

A Note on the Maximum Flow Through a Network*

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Summary—This note discusses the problem of maximizing the rate of flow from one terminal to another, through a network which consists of a number of branches, each of which has a limited capacity. The main result is a theorem: The maximum possible flow from left to right through a network is equal to the minimum value among all simple cut-sets. This theorem is applied to solve a more general problem, in which a number of input nodes and a number of output nodes are used.

CONSIDER a two-terminal network such as that of Fig. 1. The branches of the network might represent communication channels, or, more generally, any conveying system of limited capacity as, for example, a railroad system, a power feeding system, or a network of pipes, provided in each case it is possible to assign a definite maximum allowed rate of flow over a given branch. The links may be of two types, either one directional (indicated by arrows) or two directional, in which case flow is allowed in either direction at anything up to maximum capacity. At the nodes or junction points of the network, any redistribution of incoming flow into the outgoing flow is allowed, subject only to the restriction of not exceeding in any branch the capacity, and of obeying the Kirchhoff law that the total (algebraic) flow into a node be zero. Note that in the case of information flow, this may require arbitrarily large delays at each node to permit recoding of the output signals from that node. The problem is to evaluate the maximum possible flow through the network as a whole, entering at the left terminal and emerging at the right terminal.

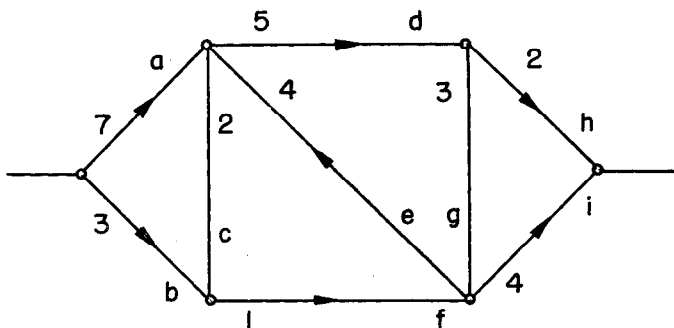


Fig. 1

The answer can be given in terms of cut-sets of the network. A *cut-set* of a two-terminal network is a set of branches such that when deleted from the network, the network falls into two or more unconnected parts with the two terminals in different parts. Thus, every path

from one terminal to the other in the original network passes through at least one branch in the cut-set. In the network above, some examples of cut-sets are (d, e, f) , and (b, c, e, g, h) , (d, g, h, i) . By a *simple cut-set* we will mean a cut-set such that if any branch is omitted it is no longer a cut-set. Thus (d, e, f) and (b, c, e, g, h) are simple cut-sets while (d, g, h, i) is not. When a simple cut-set is deleted from a connected two-terminal network, the network falls into exactly two parts, a *left part* containing the left terminal and a *right part* containing the right terminal. We assign a *value* to a simple cut-set by taking the sum of capacities of branches in the cut-set, only counting capacities, however, from the left part to the right part for branches that are unidirectional. Note that the direction of an unidirectional branch cannot be deduced from its appearance in the graph of the network. A branch is directed from left to right in a minimal cut-set if, and only if, the arrow on the branch points from a node in the left part of the network to a node in the right part. Thus, in the example, the cut-set (d, e, f) has the value $5 + 1 = 6$, the cut-set (b, c, e, g, h) has value $3 + 2 + 3 + 2 = 10$.

Theorem: The maximum possible flow from left to right through a network is equal to the minimum value among all simple cut-sets.

This theorem may appear almost obvious on physical grounds and appears to have been accepted without proof for some time by workers in communication theory. However, while the fact that this flow cannot be exceeded is indeed almost trivial, the fact that it can actually be achieved is by no means obvious. We understand that proofs of the theorem have been given by Ford and Fulkerson¹ and Fulkerson and Dantzig.² The following proof is relatively simple, and we believe different in principle.

To prove first that the minimum cut-set flow cannot be exceeded, consider any given flow pattern and a minimum-valued cut-set C . Take the algebraic sum S of flows from left to right across this cut-set. This is clearly less than or equal to the value V of the cut-set, since the latter would result if all paths from left to right in C were carrying full capacity, and those in the reverse direction were carrying zero. Now add to S the sum of the algebraic flows into all nodes in the right-hand group for the cut-set C . This sum is zero because of the Kirchhoff law constraint at each node. Viewed another way, however, we see that it cancels out each flow contributing to S , and also that each flow on a branch with both ends in the

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¹ L. Ford, Jr. and D. R. Fulkerson, *Can. J. Math.*; to be published.

