:: Solutions ::

ELEC 303 HW 8

(Due 11/16/2010)

Problem 1. Bob moves to a new house and is "fifty-percent sure" that the phone number is 2537267. To verify this, he uses the house phone to dial 2537267, obtains a busy signal, and concludes that it is indeed the right number. Assuming that the probability of a typical seven-digit number being busy at any given time is 1%, what is the probability that Bob's conclusion was correct?

Hypotheses:

H_0 : The phone num is 2537267.

H_1 : The phone num is not " ".

Priors: $P(H_0) = P(H_1) = 0.5$

Let B be the event that a busy signal is obtained. Then,

$$P(B|H_0) = 1 \quad \text{and} \quad P(B|H_0) = 0.01.$$

Probability that Bob's conclusion is correct,

$$P(H_0|B) = \frac{P(B|H_0)P(H_0)}{P(B|H_0)P(H_0) + P(B|H_1)P(H_1)}$$

$$= \frac{(1)(0.5)}{0.5 + (0.01)(0.5)} \cong \boxed{0.99}.$$

Problem 2. We have two boxes, each containing three balls: one black and two white in box 1; two black and 1 white in box 2. We choose one of the boxes at random, where the probability of choosing box 1 is equal to p, and then draw a ball.
(a) Describe the MAP rule for deciding the identity of the box based on whether the drawn ball is black or white.
(b) Assuming that $p = \frac{1}{2}$, find the probability of error if no ball had been drawn.

Hypotheses: H_1 : Box 1 , H_2 : Box 2

Priors : P(choose Box1) = p,
p(choose Box 2) = $1-p$.

Let $X=0$ be the event where ball is black AND $X=1$ be the event where the ball is white.

MAP rule
If $X=1$ (draw A white ball), declare H_1 (box 1) when:

$$p(\text{choose box } 1 \mid x=1) > p(\text{choose box } 2 \mid x=1).$$

$$p(\text{box}1 \mid x=1) = \frac{p(x=1 \mid \text{box}=1)\cdot p(\text{box}=1)}{p(x=1 \mid b=1)\,p(b=1) + p(x=1 \mid b=0)\,p(b=0)}$$

$$= \frac{2/3\,p}{2/3\,p + 1/3(1-p)} = \boxed{\frac{2p}{1+p}}.$$

$p(\text{box } 2 \mid x=1) = ?$

$$= \frac{p(x=1 \mid b=2)\, p(b=2)}{2/3\, p + 1/3(1-p)} = \boxed{\frac{1-p}{1+p}}.$$

Thus for
$X=1$ (white ball),

choose box 1 if:

$$\frac{2p}{1+p} > \frac{1-p}{1+p} \Rightarrow \boxed{p > \frac{1}{3}},$$

else choose box 2.

For $x=0$ (black ball),
choose box 1 if:

$$p(\text{box } 1 \mid x=0) > p(\text{box } 2 \mid x=0)$$

$$\Rightarrow \boxed{p > 2/3},$$

else choose Box 2.

Problem 3. Assume the interarrival times (time between the last arrival and the subsequent arrival) for the Metro at the Rice/Herman stop is exponentially distributed with parameter Θ and the prior PDF of Θ is given by,

$$p_\Theta(\theta) = \begin{cases} 12\theta, & \text{if } 0 \geq \theta \leq 1/6, \\ 0, & \text{otherwise.} \end{cases}$$

Scott has to take the metro and waits 20 minutes for it to come.

(a) Find the posterior distribution, MAP estimate, and conditional expectation estimate of Θ.

(b) Scott decides to record the amount of time that he waits for the Metro for 4 more days in order to obtain a better estimate of Θ. He records the following wait times for the next four days: 40, 15, 30, and 25. He assumes that all of the observed wait times are independent. Find the posterior PDF and the MAP estimate, along with the conditional expectation estimate of Θ given the new data.

Let X denote the wait time.

given AN OBSERVATION, $X=20$, we must compute: $p_{\Theta|X}(\theta|20)$, MAP estimate of Θ, and $E[\Theta|X=20]$.

(i) $$p_{\Theta|X}(\theta|20) = \frac{p_\Theta(\theta)\cdot p_{X|\Theta}(20|\theta)}{\int p_\Theta(\theta')\, p_{X|\Theta}(20|\theta')\,d\theta'}$$

$$= \frac{12\theta\cdot p_{X|\Theta}(20|\theta)}{\int_0^{1/6} 12\theta' p_{X|\Theta}(20|\theta')\,d\theta'}$$

$$= \begin{cases} \dfrac{12\theta\cdot\theta e^{-20\theta}}{12\int_0^{1/6}(\theta')^2 e^{-20\theta'}\,d\theta'} & \text{for } \theta\in[0,1/6], \\ 0 & \text{otherwise.} \end{cases}$$

again the posterior distribution is :

$$P_{\theta|x}(\theta|20) = \begin{cases} \dfrac{\theta^2 e^{-20\theta}}{\int_0^{1/6} (\theta')^2 e^{-20\theta'} d\theta'} , & \theta \in [0, 1/6] \\ 0 , & \text{otherwise}. \end{cases}$$

