Model reduction of large-scale systems

Lecture IV: Model reduction from measurements

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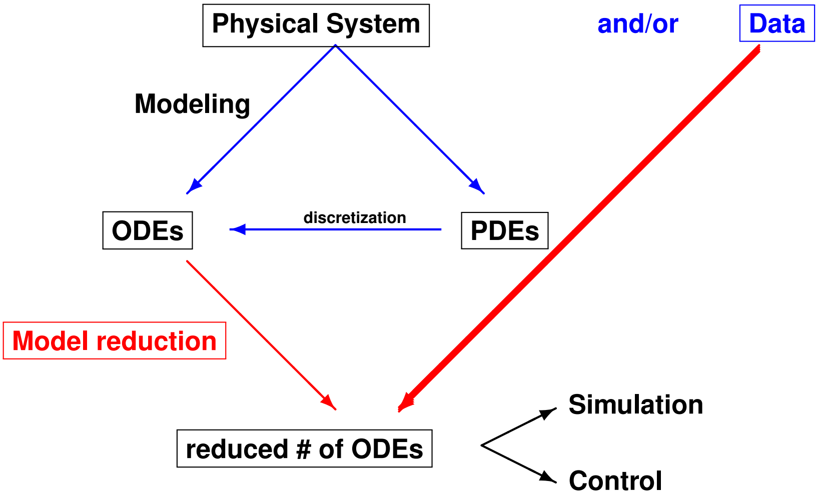
Outline

Outline

Recall: the big picture



A motivation: electronic systems

- Growth in communications and networking and demand for high data bandwidth requires streamlining of the simulation of entire complex systems from chips to packages to boards, etc.
- Thus in circuit simulation, signal intergrity (lack of signal distortion) of high speed electronic designs require that interconnect models be valid over a wide bandwidth.

An important tool: **S-parameters**

- They represent a component as a black box. Accurate simulations require accurate component models.
- In high frequencies S-parameters are important because wave phenomena become dominant.
- Advantages: $0 \leq |\mathbf{S}| \leq 1$ and can be measured using VNAs (Vector Network Analyzers).

Scattering or S-parameters

Given a system in I/O representation: $\mathbf{y}(s) = \mathbf{H}(s)\mathbf{u}(s)$,

the associated **S-paremeter representation** is

$$\bar{\mathbf{y}}(s) = \mathbf{S}(s)\bar{\mathbf{u}}(s) = \underbrace{[\mathbf{H}(s) + \mathbf{I}][\mathbf{H}(s) - \mathbf{I}]^{-1}}_{\mathbf{S}(s)}\, \bar{\mathbf{u}}(s),$$

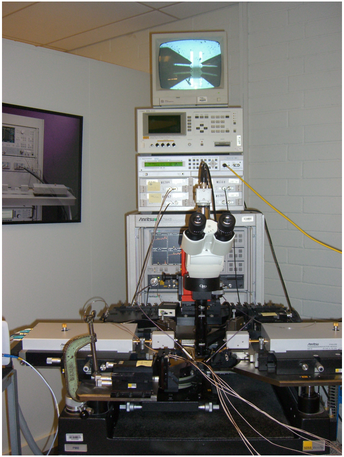
where

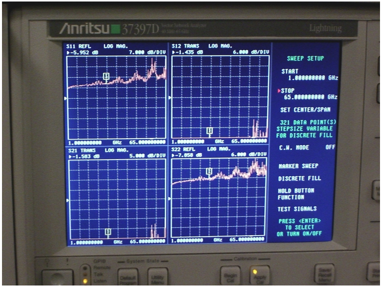
$\bar{\mathbf{y}} = \frac{1}{2}(\mathbf{y} + \mathbf{u})$ are the *transmitted waves* and,
$\bar{\mathbf{u}} = \frac{1}{2}(\mathbf{y} - \mathbf{u})$ are the *reflected waves*.

S-parameter measurements.

$\mathbf{S}(j\omega_k)$: samples of the frequency response of the **S**-parameter system representation.

Measurement of S-parameters





VNA (Vector Network Analyzer) – Magnitude of S-parameters for 2 ports

Outline

Model construction from data (at infinity): Classical realization

Given $\mathbf{h}_t \in \mathbb{R}^{p\times m}$, $t = 1, 2, \cdots$, find $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, such that

$$\mathbf{h}_t = \mathbf{C}\mathbf{A}^{t-1}\mathbf{B},\ \ t > 0$$

Main tool: *Hankel matrix*

$$\mathcal{H} = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_3 & \cdots \\ \mathbf{h}_2 & \mathbf{h}_3 & \mathbf{h}_4 & \cdots \\ \mathbf{h}_3 & \mathbf{h}_4 & \mathbf{h}_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \end{bmatrix}}_{\mathcal{O}} \underbrace{\begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \cdots \end{bmatrix}}_{\mathcal{R}}$$

Classical realization

Solvability $\Leftrightarrow \mathrm{rank}\,\mathcal{H} = n < \infty$

Solution:

- Find $\Delta \in \mathbb{R}^{n\times n}$: $\det \Delta \neq 0$.
- Let $\sigma\Delta \in \mathbb{R}^{n\times n}$: shifted matrix.
- Let $\Gamma \in \mathbb{R}^{n\times m}$: first m columns, while $\Lambda \in \mathbb{R}^{p\times n}$: the first p rows.