² G. B. Dantzig and D. R. Fulkerson, "On the Max-Flow Min-Cut Theorem of Networks," in "Linear Inequalities," *Ann. Math. Studies*, no. 38, Princeton, New Jersey, 1956.

right hand group appears with both plus and minus signs and therefore cancels out. The only term left, therefore, which is not cancelled is the flow out of the right hand terminal, that is to say, the total flow F through the network. We conclude, then that $F \leq V$.

We now prove the more interesting positive assertion of the theorem: That a flow pattern can be found which actually achieves the rate V . From any given network with minimum cut-set value V it is possible to construct what we will call a *reduced* network with the properties listed below.

- 1) The graph of the reduced network is the same as that of the original network except possibly that some of the branches of the original network are missing (zero capacity) in the reduced network.
- 2) Every branch in the reduced network has a capacity equal to or less than the corresponding branch of the original network.
- 3) Every branch of the reduced network is in at least one cut-set of value V , and V is the minimum value cut-set for the reduced network.

A reduced network may be constructed as follows. If there is any branch which is not in some minimum cut-set, reduce its capacity until either it is in a minimum cut-set or the value reaches zero. Next, take any other branch not in a minimum cut-set and perform the same operation. Continue in this way until no branches remain which are not in minimum cut-sets. The network then clearly satisfies the condition. In general, there will be many different reduced networks obtainable from a given network depending on the order in which the branches are chosen. If a satisfactory flow pattern can be found for a reduced network, it is clear that the same flow pattern will be satisfactory in the original network, since both the Kirchhoff condition and the capacity limitation will be satisfied. Hence, if we prove the theorem for reduced networks, it will be true in general.

The proof will proceed by an induction on the number of branches. First note that if every path through a reduced network contains only two or less elements, the network is of the form shown typically in Fig. 2. In

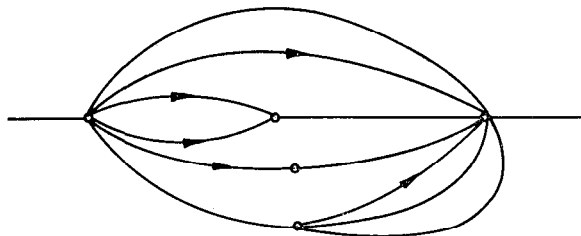


Fig. 2

general, such a network consists of a paralleling of series subnetworks, these series combinations being at most two long with or without arrows from left to right. It is

obvious that for such a reduced network, the theorem is true. It is only necessary to load up each branch to capacity. Now suppose the theorem true for all reduced networks with less than n nodes. We will then show that it is true for any reduced network with n nodes.

Either the given reduced network with n nodes has a path from left to right of length at least three, or it is of the type just described. In the latter case the theorem is true, as mentioned. In the former case, taking the second branch on a path of length three, we have an element running between internal nodes. There exists (since the network is reduced) a minimum cut-set containing this branch. Replace each branch in the cut-set by two branches in series, each with the same capacity as the original branch. Now identify (or join together) all of these newly-formed middle nodes as one single node. The network then becomes a series connection of two simpler networks. Each of these has the same minimum cut-set value V since they each contain a cut-set corresponding to C , and furthermore neither can contain higher-valued cut-sets since the operation of identifying nodes only eliminates and cannot introduce new cut-sets.

Each of the two networks in series contains a number of branches smaller than n . This is evident because of the path of length at least three from the left terminal to the right terminal. This path implies the existence of a branch in the left group which does not appear in the right group and conversely. Thus by inductive assumption, a satisfactory flow pattern with total flow V can be set up in each of these networks. It is clear, then, that when the common connecting node is separated into its original form, the same flow pattern is satisfactory for the original network. This concludes the proof.

It is interesting that in a reduced network each branch is loaded to its full capacity and the direction of flow is determined by any minimum cut-set through a branch. In nonreduced networks there is, in general, some freedom in the amount of flow in branches and even, sometimes, in the direction of flow.