The MAP estimate of $\hat{\theta}$, selects the value of θ that maximizes the posterior. Thus (since the denominator is a constant) ~~we~~ we must take a derivative of the numerator & set to zero.

$$\frac{d}{d\theta}(\theta^2 e^{-20\theta}) = 2\theta e^{-20\theta} + \theta^2(-20)e^{-20\theta}$$

$$= e^{-20\theta}(2\theta - 20\theta^2) = 0$$

$$\Rightarrow \boxed{\hat{\theta} = \frac{1}{10}}.$$

The conditional expectation estimate:

$$E[\theta | X=20] = \boxed{\frac{\int_0^{1/6} \theta^3 e^{-20\theta} d\theta}{\int_0^{1/6} (\theta')^2 e^{-20\theta'} d\theta'}}.$$

Let X_i denote the wait time for the ith Day, we have observations:

$X = [x_1, \ldots, x_4] = [40, 15, 30, 25]$.

Because all X_i's are assumed to be independent,

$$p_{X|\Theta}(x|\theta) = p_{X_1|\Theta}(x_1|\theta) \cdots p_{X_4|\Theta}(x_4|\theta)$$

$$= \theta^4 e^{-(40+15+30+25)\theta} = \boxed{\theta^4 e^{-110\theta}}.$$

Thus the posterior is given by:

$$p_{\Theta|X}(\theta|x) = \begin{cases} \dfrac{\theta^4 e^{-110\theta} \cdot 12\theta}{12\int_0^{1/6} (\theta')^5 e^{-110\theta'} d\theta'}, & \theta \in [0, 1/6] \\ 0, & \text{otherwise.} \end{cases}$$

MAP rule:

$$\frac{d}{d\theta}\left(\theta^5 e^{-110\theta}\right) = 5\theta^4 e^{-110\theta} + \theta^5(-110)e^{-110\theta} = 0$$

$$\Rightarrow 5\theta^4 - 110\theta^5 = 0$$

$$\Rightarrow \boxed{\hat{\theta} = \frac{1}{22}}.$$

$$\boxed{E[\Theta \mid X = 40, 15, 30, 25] = \frac{\int_0^{1/6} \theta^6 e^{-110\theta} d\theta}{\int_0^{1/6} (\theta')^5 e^{-110\theta'} d\theta'}}$$