Then

$$\mathbf{A} = \Delta^{-1}\sigma\Delta,\quad \mathbf{B} = \Delta^{-1}\Gamma,\quad \mathbf{C} = \Lambda.$$

Consequence. In terms of the data:

$$\mathbf{H}(s) = \Lambda(s\Delta - \sigma\Delta)^{-1}\Gamma$$

Outline

Model construction from data at finite points: interpolation

Assume for simplicity that the given data are scalar:

$$(s_i, \phi_i),\ i = 1, 2, \cdots, N,\ \ s_i \neq s_j,\ i \neq j$$

Find $\mathbf{H}(s) = \frac{\mathbf{n}(s)}{\mathbf{d}(s)}$ such that $\mathbf{H}(s_i) = \phi_i,\ i = 1, 2, \cdots, N$,
and $\mathbf{n}, \mathbf{d}$: coprime polynomials.

A solution always exists (e.g. *Lagrange interpolating polynomial*).

Additional constraints for $\mathbf{H}$: *minimality*, *stability*, *bounded realness* etc.

Main tool: **Loewner matrix**. Divide the data in disjoint sets:
(λ_i, w_i), $i = 1, 2, \cdots, k$, (μ_j, v_j), $j = 1, 2, \cdots, q$, $k + q = N$:

$$\mathbb{L} = \begin{bmatrix} \frac{v_1 - w_1}{\mu_1 - \lambda_1} & \cdots & \frac{v_1 - w_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{v_q - w_1}{\mu_q - \lambda_1} & \cdots & \frac{v_q - w_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

Model construction from data at finite points: interpolation

Main result (1986). The rank of $\mathbb{L}$ encodes the information about the minimal degree interpolants: $\text{rank}\,\mathbb{L}$ or $N - \text{rank}\,\mathbb{L}$.

Remarks.

(a) In this framework the strict properness assumption has been dropped. Thus rational functions with polynomial part can be recovered from input-output data.

(b) The construction of interpolants will be deferred until later.

(c) If $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$, then

$$\mathbb{L} = -\underbrace{\begin{bmatrix} \mathbf{C}(\lambda_1\mathbf{I} - \mathbf{A})^{-1} \\ \mathbf{C}(\lambda_2\mathbf{I} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\lambda_k\mathbf{I} - \mathbf{A})^{-1} \end{bmatrix}}_{\mathcal{O}} \underbrace{\begin{bmatrix} (\mu_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} & \cdots & (\mu_q\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \end{bmatrix}}_{\mathcal{R}}$$

Scalar interpolation – multiple points

Special case. single point with multiplicity: $(s_0; \phi_0, \phi_1, \cdots, \phi_{N-1})$, i.e. the value of the function and that of a number of derivatives is provided.The **Loewner matrix** becomes:

$$\mathbb{L} = \begin{bmatrix} \frac{\phi_1}{1!} & \frac{\phi_2}{2!} & \frac{\phi_3}{3!} & \frac{\phi_4}{4!} & \cdots \\ \frac{\phi_2}{2!} & \frac{\phi_3}{3!} & \frac{\phi_4}{4!} & \cdots & \\ \frac{\phi_3}{3!} & \frac{\phi_4}{4!} & & & \\ \frac{\phi_4}{4!} & \vdots & & \ddots & \\ \vdots & & & & \end{bmatrix} \quad \Rightarrow \quad \textbf{HANKEL MATRIX}$$

Thus the **Loewner matrix generalizes** the **Hankel matrix** when general interpolation replaces realization.

Outline

General framework – tangential interpolation

Given: • right data: $(\lambda_i; \mathbf{r}_i, \mathbf{w}_i)$, $i = 1, \cdots, k$

• left data: $(\mu_j; \ell_j, \mathbf{v}_j)$, $j = 1, \cdots, q$.

We assume for simplicity that all points are distinct.

Problem: Find rational $p \times m$ matrices $\mathbf{H}(s)$, such that

$$\boxed{\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i} \qquad \boxed{\ell_j \mathbf{H}(\mu_j) = \mathbf{v}_j}$$

Right data:

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \in \mathbb{C}^{k \times k}, \quad \begin{array}{l} \mathbf{R} = [\mathbf{r}_1 \;\; \mathbf{r}_2, \;\; \cdots \;\; \mathbf{r}_k] \in \mathbb{C}^{m \times k}, \\ \mathbf{W} = [\mathbf{w}_1 \;\; \mathbf{w}_2 \;\; \cdots \;\; \mathbf{w}_k] \in \mathbb{C}^{p \times k} \end{array}$$

Left data:

$$M = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_q \end{bmatrix} \in \mathbb{C}^{q \times q}, \mathbf{L} = \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_q \end{bmatrix} \in \mathbb{C}^{q \times p}, \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_q \end{bmatrix} \in \mathbb{C}^{q \times m}$$

Problems

- **Model Construction**
  - Data Λ, M, $\mathbf{R}$, $\mathbf{L}$, $\mathbf{W}$, $\mathbf{V}$
  - Construct $[\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}]$, such that $\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i, \;\; \ell_j \mathbf{H}(\mu_j) = \mathbf{v}_j$.
- **Model Reduction**
  - Data M, Λ, $\mathbf{R}$, $\mathbf{L}$ and high-order system $[\hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}}]$
  - Contrsuct $[\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}]$ of lower order such that
    $$\hat{\mathbf{H}}(\lambda_i)\mathbf{r}_i = \mathbf{H}_i(\lambda_i)\mathbf{r}_i, \;\; \ell_j \hat{\mathbf{H}}(\mu_j) = \ell_j \mathbf{H}(\mu_j).$$
- **Model Construction and Reduction (MR from data)**
  - Data Λ, M, $\mathbf{R}$, $\mathbf{L}$, $\mathbf{W}$, $\mathbf{V}$
  - First step: construct high-order $[\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}]$
  - Second step: reduce its dimension appropriately

