ELEC 303 HW 9
(Due 12/2/2010)

Solutions

fall 2010

1-1

Problem 1. Let A and B be random variables and let the random process be defined as:

$$X(t) = A\cos(2\pi f_c t) + B\sin(2\pi f_c t),$$

where f_c is a constant.

(a) Show that if A and B are uncorrelated and have zero mean and equal variances, then $X(t)$ is WSS.

(b) Find the autocorrelation function of $X(t)$.

(c) Find the power spectral density.

WSS

(1) constant mean

$$E[X(t)] = E[A\cos(2\pi fct)] + E[B\sin(2\pi fct)]$$

$$= E[A]\cos(2\pi fct) + E[B]\sin(2\pi fct)$$

to ensure constant mean,

$$E[A] = E[B] = 0$$

(2) $R_X(\tau)$ autocorrelation only depends on time lag τ.

$$R_X(t_1, t_2)\ E[X(t_1)X(t_2)]$$

$$= E[(A\cos(2\pi fct_1) + B\sin(2\pi fct_1))(A\cos(2\pi fct_2) + B\sin(2\pi fct_2))]$$

$$= E[A^2]\cos(2\pi fct_1)\cos(2\pi fct_2) + E[B^2]\sin(2\pi fct_1)\sin(2\pi fct_2)$$

$$+ E[AB]\sin(\cdot)\cos(\cdot) + E[AB]\sin(\cdot)\cos(\cdot)$$

Using the fact that A, B are uncorrelated, zero mean,

$\Downarrow$ $E[AB]=E[A]E[B]$ 1-2

$=0$.

$R_x(t_1,t_2)=$

$E[A^2]\left(\cos(2\pi f_c t_1)\cos(2\pi f_c t_2)\right)$

$+E[B^2]\left(\sin(2\pi f_c t_1)\sin(2\pi f_c t_2)\right)$

$+\cancel{E[AB]}(\qquad)+\cancel{E[AB]}(\qquad)$

Now, using the fact that $E[A^2]=E[B^2]$ (equal variance),

$$R_x(t_1,t_2)=E[A^2]\left(\cos(2\pi f_c t_1)\cos(2\pi f_c t_2)+\sin(2\pi f_c t_1)\sin(2\pi f_c t_2)\right)$$

$$=E[A^2]\left(\frac{1}{2}\left(\cos(2\pi f_c(t_1-t_2))+\cancel{\cos(2\pi f_c(t_1+t_2))}\right)+\frac{1}{2}\left(\cos(2\pi f_c(t_1-t_2))-\cancel{\cos(2\pi f_c(t_1+t_2))}\right)\right)$$

$$=E[A^2]\cos(2\pi f_c(t_1-t_2)).$$

only depends on $t_1-t_2=\tau$.

$\Rightarrow$ if $E[AB]=E[A]E[B]=0$ & $E[A^2]=E[B^2]$ then $X(t)$ is WSS!

(b) Find $R_x(\tau)$

From part (a),

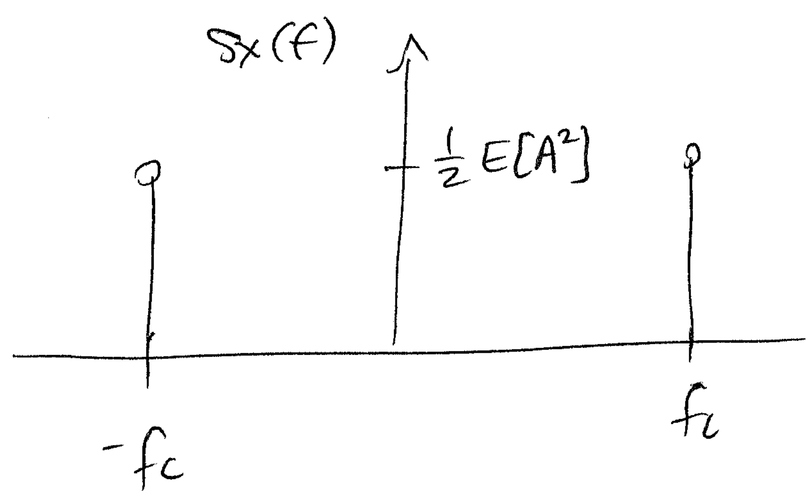
$$R_x(\tau) = E[A^2]\cos(2\pi f_c$$

(c) $S_x(f) \xleftrightarrow{\mathcal{F}} R_x(\tau)$

$$R_x(\tau) = \frac{1}{2}E[A^2]\left(e^{j2\pi f_c\tau} + e^{-j2\pi f_c\tau}\right)$$