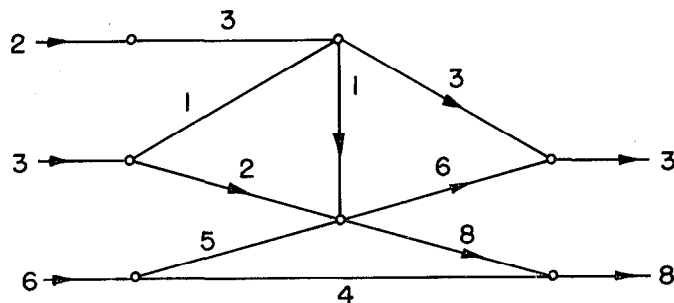


Fig. 3

A more general problem concerning flow through a network can be readily reduced to the above result. Suppose we have a network with a number of input nodes and a number of output nodes as in Fig. 3. The three nodes

on the left are inputs and it is desired to introduce two, three, and six units of flow at these points. The nodes on the right are outputs and it is desired to deliver three and eight units at these points. The problem is to find conditions under which this is possible.

This problem may be reduced to the earlier one by adding a channel for each input to a common left-hand node, the capacity of the channel being equal to the input flow, and also introducing channels from the outputs to a common right-hand node with capacities equal to the output flow. In the particular case this leads to Fig. 4. The

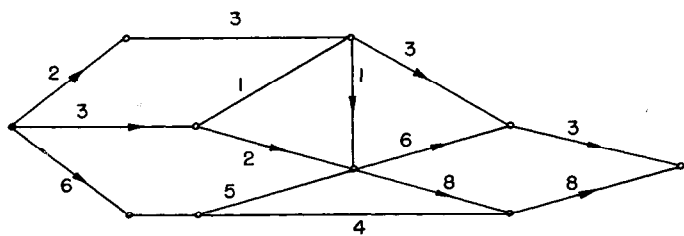


Fig. 4

network obtained in this way from the original problem will be called the *augmented* network.

It is easy to show that necessary and sufficient conditions for solving this multiple input multiple output problem are the following:

- 1) The sum of the input flows must equal the sum of the output flows. Let this sum be C .
- 2) The minimum cut-set in the augmented network must have a value C .

To prove these, note that the necessity of 1 is obvious and that of 2 follows by assuming a flow pattern in the original network satisfying the conditions. This can be translated into a flow pattern in the augmented network, and using the theorem, this implies no cut-set with value less than C . Since there are cut-sets with value C (those through the added branches), the minimum cut-set value is equal to C .

The sufficiency of the conditions follows from noting that 2 implies, using the theorem, that a flow pattern can be set up in the augmented network with C in at the left and out at the right. Now by Kirchhoff's law at the right and left terminals and using condition 1, each added output branch and input branch is carrying a flow equal to that desired. Hence, this flow pattern in the original network solves the problem.

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Rectification of Two Signals in Random Noise*

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Summary—The spectrum of the output of a half-wave rectifier is derived for an input which is the sum of random noise and two sinusoidal signals of different frequencies. The method used is the characteristic function method described by Rice. The components of the output spectrum are given as infinite series of hypergeometric functions. If both the input signals are small compared with the noise, it is shown that the ratio of the output signal power at the difference frequency to the output noise power is proportional to the product of the input signal-to-noise power ratios at the two frequencies. If one of the input signals is very large compared with the noise, it is shown that the other signal and the noise are translated in frequency without alteration of the signal-to-noise ratio. A correction factor is obtained for the case where the large signal is not quite large enough. Finally, the output signal-to-noise ratio of a single-sideband detector is calculated as a function of the input signal-to-noise ratio, when the sideband amplitude is one-half the carrier amplitude.

LIST OF PRINCIPAL SYMBOLS

- P, Q —amplitudes of input signals of frequencies $p/2\pi$, $q/2\pi$.
 $V(t)$ —total input voltage.
 $V_s(t) = P \cos pt + Q \cos qt$ —total input signal.
 $V_N(t)$ —input noise voltage.
 α, ν —rectifier parameters, defined by $I = \alpha V^\nu$ for $V > 0$, $I = 0$ for $V < 0$ where V is the input and I the output.
 $w(f)$ —power spectrum of input noise.
 ψ_0 —input noise power.
 $\psi_\tau = \int_0^\infty w(f) \cos 2\pi f \tau df$ —autocorrelation function of input noise.
 $\Psi(\tau)$ —autocorrelation function of the rectifier output.
 $\Psi_\infty(\tau)$ —autocorrelation function of the portion of the rectifier output with a discrete spectrum.

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