Outline

General framework – tangential interpolation

Input-output data. The Loewner matrix is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1\mathbf{r}_1-\ell_1\mathbf{w}_1}{\mu_1-\lambda_1} & \cdots & \frac{\mathbf{v}_1\mathbf{r}_k-\ell_1\mathbf{w}_k}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q\mathbf{r}_1-\ell_q\mathbf{w}_1}{\mu_q-\lambda_1} & \cdots & \frac{\mathbf{v}_q\mathbf{r}_k-\ell_q\mathbf{w}_k}{\mu_q-\lambda_k} \end{bmatrix} \in \mathbb{C}^{q\times k}$$

Recall:

$$\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i, \;\; \ell_j\mathbf{H}(\mu_j) = \mathbf{v}_j$$

Therefore $\mathbb{L}$ satisfies the Sylvester equation

$$\mathbb{L}\Lambda - M\mathbb{L} = \mathbf{VR} - \mathbf{LW}$$

General framework – tangential interpolation

State space data. Suppose that $\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}$ of minimal degree n are given such that $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$.
Let $\mathbf{X}, \mathbf{Y}$ satisfy the following **Sylvester equations**

$$\boxed{\mathbf{EX\Lambda} - \mathbf{AX} = \mathbf{BR}} \quad \text{and} \quad \boxed{\mathbf{MYE} - \mathbf{YA} = \mathbf{LC}}$$

If the generalized eigenvalues of $(\mathbf{E}, \mathbf{A})$ are distinct from λ_i and μ_j, $\mathbf{X}, \mathbf{Y}$ are unique solutions of these equations. Actually

$\mathbf{x}_i = (\lambda_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\mathbf{r}_i \;\Rightarrow \mathbf{X}$: *generalized* reachability matrix
$\mathbf{y}_j = \ell_j\mathbf{C}(\mu_j\mathbf{E} - \mathbf{A})^{-1} \;\Rightarrow \mathbf{Y}$: *generalized* observability matrix.

$$\Rightarrow \boxed{\mathbb{L} = -\mathbf{YEX}}$$

Construction of Interpolants

Suggested construction procedure for an interpolant of McMillan degree n:

1. Factor $\mathbb{L} = -\mathbf{YEX}$, so that $\mathbf{E}$ has rank n.
2. Construct $\mathbf{A}, \mathbf{B}, \mathbf{C}$ to satisfy the Sylvester equations above.
3. Define $\mathbf{D} := \mathbf{w}_j - \mathbf{C}(s_j\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$.

Steps 2 and 3 are easy; the first step is **problematic**: **how** do we choose $\mathbf{E}$?

- If the system is proper, then $\mathrm{size}\,(\mathbf{E}) = \mathrm{rank}\,(\mathbf{E})$, and we could use, for example, the SVD to factor $\mathbb{L} \Rightarrow$ proper systems are easy.
- If the system is singular, then $\mathrm{size}\,(\mathbf{E}) > \mathrm{rank}\,(\mathbf{E})$, and we're stuck.

Solution: use the **shifted Loewner matrix** $\sigma\mathbb{L}$

The shifted Loewner matrix

- The shifted Loewner matrix, $\sigma\mathbb{L}$, is the Loewner matrix associated to $s\mathbf{H}(s)$.

$$\sigma\mathbb{L} = \begin{bmatrix} \frac{\mu_1 \mathbf{v}_1 \mathbf{r}_1 - \ell_1 \mathbf{w}_1 \lambda_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1 \mathbf{v}_1 \mathbf{r}_k - \ell_1 \mathbf{w}_k \lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q \mathbf{v}_q \mathbf{r}_1 - \ell_q \mathbf{w}_1 \lambda_1}{\mu_q - \lambda_1} & \cdots & \frac{\mu_q \mathbf{v}_q \mathbf{r}_k - \ell_q \mathbf{w}_k \lambda_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

- $\sigma\mathbb{L}$ satisfies the Sylvester equation

$$\boxed{\sigma\mathbb{L}\Lambda - M\sigma\mathbb{L} = \mathbf{VR}\Lambda - M\mathbf{LW}}$$

- $\sigma\mathbb{L}$ can be factored as

$$\Rightarrow \quad \boxed{\sigma\mathbb{L} = -\mathbf{YAX}}$$

Construction of Interpolants (Models)

Assume that $k = \ell$, and let

$$\det(x\mathbb{L} - \sigma\mathbb{L}) \neq 0, \quad x \in \{\lambda_i\} \cup \{\mu_j\}$$

Then

$$\mathbf{E} = -\mathbb{L}, \;\; \mathbf{A} = -\sigma\mathbb{L}, \;\; \mathbf{B} = \mathbf{V}, \;\; \mathbf{C} = \mathbf{W}$$

is a minimal realization of an interpolant of the data, i.e., the function

$$\boxed{\mathbf{H}(s) = \mathbf{W}(\sigma\mathbb{L} - s\mathbb{L})^{-1}\mathbf{V}}$$

interpolates the data.

Proof. Multiplying the first equation by s and subtracting it from the second we get

$$(\sigma\mathbb{L} - s\mathbb{L})\Lambda - M(\sigma\mathbb{L} - s\mathbb{L}) = \mathbf{LW}(\Lambda - s\mathbf{I}) - (M - s\mathbf{I})\mathbf{VR}.$$

Multiplying this equation by $\mathbf{e}_i$ on the right and setting $s = \lambda_i$, we obtain

$$(\lambda_i\mathbf{I} - M)(\sigma\mathbb{L} - \lambda_i\mathbb{L})\mathbf{e}_i = (\lambda_i\mathbf{I} - M)\mathbf{V}\mathbf{r}_i \;\; \Rightarrow \;\; (\lambda_i\mathbb{L} - \sigma\mathbb{L})\mathbf{e}_i = \mathbf{V}\mathbf{r}_i \;\; \Rightarrow \;\; \mathbf{W}\mathbf{e}_i = \mathbf{W}(\lambda_i\mathbb{L} - \sigma\mathbb{L})^{-1}\mathbf{V}$$