$\mathcal{F}$

$$S_x(f) = \frac{1}{2}E[A^2]\left(\delta(f-f_c) + \delta(f+f_c)\right)$$



2-0

Problem 2. A white noise process $X(t)$ with power spectral density $S_X(f) = N_0/2$ is passed through two real LTI systems with transfer functions, $h_Y(t)$ and $h_Z(t)$ as shown below.

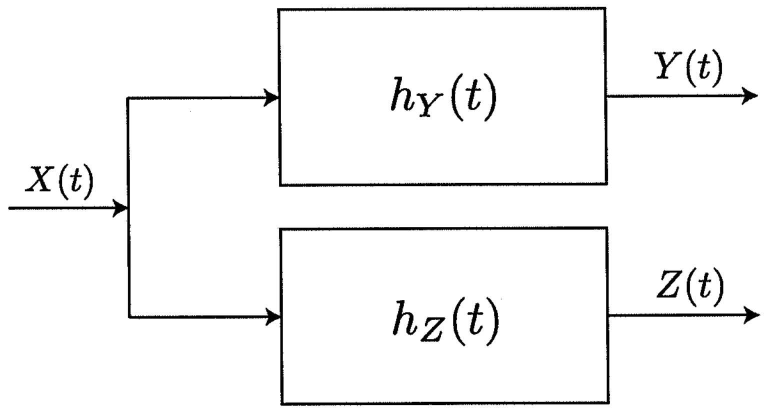


Figure 1: Parallel Filtering of a Noise Process.

As a result of this parallel filtering procedure, random processes $Y(t)$ and $Z(t)$ are obtained. Determine the cross power spectral densities $S_{YZ}(f)$ and $S_{ZY}(f)$.

$$R_{YZ}(t_1,t_2) = E[Y(t_1)Z(t_2)]$$

$$= E[Y(t)Z(t-\tau)], \quad \underline{\tau = t_1 - t_2, \ t_2 = t_1 - \tau}$$

$$= E\left[\left(\int_{-\infty}^{\infty} X(t-s)\, h_Y(s)\, ds\right)\left(\int_{-\infty}^{\infty} X(t-\tau-r)\, h_Z(r)\, dr\right)\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_Y(s)\, h_Z(r)\, \underbrace{E[X(t-s)X(t-\tau-r)]}_{\eta = t-s,\ \gamma = s-\tau-r. \ = E[X(\eta)X(\eta+\gamma)]}\, ds\, dr$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_Y(s)\, h_Z(r)\, R_X(\tau + r - s)\, ds\, dr$$

$$R_{YZ}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_Y(s)\, h_Z(r)\, R_X(\tau + r - s)\, ds\, dr$$

$X(t)$ is a white noise process with
$R_X(\tau) = \frac{N_0}{2}\delta(\tau)$, thus

$$R_{YZ}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_Y(s)\left(\frac{N_0}{2}\delta(\tau + r - s)\right) h_Z(r)\, ds\, dr$$

SPEARE $h_Y(s)$
@ $s = \tau + r$.

$$R_{YZ}(\tau) = \frac{N_0}{2}\int_{-\infty}^{\infty} h_Y(\tau + r)\, h_Z(r)\, dr$$

$\mathcal{F}$ $\updownarrow$

$$S_{YZ}(f) = \frac{N_0}{2}\left(H_Y^*(f)\, H_Z(f)\right)$$

in general, $S_{YZ}(f) = S_X(f)\, H_Y^*(f)\, H_Z(f)$.

