathbf{A}\mathbf{V}_j,\ \mathbf{B}_j = (\mathbf{W}_j^*\mathbf{V}_j)^{-1}\mathbf{W}_j^*\mathbf{B},\ \mathbf{C}_j = \mathbf{C}\mathbf{V}_j,$$

where

$$\mathbf{V}_j = \left[(\lambda_1\mathbf{I}-\mathbf{A})^{-1}\mathbf{B},\ \cdots\ ,(\lambda_k\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\right],$$
$$\mathbf{W}_j^* = \left[\mathbf{C}(\lambda_1\mathbf{I}-A)^{-1},\ \cdots\ ,\mathbf{C}(\lambda_k\mathbf{I}-\mathbf{A})^{-1}\right],$$

and: $-\lambda_1,\cdots,-\lambda_k\ \in\ \sigma(\mathbf{A}_{j-1})$,
i.e., $-\lambda_i$ are the eigenvalues of the $(j-1)^{\rm st}$ iterate $\mathbf{A}_{j-1}$.

The Newton step: can be computed explicitly

$$\begin{bmatrix}\lambda_1^{(k)}\\ \lambda_2^{(k)}\\ \vdots\end{bmatrix} \longleftarrow \begin{bmatrix}\lambda_1^{(k)}\\ \lambda_2^{(k)}\\ \vdots\end{bmatrix} - \mathbf{J}^{-1}\begin{bmatrix}\lambda_1^{(k-1)}\\ \lambda_2^{(k-1)}\\ \vdots\end{bmatrix}$$

$\Rightarrow$ **local convergence guaranteed.**

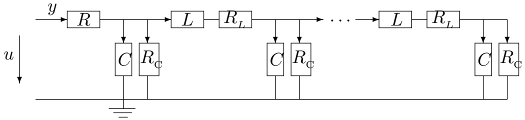
An iterative rational Krylov algorithm (IRKA)

The proposed algorithm produces a reduced order model $\hat{\mathbf{H}}(s)$ that satisfies the interpolation-based conditions, i.e.

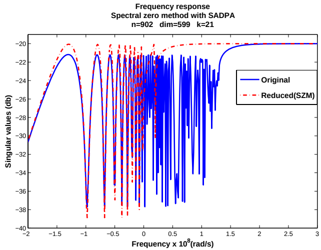
$$\mathbf{H}(-\hat{\lambda}_i^*) = \hat{\mathbf{H}}(-\hat{\lambda}_i^*) \text{ and } \left.\frac{d}{ds}\mathbf{H}(s)\right|_{s=-\hat{\lambda}_i^*} = \left.\frac{d}{ds}\hat{\mathbf{H}}(s)\right|_{s=-\hat{\lambda}_i^*}$$

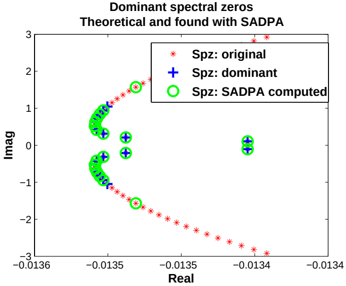
1. Make an initial selection of σ_i, for $i = 1, \ldots, k$
2. $\mathbf{W} = [(\sigma_1\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{C}^*, \cdots, (\sigma_k\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{C}^*]$
3. $\mathbf{V} = [(\sigma_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}, \cdots, (\sigma_k\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}]$
4. while (not converged)
   - $\hat{\mathbf{A}} = (\mathbf{W}^*\mathbf{V})^{-1}\mathbf{W}^*\mathbf{A}\mathbf{V}$,
   - $\sigma_i \longleftarrow -\lambda_i(\hat{\mathbf{A}})$ + Newton correction, $i = 1, \ldots, k$
   - $\mathbf{W} = [(\sigma_1\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{C}^*, \cdots, (\sigma_k\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{C}^*]$
   - $\mathbf{V} = [(\sigma_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}, \cdots, (\sigma_k\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}]$
5. $\hat{\mathbf{A}} = (\mathbf{W}^*\mathbf{V})^{-1}\mathbf{W}^*\mathbf{A}\mathbf{V}$, $\hat{\mathbf{B}} = (\mathbf{W}^*\mathbf{V})^{-1}\mathbf{W}^*\mathbf{B}$, $\hat{\mathbf{C}} = \mathbf{C}\mathbf{V}$

Moderate-dimensional example



- total system variables $\mathrm{n} = 902$, independent variables $\mathrm{dim} = 599$, reduced dimension $\mathrm{k} = 21$
- reduced model captures dominant modes





$\mathcal{H}_\infty$ and $\mathcal{H}_2$ error norms

Relative norms of the error systems

Reduction Method $n = 902, dim = 599, k = 21$	$\mathcal{H}_\infty$	$\mathcal{H}_2$
PRIMA	1.4775	-
Spectral Zero Method with SADPA	0.9628	0.841
Optimal $\mathcal{H}_2$	**0.5943**	**0.4621**
Balanced truncation (BT)	0.9393	0.6466
Riccati Balanced Truncation (PRBT)	0.9617	0.8164

Outline

Approximation methods: Summary

Krylov

- Realization
- Interpolation
- Lanczos
- Arnoldi

Properties

- **numerical efficiency**
- **n $\gg 10^3$**
- **choice of matching moments**

SVD

Nonlinear systems	**Linear systems**
• POD methods	• Balanced truncation
• Empirical Gramians	• Hankel approximation

Properties

- **Stability**
- **Error bound**
- **n $\approx 10^3$**

Krylov/SVD Methods

Complexity considerations

- **Dense problems**

Major cost **Balanced Truncation**:

Compute gramians $\approx 30N^3$ (eigenvalue decomp.)

Perform balancing $\approx 25N^3$ (sing. value decomp.)

Rational Krylov approximation:

Decompose $(\mathbf{A} - \sigma_i \mathbf{E})$ for k points $\approx \frac{2}{3}kN^3$

Remark : Iterations (Sign, Smith) can accelerate the computation of gramians (esp. on parallel machines)

- **Approximate and/or sparse decompositions**

Major cost **Balanced Truncation**:

Compute gramians $\approx c_1 \alpha k N$

Perform balancing $O(n^3)$

Rational Krylov approximation:

Iterative solves for $(\mathbf{A} - \sigma_i \mathbf{E})\mathbf{x} = \mathbf{b} \approx c_2 k \alpha N$, where k = number of expansion points; α = average number of non-zero elements per row in $\mathbf{A}$, $\mathbf{E}$.