Therefore $\mathbf{w}_i = \mathbf{H}(\lambda_i)\mathbf{r}_i$. This proves the right tangential interpolation property. To prove the left tangential interpolation property, we multiply the above equation by $\mathbf{e}_j^*$ on the left and set $s = \mu_j$:

$$\mathbf{e}_j^*(\sigma\mathbb{L} - \mu_j\mathbb{L})(\Lambda - \mu_j\mathbf{I}) = \mathbf{e}_j^*\mathbf{LW}(\mu_j\mathbf{I} - \Lambda) \;\; \Rightarrow \;\; \mathbf{e}_j^*(\sigma\mathbb{L} - \mu_j\mathbb{L}) = \ell_j\mathbf{W} \;\; \Rightarrow \;\; \mathbf{e}_j^*\mathbf{V} = \ell_j\mathbf{W}(\sigma\mathbb{L} - \mu_j\mathbb{L})^{-1}\mathbf{V}$$

Therefore $\mathbf{v}_j = \ell_j\mathbf{H}(\mu_j)$. ∎

Construction of interpolants: **New procedure**

Main assumption:

$$\boxed{\mathrm{rank}\,(x\mathbb{L}-\sigma\mathbb{L}) = \mathrm{rank}\,\begin{pmatrix} \mathbb{L} & \sigma\mathbb{L} \end{pmatrix} = \mathrm{rank}\begin{pmatrix} \mathbb{L} \\ \sigma\mathbb{L} \end{pmatrix} =: k,\ \ x \in \{\lambda_i\} \cup \{\mu_j\}}$$

Then for some $x \in \{\lambda_i\} \cup \{\mu_j\}$, we compute the *SVD*

$$x\mathbb{L} - \sigma\mathbb{L} = \mathbf{Y}\Sigma\mathbf{X}$$

with $\mathrm{rank}\,(x\mathbb{L}-\sigma\mathbb{L}) = \mathrm{rank}\,(\Sigma) = \mathrm{size}\,(\Sigma) =: k$, $\mathbf{Y} \in \mathbb{C}^{\nu\times k}$, $\mathbf{X} \in \mathbb{C}^{k\times\rho}$.

Theorem. A realization $[\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}]$, of an interpolant is given as follows:

$\mathbf{E} = -\mathbf{Y}^*\mathbb{L}\mathbf{X}^*$	$\mathbf{B} = \mathbf{Y}^*\mathbf{V}$
$\mathbf{A} = -\mathbf{Y}^*\sigma\mathbb{L}\mathbf{X}^*$	$\mathbf{C} = \mathbf{W}\mathbf{X}^*$

Remark. The system $[\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}]$ can now be further reduced using any of the usual reduction methods.

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{E} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Thus the transfer function is

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} s & -1 \\ -1 & \frac{1}{s} \end{bmatrix}.$$

We now define the interpolation data. $\Lambda = diag([1,\ 2,\ 3])$, $M = diag([-1,\ -2,\ -3,\ -4])$, while

$$\mathbf{L} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \\ 1 & -3 \\ 1 & 0 \end{bmatrix},\ \mathbf{R} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}.$$

These imply:

$$\mathbf{V} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -4 & -1 \end{bmatrix},\ \mathbf{W} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

Example

It follows that the tangential controllability and observability matrices associated with this data are:

$$\mathbf{X} = \left[\begin{array}{ccc} (\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}\mathbf{R}(:,1), & (2\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}\mathbf{R}(:,2), & (3\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}\mathbf{R}(:,3)] \end{array} \right] = \left[\begin{array}{ccc} 0 & 0 & -1 \\ -1 & -1 & 0 \\ 1 & 1 & \frac{1}{3} \end{array} \right],$$

$$\mathbf{Y} = \left[\begin{array}{c} \mathbf{L}(1,:)\mathbf{C}(-1\mathbf{E}-\mathbf{A})^{-1} \\ \mathbf{L}(2,:)\mathbf{C}(-2\mathbf{E}-\mathbf{A})^{-1} \\ \mathbf{L}(3,:)\mathbf{C}(-3\mathbf{E}-\mathbf{A})^{-1} \\ \mathbf{L}(4,:)\mathbf{C}(-4\mathbf{E}-\mathbf{A})^{-1} \end{array} \right] = \left[\begin{array}{ccc} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & -4 & 0 \end{array} \right].$$

We notice that $\mathrm{rank}\,\mathbf{X} = \mathrm{rank}\,\mathbf{Y} = 2$. Thus the Loewner and shifted Loewner matrices are

$$\mathbb{L} = \left[\begin{array}{ccc} 0 & 0 & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} \\ 1 & 1 & 0 \end{array} \right], \quad \sigma\mathbb{L} = \left[\begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \\ -4 & -4 & -1 \end{array} \right].$$

We check the assumption for $\delta = 0$:

$$\begin{array}{l} \mathrm{rank}\,(1\cdot\mathbb{L}-\sigma\mathbb{L}) = 2,\ \mathrm{rank}\,(2\cdot\mathbb{L}-\sigma\mathbb{L}) = 2,\ \mathrm{rank}\,(3\cdot\mathbb{L}-\sigma\mathbb{L}) = 1 \\ \mathrm{rank}\,(-1\cdot\mathbb{L}-\sigma\mathbb{L}) = 2,\ \mathrm{rank}\,(-2\cdot\mathbb{L}-\sigma\mathbb{L}) = 2 \\ \mathrm{rank}\,(-3\cdot\mathbb{L}-\sigma\mathbb{L}) = 2,\ \mathrm{rank}\,(-4\cdot\mathbb{L}-\sigma\mathbb{L}) = 1 \\ \mathrm{rank}\,([\mathbb{L}\ \ \sigma\mathbb{L}]) = 2,\ \mathrm{rank}\,([\mathbb{L};\ \ \sigma\mathbb{L}]) = 2, \end{array}$$