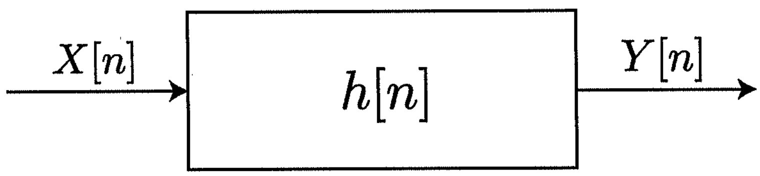
PROBLEM 2

Similarly for

$$S_{ZY}(f) = \underline{S_X(f)\, H_Z^*(f)\, H_Y(f)}.$$

3-0

Problem 3. Let $X[n]$ be white noise with unit power. Suppose that we put $X[n]$ through an LTI filter



whose input-output relationship is described by the difference equation:

$$Y[n] = \frac{1}{3}Y[n-1] + X[n].$$

(a) What is the impulse response $h[n]$ of this LTI system?

(b) Compute the power spectral density of the output, $S_Y(f)$.

(c) Now, we wish to design an inverse whitening filter to reverse the effects of $h[n]$. In other words, we must design a filter $h_I[n]$ that will produce white noise when $Y[n]$ is passed through it. Does such a LTI system exist? If so, what is its impulse response? When does such a LTI system exist in general?

(a) impulse response $h[n] = ?$

$$y[n] - \frac{1}{3} y[n-1] = x[n]$$

Set $x[n] = \delta[n]$ to get impulse resp,

$$\Rightarrow h[n] - \frac{1}{3} h[n-1] = \delta[n].$$

By inspection,

$$h[n] = \left(\frac{1}{3}\right)^n u[n]$$

Alternatively,

$$y[n] - \frac{1}{3} y[n-1] = x[n] \xleftrightarrow{z} Y(z)\left(1 - \frac{1}{3} z^{-1}\right) = X(z)$$

$$\Rightarrow H(z) = \frac{1}{1 - \frac{1}{3} z^{-1}} = \boxed{\frac{z}{z - \frac{1}{3}}}$$

$$\Big\updownarrow z^{-1}$$

$$h[n] = \left(\frac{1}{3}\right)^n u[n]$$

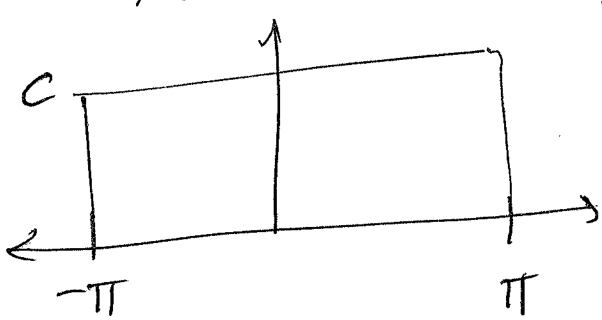
(b) $S_Y(f) = ?$

$$S_Y(f) = H(f)\,H^*(f)\,S_X(f) =$$

$S_X(f) = ?$

unit power for discrete-time RP, $X(t)$,

$S_X(f)$



$$\text{Power} = \int_{-\infty}^{\infty} S_X(f)\,df = \int_{-\pi}^{\pi} c\,df$$

$$= c\cdot 2\pi$$

To ensure unit power,

$$\boxed{c = \frac{1}{2\pi}}.$$

$$\Rightarrow \boxed{S_X(f) = \frac{1}{2\pi}}$$

from part (a),

$$H(z^{-1})\,(z) = \left(\frac{z}{z-\frac{1}{3}}\right)\left(\frac{z^{-1}}{z^{-1}-\frac{1}{3}}\right) = \frac{1}{1-\frac{1}{3}z^{-1}-\frac{1}{3}z+\frac{1}{9}}$$

$$= \frac{1}{\frac{10}{9}-\frac{1}{3}(z+z^{-1})}$$

Now finally,

$$S_Y(f) = S_X(f)\cdot H(z^{-1})H(z)\Big|_{z=e^{j2\pi f}}$$

$$= \frac{1}{2\pi}\left(\frac{1}{10/9 - \frac{1}{3}\left(e^{-j2\pi f} + e^{j2\pi f}\right)}\right)$$

$$= \frac{1}{2\pi}\left(\frac{1}{\frac{10}{9} - \frac{1}{3}\left(2\cos(2\pi f)\right)}\right)$$

$$= \boxed{\frac{1}{2\pi}\left(\frac{1}{\frac{10}{9} - \frac{2}{3}\cos(2\pi f)}\right)}.$$