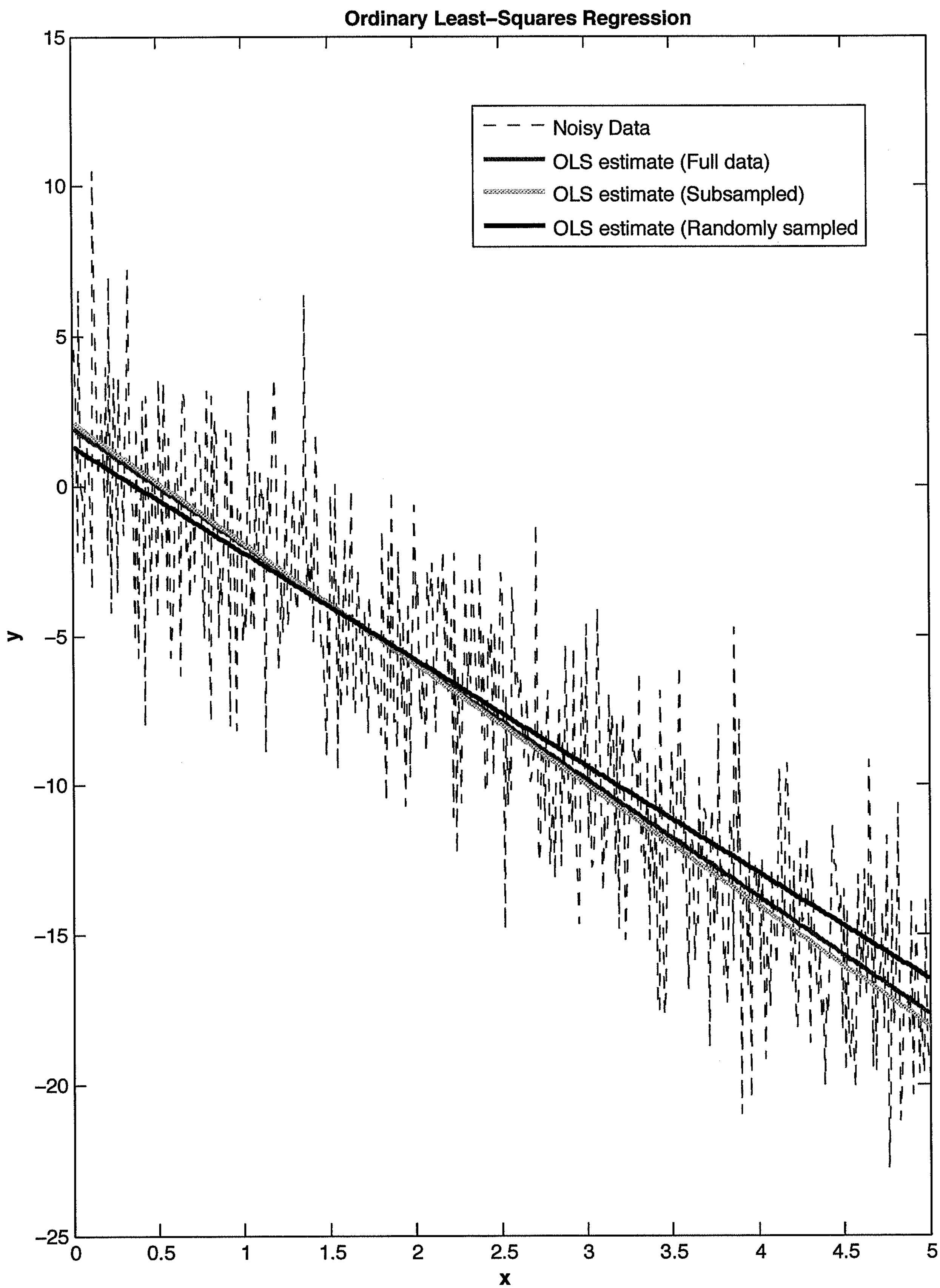
Problem 4 (MATLAB). This problem deals with solving a linear regression problem for a set of observed data. You can find the observed data in the matfile, hw7data.mat, located on the course webpage. To load this data into MATLAB, type **load hw7data** into the command line. After loading the data, you should have two vectors x and y in your workspace, each of size 500×1.

Let $x \in \mathbb{R}^500$ be defined as $x = [0.01, .02, \ldots, 5]$ and let $y \in \mathbb{R}^500$ be a sequence of 500 real-valued observations, where the i^{th} observation can be expressed as $y_i = a_1 + a_2 x_i + \eta_i$, and η_i is a realization of a zero mean Gaussian random variable N with $E[N] = 0$ and $var(N) = \sigma^2$.

(a) Compute the coefficients a_1 and a_2 for: (i) all 500 observations in Y, (ii) 50 uniform samples (downsample Y by a factor of 10), and (iii) 100 observations selected at random from Y.

(b) With the coefficients computed in part (a), find the estimated signal component s, where $s_i = a_1 + a_2 x_i$. Subtract this signal component from the observations y to obtain a residual component $r = y - s$. Compute the ℓ_2-norm (sum of squares) and variance σ^2 of the residual for each of the three cases.

(c) Which estimate is the best and why? Comment on the results obtained in all three cases.


Ordinary Least-Squares Regression
Noisy Data
OLS estimate (Full data)
OLS estimate (Subsampled)
OLS estimate (Randomly sampled
15
10
5
0
−5
−10
−15
−20
−25
y
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
x

```
%% Load Data and Initialize Variables

load HW8data

%% Estimate Slope and Intercept with Ordinary Least Squares

% (a) Using all 500 observations

N = length(y);

Dmat0 = [ones(N,1) x];

aest = pinv(Dmat0)*y;

yest = Dmat0*aest;

% (b) Using 50 uniform samples

newsamp = [1:10:500];

newy = y(newsamp);

Dmat = [ones(50,1) x(newsamp)];

subsamp_aest = pinv(Dmat)*newy;

subsamp_yest = Dmat0*subsamp_aest;

% (c) Using 100 samples selected at random

whichsamp =[]; L = length(whichsamp);
while L<100
    tmp = [whichsamp; ceil(rand((100-L),1)*(N-1))];
    whichsamp = unique(tmp);
    L = length(whichsamp);
end

newy2 = y(whichsamp);

Dmat = [ones(100,1) x(whichsamp)];

random_aest = pinv(Dmat)*newy2;

random_yest = Dmat0*random_aest;

%% Plot Results

figure; plot(x,y,'--'); hold on;
plot(x,yest,'r','LineWidth',2);
```

```
plot(x,subsamp_yest,'g','LineWidth',2);
plot(x,random_yest,'k','LineWidth',2);
legend('Noisy Data', 'OLS estimate (Full data)', 'OLS estimate (Subsampled)', 'OLS
estimate (Randomly sampled')
```