Example

and for $\delta = 1$:

$$\begin{aligned}
&\text{rank}\,(\mathbb{L} - \sigma\mathbb{L} + \mathbf{L}\cdot\mathbf{R}) = 3\\
&\text{rank}\,(2\cdot\mathbb{L} - \sigma\mathbb{L} + \mathbf{L}\cdot\mathbf{R}) = 3\\
&\text{rank}\,(3\cdot\mathbb{L} - \sigma\mathbb{L} + \mathbf{L}\cdot\mathbf{R}) = 3\\
&\text{rank}\,(-1\cdot\mathbb{L} - \sigma\mathbb{L} + \mathbf{L}\cdot\mathbf{R}) = 3\\
&\text{rank}\,(-2\cdot\mathbb{L} - \sigma\mathbb{L} + \mathbf{L}\cdot\mathbf{R}) = 3\\
&\text{rank}\,(-3\cdot\mathbb{L} - \sigma\mathbb{L} + \mathbf{L}\cdot\mathbf{R}) = 3\\
&\text{rank}\,(-4\cdot\mathbb{L} - \sigma\mathbb{L} + \mathbf{L}\cdot\mathbf{R}) = 3\\
&\text{rank}\,([\mathbb{L}\ \ \sigma\mathbb{L} - \mathbf{L}\cdot\mathbf{R}]) = 3\\
&\text{rank}\,([\mathbb{L};\ \ \sigma\mathbb{L} - \mathbf{L}\cdot\mathbf{R}]) = 3
\end{aligned}$$

Since the assumption is violated for $\delta = 0$, but is not violated for $\delta = 1$, we will compute interpolants with $\delta \neq 0$.
For non-zero **D**, the dimension is 3. From $\mathbb{L} - \sigma\mathbb{L} + \mathbf{LR}$, we obtain the projectors

$$\pi_l = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}, \quad \pi_r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The resulting transfer function is

$$\hat{\mathbf{H}}(s) = (\mathbf{W} - \delta\mathbf{R})\pi_r(\pi_l(s\mathbb{L} - \sigma\mathbb{L} + \delta\mathbf{LR})\pi_r)^{-1}\pi_l(\mathbf{V} - \delta\mathbf{L}) - \delta\mathbf{I}_2) = \begin{bmatrix} -\frac{\delta s^2}{s^2+\delta s+s-12} & \frac{\delta s}{s^2+\delta s+s-12} \\ \frac{\delta s}{s^2+\delta s+s-12} & -\frac{\delta}{s^2+\delta s+s-12} \end{bmatrix}$$

Notice that the original rational function is obtained for $\delta \to \infty$.

Example

It can be readily checked that all seven interpolation conditions are satisfied. A realization of this interpolant is as follows

$$s\hat{\mathbf{E}} - \hat{\mathbf{A}} = \pi_l \cdot (s\mathbb{L} - \sigma\mathbb{L} + \delta\mathbf{LR}) \cdot \pi_r = \begin{bmatrix} 2\cdot d & 4\cdot d & 2\cdot d \\ s+4+d & s+4+d & 1 \\ -5\cdot d & -13\cdot d & 3-s-8\cdot d \end{bmatrix}$$

and thus $\hat{\mathbf{E}} = \pi_l \cdot \mathbb{L} \cdot \pi_r$, $\hat{\mathbf{A}} = \pi_l \cdot \sigma\mathbb{L} \cdot \pi_r$, turn out to be

$$\hat{\mathbf{E}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \ \hat{\mathbf{A}} = \begin{bmatrix} -2\cdot\delta & -4\cdot\delta & -2\cdot\delta \\ -4-\delta & -4-\delta & -1 \\ 5\cdot\delta & 13\cdot\delta & -3+8\cdot\delta \end{bmatrix}.$$

Furthermore $\hat{\mathbf{B}} = \pi_l \cdot (\mathbf{V} - \delta\mathbf{L})$ and $\hat{\mathbf{C}} = (\mathbf{W} - \delta\mathbf{R}) \cdot \pi_r$:

$$\hat{\mathbf{B}} = \begin{bmatrix} 0 & -2\cdot d \\ -4-d & -1 \\ -3\cdot d & 8\cdot d \end{bmatrix}, \ \hat{\mathbf{C}} = \begin{bmatrix} -d & -d & -1 \\ -d & -2\cdot d & 1/3-d \end{bmatrix}.$$

Finally we need to check that the characteristic polynomial of the system is non zero at all interpolation points:

$$\chi(s) = \det(s\hat{\mathbf{E}} - \hat{\mathbf{A}}) = 2\delta(s^2 + (\delta+1)s - 12)$$

Therefore

$$\begin{aligned} &\chi(1) = 2\delta(\delta-10),\ \chi(2) = 4\delta(\delta-3),\ \chi(3) = 6\delta^2 \\ &\chi(-1) = -2\delta(\delta+12),\ \chi(-2) = -4\delta(\delta+5) \\ &\chi(-3) = -6\delta(\delta+2),\ \chi(-4) = -8\delta^2 \end{aligned}$$

Thus δ must be different from $-12, -5, -2, 0, 3, 10$. ■

Example: revisited

$$H = \begin{bmatrix} s & 1 \\ 1 & s^{-1} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad M = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$W = \begin{bmatrix} 1 & 1 & 4 & 3 \\ 1 & 1/2 & 4/3 & 3/4 \end{bmatrix}, \quad V = \begin{bmatrix} -1 & 1 \\ 1 & -1/2 \\ -2 & 2/3 \\ -5 & 5/4 \end{bmatrix}$$

