Problem [1]. As in problem [1] of the previous assignment, consider the RLC system with the following parameters: \( C_1 = C_2 = 1, R = R_1 = R_2 = 1, L_1 = L_2 = 1 \). Consider also the states:

\[
\hat{x} = [1 \ 0 \ 0 \ 0]^T, \ \check{x} = [0 \ 0 \ 0 \ 1]^T
\]

(a) Compute the infinite reachability gramian. (b) Hence find the state which is the easiest and most difficult to reach. (c) Find an input which will steer the system from 0 to \( \hat{x} \) in an infinite amount of time. Repeat for \( \check{x} \). What is the required energy in each case? Plot the two inputs on the same graph.

Problem [2]. (Problem [2], HW #4) Aircraft AFTI-16 lateral motion. The state equations are given by 

\[
\begin{align*}
\dot{x} &= Ax + Bu, \\
A &= \begin{bmatrix} -0.746 & 0.006 & -0.999 & 0.0369 \\
-12.9 & -0.746 & 0.387 & 0 \\
4.31 & 0.024 & -0.174 & 0 \\
0 & 1 & 0 & 0 \\
\end{bmatrix}, \\
B &= \begin{bmatrix} 0.0012 & 0.0092 \\
6.05 & 0.952 \\
-0.416 & -1.76 \\
0 & 0 \\
\end{bmatrix}
\end{align*}
\]

The state is \( x = [\beta \ \dot{\phi} \ r \ \phi]^T \), where \( \beta \) is the slip-angle, \( \dot{\phi} \) is the roll rate, \( r \) is the yaw rate and \( \phi \) is the roll angle. The control inputs are \( u = [u_1 \ u_2]^T \), where \( u_1 \) is the aileron deflection and \( u_2 \) is the rudder deflection.

Given a reachable state \( x \), the minimal energy required to reach it in \( T \) units of time is \( x^*P^{-1}(T)x \). Compute the infinite reachability gramians \( P_1, P_2, \) and \( P \) for inputs: \( u_1, u_2, u \), respectively. Show that \( P = P_1 + P_2 \); therefore verify that the eigenvalues of \( P_i \) are less than those of \( P, i = 1, 2 \) (this can be expressed compactly by means of the inequality \( P \geq P_i \Rightarrow P - P_i, i = 1, 2, \) is positive semi-definite).

Hence determine the energy required to reach each one of the states \( x = e_i, i = 1, 2, 3, 4 \), where \( e_i \) is the \( i \)th unit vector. Comment on the relative difficulty of reaching each one of these states.

Suggestion: the use of Matlab for the preceding problems is recommended. Please attach the m-files used and the corresponding diary.

Problem [3]. (Extra Credit 25%) Given the system \( \dot{x} = Ax + Bu \), let \( \mathcal{P}(T) \) be the corresponding gramian at time \( T \), \( \hat{x} \) a reachable state, and \( \xi \) be such that \( \dot{\xi} = \mathcal{P}(T)\xi \). Show that among all inputs \( u \) which steer the state of the system from 0 at time 0, to \( \hat{x} \) at time \( T \), the input \( \hat{u}(t) = B^*e^{A^*(T-t)}\xi, 0 \leq t \leq T \), has the smallest energy, i.e.

\[
< u, u > \geq < \hat{u}, \hat{u} > \text{ where } < f, g > := \int_0^T f^*(t)g(t)dt.
\]

Hint: show that \( < \hat{u}, u - \hat{u} > = 0 \).