(c) $H(z) = \dfrac{z}{z-\frac{1}{3}} \Rightarrow \boxed{H^{-1}(z) = \dfrac{z-\frac{1}{3}}{z}}.$

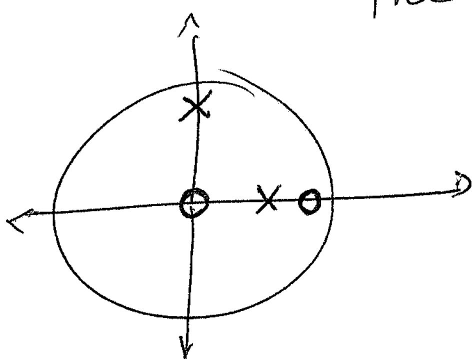
An inverse filter exists as long as:

(1) $\underline{H(z)} \xleftrightarrow{z^{-1}} \underline{h[n]}$ does not equal zero anywhere (else the inverse filter will <u>blow</u> up!)

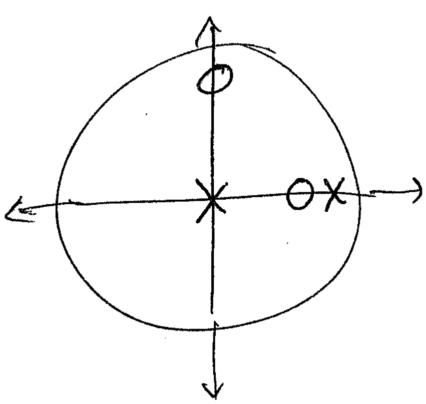
(2) the filter $H(z) \overset{z^{-1}}{\longleftrightarrow} h[n]$ is minimum-phase
[all of its zeros lie w/in the unit circle on the complex PLANE]

Why?

if $H(z)$ has poles AND zeros inside the unit circle,



then $H^{-1}(z)$ (REPLACES each pole in $H(z)$ w/ A zero & each zero w/ A pole)

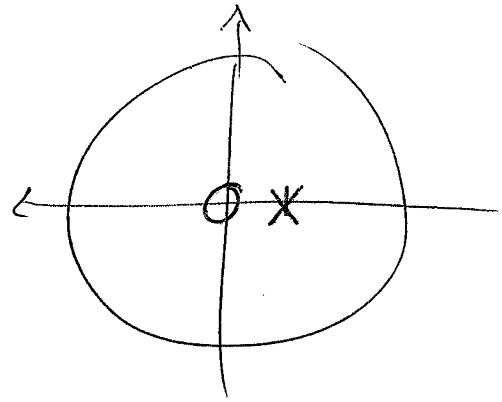


will be STABLE!

- if A zero lies outSIDE of unit circle ($H(z)$) then one of $H^{-1}(z)$ poles will lie outSIDE unit circle
$\Rightarrow H^{-1}(z)$ is UNSTABLE!!

For our problem,

$$H(z) = \underbrace{\frac{z}{z-\frac{1}{3}}}_{\substack{\text{zero at } 0, \\ \text{pole at } \frac{1}{3}}}$$



$H(z)$ is minimum phase

$\Rightarrow H^{-1}(z)$ exists & is STABLE!

$h_I[n] = ?$

$$H^{-1}(z) = \frac{z-\frac{1}{3}}{z} = \frac{1-\frac{1}{3}z^{-1}}{1} = \frac{Y(z)}{X(z)}$$

$$\Rightarrow Y(z) = X(z)\left(1-\frac{1}{3}z^{-1}\right)$$

$$\Rightarrow \boxed{h_I[n] = \delta[n] - \frac{1}{3}\delta[n-1]}.$$