Example: revisited

$$P = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1/4 & 1/6 & -1/8 \\ 1 & 1/6 & \frac{10}{9} & \frac{11}{12} \\ 1 & -1/8 & \frac{11}{12} & \frac{17}{16} \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 2 & 1 \\ -4 & 1 & -1 & -2 \end{bmatrix}$$

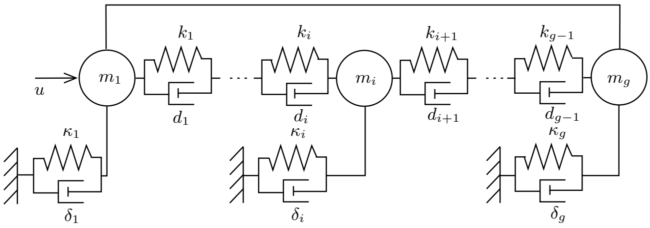
$$\Pi = \begin{bmatrix} 0 & 0 & 10 \\ 10 & 0 & 0 \\ 0 & 10 & 0 \\ -19 & 21 & -11 \end{bmatrix} \Rightarrow \Pi^* P \Pi \begin{bmatrix} \frac{7297}{16} & -\frac{9723}{16} & \frac{733}{16} \\ -\frac{9723}{16} & \frac{138913}{144} & -\frac{1741}{48} \\ \frac{733}{16} & -\frac{1741}{48} & \frac{137}{16} \end{bmatrix}, \Pi^* Q \Pi = \begin{bmatrix} -1102 & 1298 & 332 \\ 918 & -682 & -588 \\ -808 & 1292 & -22 \end{bmatrix}$$

Notice rank $\Pi^* P \Pi = 2 =$ McMillan degree of H.

$$\Rightarrow \underbrace{W\Pi}_{C} \left[s \underbrace{\Pi^* P \Pi}_{E} - \underbrace{\Pi^* Q \Pi}_{A} \right]^{-1} \underbrace{\Pi^* V}_{B} = C(sE - A)^{-1} B = H(s) = \begin{bmatrix} s & 1 \\ 1 & s^{-1} \end{bmatrix}$$

Example: mechanical system

Mechanical example: Stykel, Mehrmann

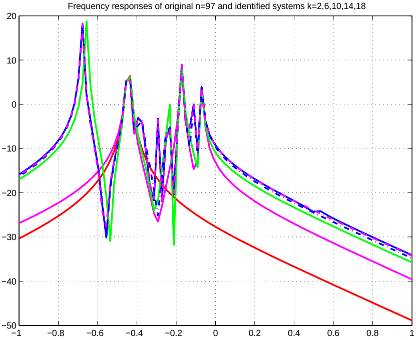


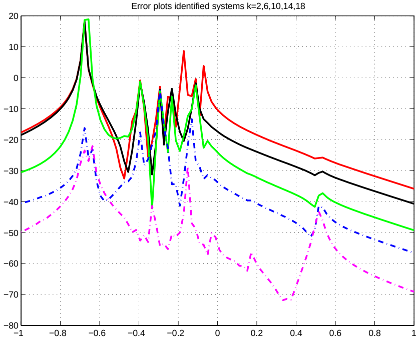
The vibration of this system is described in generalized state space form as:

$$\begin{aligned} \dot{\mathbf{p}}(t) &= \mathbf{v}(t) \\ \mathbf{M}\dot{\mathbf{v}}(t) &= \mathbf{K}\mathbf{p}(t) + \mathbf{D}\mathbf{v}(t) - \mathbf{G}^*\lambda(t) + \mathbf{B}_2\mathbf{u}(t) \\ \mathbf{0} &= \mathbf{G}\mathbf{p}(t) \\ \mathbf{y}(t) &= \mathbf{C}_1\mathbf{p}(t) \end{aligned}$$

Measurements: 500 frequency response data between $[-2i, +2i]$.

Mechanical system: plots



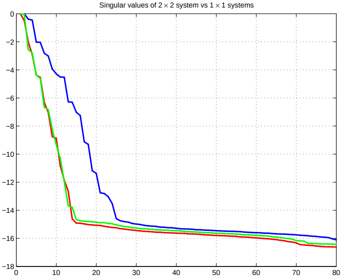


Left: Frequency responses of original system and approximants (orders 2, 6, 10, 14, 18)

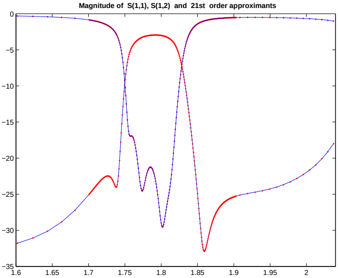
Right: Frequecy responses of error systems

Example: Four-pole band-pass filter

• **1000 measurements between 40 and 120 GHz; S-parameters** 2×2, **MIMO (approximate) interpolation** $\Rightarrow \mathbb{L}, \sigma\mathbb{L} \in \mathbb{R}^{2000\times 2000}$.



