1. (5 points) Consider $X \sim \mathcal{N}(H\theta, \Sigma)$ where $H$ and $\Sigma$ are known. Assume $X$ is $N \times 1$ and $\theta$ is $p \times 1$. Vector calculus can be used to show that the Fisher information matrix is

$$I(\theta) = H^T \Sigma^{-1} H.$$ 

Show that the MLE of $\theta$ is efficient.

2. (5 points) The data $x(n) = Ar^n + w(n)$, $n = 0, \ldots, N-1$, are observed where $w(n)$ is Gaussian white noise with variance $\sigma^2$ and $r > 0$ is known. What is the MVUE? Is it efficient? What happens to the variance as $N \to \infty$ for various values of $r$?

3. (7 points) If $x(n) = r^n + w(n)$, $n = 0, \ldots, N-1$, are observed, where $w(n)$ is Gaussian white noise with variance $\sigma^2$ (known) and $r > 0$ is to be estimated, find the CRLB. Does an efficient estimator exist and if so find its variance?

4. (8 points) Consider the problem of line fitting to noisy observations:

$$x_n = A + Bn + w_n, \quad n = 1, \ldots, N$$

where $w_n \sim N(0, \sigma^2)$. In this case, the parameter is the vector $\theta = [A B]^T$.

a. Determine the Fisher Information Matrix for $\theta$ and the CRLBs for $\hat{A}$ and $\hat{B}$.
b. Compare the CRLB of $\hat{A}$ in this case, with the CRLB for $\hat{A}$ when $B$ is known.

5. (5 points) Extend the bias-variance decomposition of the MSE to a vector parameter.

6. (Challenge-Optional) Consider the setting of problem 1.a on HW 3. Find the CRLB. Does an efficient estimator exist? 

**HINT 1:** If $Z \sim \chi^2_\nu$, then $\mathbb{E}[Z] = \nu$ and $\text{Var}(Z) = 2\nu$. 

**HINT 2:** The sample mean and sample variance of IID Gaussian data are independent. 

**HINT 3:** If $S^2 = 1/(N-1) \sum (x_i - \bar{x})^2$ is the sample variance, then $(N-1)S^2/\sigma^2 \sim \chi^2_{N-1}$. 