The singular values of $\mathbb{L}$, $\sigma\mathbb{L}$

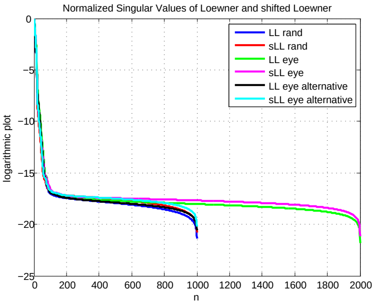


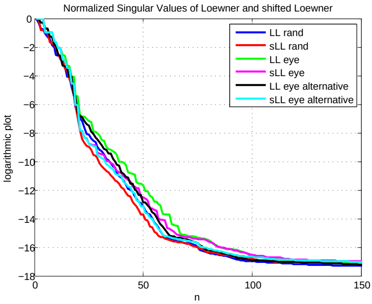
The $S(1,1)$ and $S(1,2)$ parameter data

17-th order model

Example: co-axial cable

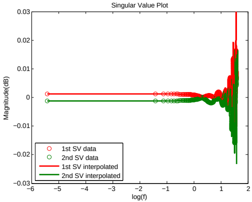
- **1000 measurements between 0 and 40 MHz; S-parameters** 2×2**,**
  tangential interpolation $\Rightarrow$ $\mathbb{L}$, $\sigma\mathbb{L} \in \mathbb{R}^{1000\times 1000}$.



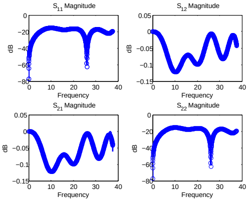


singular values of $\mathbb{L}$, $\sigma\mathbb{L}$ **detail**

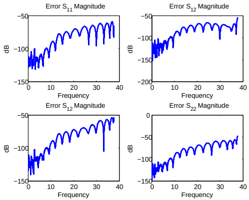
Example: co-axial cable



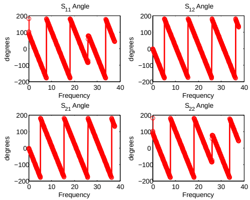
singular values



freq. resp. magnitude



freq. resp. magnitude: error



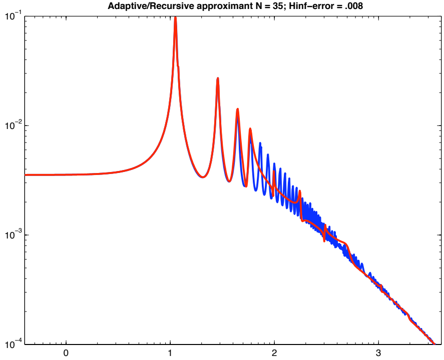
freq. resp. angle

Example: delay system

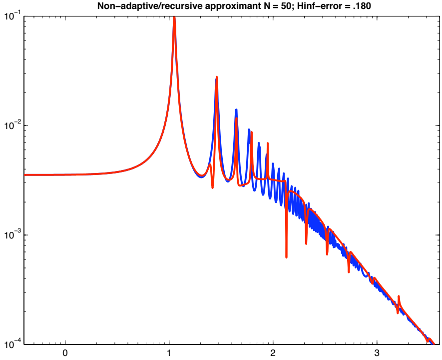
$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}_0\mathbf{x}(t) + \mathbf{A}_1\mathbf{x}(t-\tau) + \mathbf{B}\mathbf{u}(t),\ \ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),$$

where $\mathbf{E}$, $\mathbf{A}_0$, $\mathbf{A}_1$ are 500×500 and $\mathbf{B}$, $\mathbf{C}^*$ are 500-vectors.

Procedure: compute 1000 frequency response samples. Then apply **recursive/adaptive** Loewner-framework procedure. (Blue: original, red: approximants.)



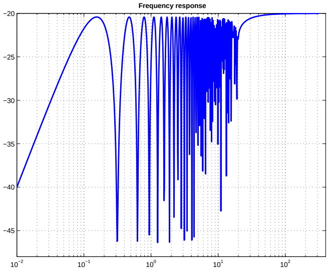
35-th order adaptively constructed model;
H_∞ norm of error: 0.008.

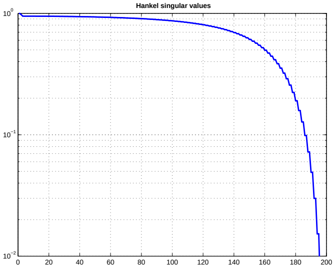


50-th order non-adaptively constructed model;
H_∞ norm of error: 0.180.

Outline

Motivation





Frequency response and Hankel singular values

Dominant system poles

Goal: compute a reduced order model by evaluating the transfer function of the original high-order system, in as few frequencies as possible.

Given a rational function $\mathbf{H}(s)$, let $s_0 \in \mathbb{C}$ be a (simple) pole, The corresponding *residue* is defined as

$$\rho_0 = (s - s_0)\mathbf{H}(s)|_{s=s_0}.$$

Concept of dominance:

$$\mu_i := \frac{|\rho_i|}{|\mathcal{R}e(s_i)|}.$$

Remark: for poles s_i, s_j which are not too close to each other, μ_i, μ_j, give the (local) maxima of the amplitude Bode plot; these are attained at the frequencies $\omega_i = \mathcal{I}m(s_i)$, $\omega_j = \mathcal{I}m\,(s_j)$, respectively.

The procedure

The proposed method (sometime referred to as FSS - Fast Frequency Sweep) consists of the following steps.

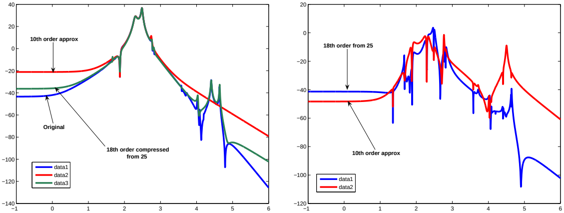
1. Compute some of the most dominant system poles $s_i = \sigma_i + j\omega_i$.
2. Evaluate the transfer function of the system at the frequencies ω_i, that is at the imaginary parts of the computed dominant poles.
3. Finally, compute a low order model of the underlying system using the Loewner matrix-based, data-to-reduced-model method described above.

Three examples

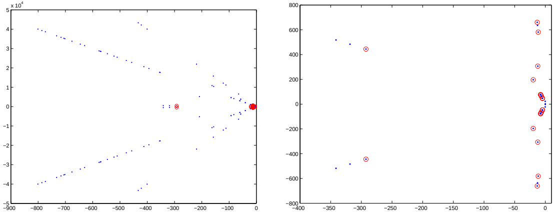
Goal: construction of reduced models for three systems given by means of $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$, using the proposed interpolation approach.

- A CD player model, with order $n = 120$.
- The model of a flexible beam, fixed at one, with oder $n = 348$.
- The model of a transmission line, with order $n = 199$.

CD player

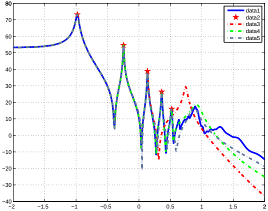


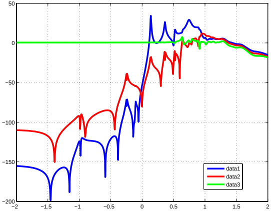
Left pane: amplitude Bode plot of the original model (blue), 8th order reduced (red) and 18th order reduced models.
Right pane: amplitude Bode plots of the corresponding error systems. Notice the local character of the approximants: the first one captures the first main peak, while the second reproduces both.



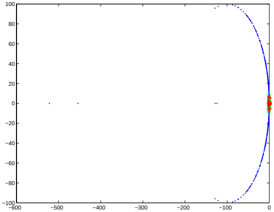
Left pane: the poles of the system (blue dots) and the dominant poles (red circles). **Right pane**: detail of the pole plot.

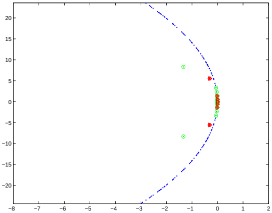
Clamped beam





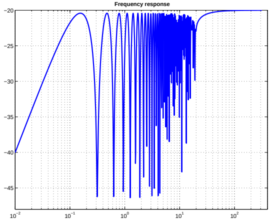
Left pane: amplitude Bode plots of original system (blue), 8th order approximant (red dash-dot), 17th order approximant (blue dash-dot), and 12th oder approximant obtained by balanced truncation (green dash-dot). The stars indicate the value of the dominance index μ_j, for the first 5 dominant poles.
Right pane: amplitude Bode plots of error systems for the 8th order approximant (blue), for the 17th order approximant (red), and for the 12th order approximant obatined by balanced truncation.



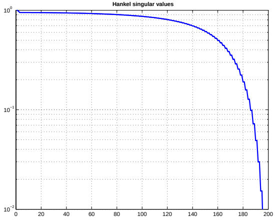


Left pane: system poles (blue) and some dominant poles (red). **Right pane**: close-up look.

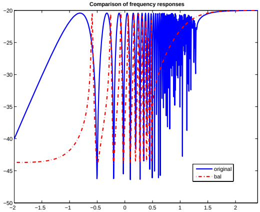
Transmission line



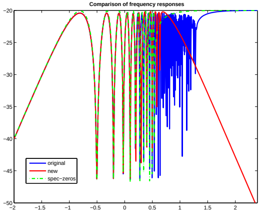
Frequency response



Hankel singular values

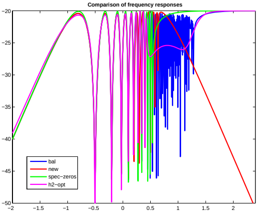


original=blue, bal=red

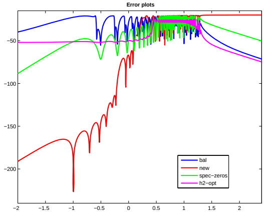


original=blue, new=red, spec-zeros=green

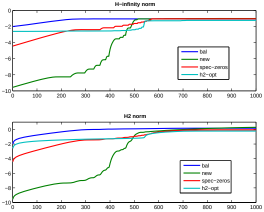
Transmission line – continued



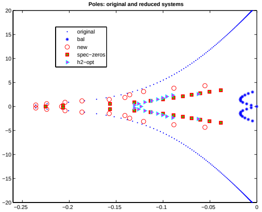
original=blue, new=red, **H2-opt=mag**



Error plots



H_2 and H_∞ errors



Poles

Outline

Summary and conclusions

- Given frequency response data, we can construct with no computation, a high order system in generalized state space form $\Rightarrow$

  let the data reveal the underlying system

- The system is such that (**A**, **E**) is neither regular nor stable and hence (**E**, **A**, **B**, **C**) is not passive.
- An SVD of $\mathbb{L}$, $(\sigma\mathbb{L})$, will produce a regular (and stable) system.
- At this stage all usual model reduction methods are applicable.
- Approach is the natural way to construct models and reduced models from data as it does **not** require inversion of **E**.
- For small systems described by (**E**, **A**, **B**, **C**), one can compute all poles s_i, and their dominance indices μ_i. The major issue for the applicability of the proposed method to large-scale systems is the determination of a few dominant poles of (**A**, **E**), without computing all the poles first. This can be achieved using the iterative method known as SADPA (subspace accelerated dominant pole algorithm) developed by Joost Rommes.
- The proposed method reduces systems at specific frequency ranges.
- This in **not** a Krylov method. It is an interpolation method.
- It exhibits similarities with Krylov methods, like the spectral zero method and the optimal H_2 method.

References

- Mayo, Antoulas, LAA 2007
- Lefteriu, Antoulas, Tech. Report 2008
- Vandendorpe, Van Dooren, SIMAX 2005
- Antoulas: SIAM book 2005